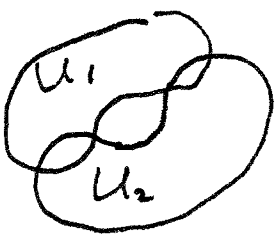
The deRham Isomorphism Theorem for 1-Cohomology

The deRham isomorphism theorem asserts that $H^k_{deRham}(M, \mathbb{R})$ is isomorphic to something else that is more overtly topological-looking. To explain what this something else is, we need some definitions.

Let $\{U_i : i \in \Lambda\}$ be a locally finite open cover of M by connected open sets such that, if $i, j \in \Lambda$, then $U_i \cap U_j$ is either empty or connected.



is not allowed.

Definitions: A __Cech__ __1-cochain__ relative to the cover is an assignment of numbers α_{ij} to each ordered pair (i, j), $i, j \in \Lambda$, for which $U_i \cap U_j \neq \phi$ satisfying $\alpha_{ij} = -\alpha_{ji}$.

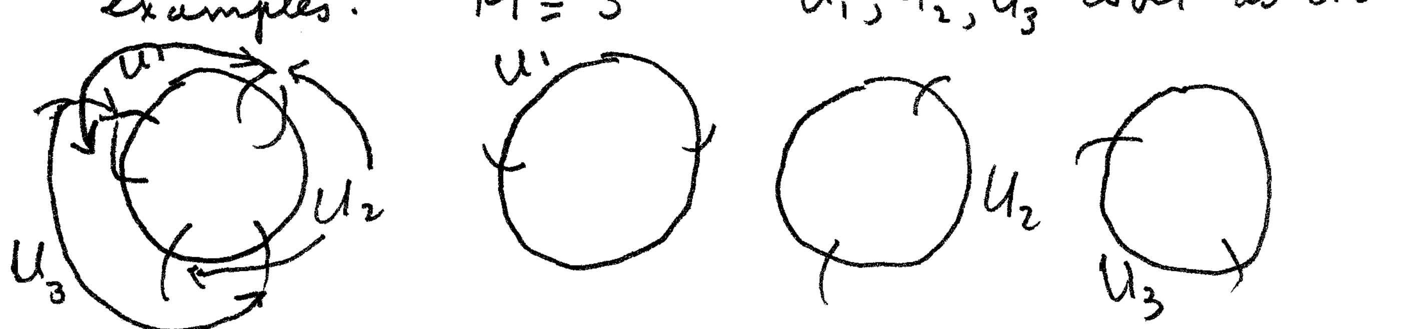
A __Cech__ __1-cocycle__ is a Cech 1-cochain satisfying, for all $i, j, k \in \Lambda$ such that $U_i \cap U_j \cap U_k \neq \phi$: $\alpha_{ij} + \alpha_{jk} + \alpha_{ki} = 0$. A __Cech__ __1-coboundary__ is a Cech 1-cochain $\{\alpha_{ij}\}$ such that $\exists$ numbers β_i, $i \in \Lambda$, satisfying $\alpha_{ij} = \beta_i - \beta_j$ for $\forall\, i, j \in \Lambda \ni U_i \cap U_j \neq \phi$.

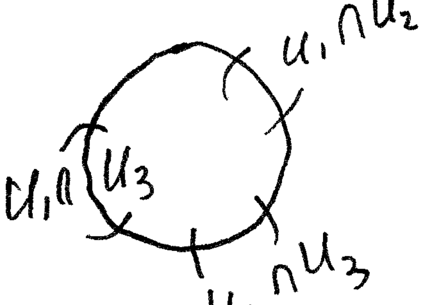
Note that all 1-coboundaries are automatically 1-cocycles.

The Cech 1-cohomology of M relative to the cover $\{U_i\}$

$$= \frac{\text{the vector space of 1-cocycles}}{\text{the vector space of 1-coboundaries}}.$$

To put this in some visualizable context, we give two examples: $M = S^1$ $\quad$ U_1, U_2, U_3 cover as shown





triple intersection empty (so "cocycle condition" $\alpha_{ij} + \alpha_{jk} + \alpha_{ki} = 0$ when $U_i \cap U_j \cap U_k \neq \phi$ never happens — the condition always satisfied "vacuously")

All 1-cochains are cocycles

$\alpha_{12}, \alpha_{23}, \alpha_{13}$ arbitrary

$\alpha_{21} = -\alpha_{12}$ $\quad \alpha_{32} = -\alpha_{23}$ $\quad \alpha_{31} = -\alpha_{13}$.

When is a 1-cocycle $\{\alpha_{ij}\}$ a coboundary?

Being a coboundary is the same as solving for $\beta_1, \beta_2, \beta_3$:

$$\beta_1 - \beta_2 = \alpha_{12} \qquad \beta_2 - \beta_3 = \alpha_{23} \qquad \beta_3 - \beta_1 = \alpha_{31}$$

Linear algebra shows this is solvable if and only if $\alpha_{12} + \alpha_{23} + \alpha_{31} = 0$. Note that this is not automatically true since $\alpha_{12}, \alpha_{23}, \alpha_{13} (= -\alpha_{31})$ are arbitrary — they can be any three numbers.

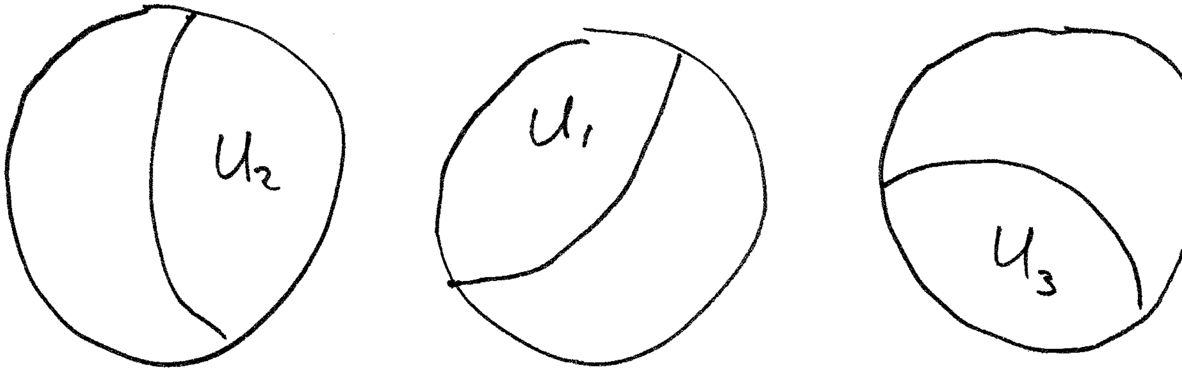
So $\dfrac{\text{1-cocycle}}{\text{1-coboundaries}}$ is $\dfrac{\mathbb{R}^3}{\mathbb{R}^2}$ (set of points of $\mathbb{R}^3$: the three $\{\alpha_{12}, \alpha_{23}, \alpha_{31}\}$; $\mathbb{R}^2$ = kernel of $(\alpha_{12}, \alpha_{23}, \alpha_{31}) \to \alpha_{12} + \alpha_{23} + \alpha_{31}$, i.e. $(\alpha_{12}, \alpha_{23}, \alpha_{31}) \ni \alpha_{12} + \alpha_{23} + \alpha_{31} = 0$)

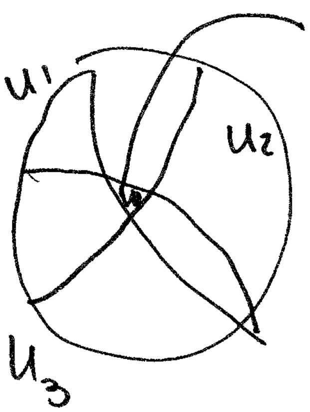
that is

$$\text{Čech 1-cohomology} \cong \mathbb{R}$$

the map being $[(\alpha_{12}, \alpha_{23}, \alpha_{13})] \to \alpha_{12} + \alpha_{23} - \alpha_{13}$

This should be contrasted with the cover of the open disc (M = disc) shown



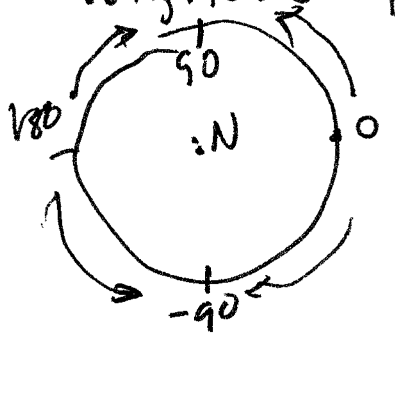


$U_1 \cap U_2 \cap U_3 \neq \phi$. Here α_{12} α_{23} α_{13} cannot be chosen arbitrarily for a cocycle. Because $U_1 \cap U_2 \cap U_3 \neq \phi$, they have to be, if $\{\alpha_{ij}\}$ is a cocycle, solutions of $\alpha_{12} + \alpha_{23} + \alpha_{31}$ $(= \alpha_{12} + \alpha_{23} - \alpha_{13}) = 0$.

So in this case 1-cocycle $\Leftrightarrow$ 1 coboundary (see linear algebra of first example) and

Cech 1-cohomology of disc relative to this cover $= 0$.

The determined(!) reader is invited to examine the situation of $M = S^2$ cover by northern, southern hemispheres, eastern (long. 0 to 180) and western (0 to −180) hemispheres, and two more hemispheres: longitude −90 to +90, or longitude +90 to −90 (opposite one)



As shown (for last two) viewed from north pole.

Clearly, the combinatorics and linear algebra here can become complicated.

For certain covers (which always exist), there is a simplification of the situation in that Cech cohomology for the cover is necessarily isomorphic to deRham cohomology: (9)

deRham Isomorphism Theorem for 1-cohomology:

Suppose the cover $\{U_i, i \in \Lambda\}$ satisfies (in addition to U_i and $U_i \cap U_j$ being connected as usual):
$H^1_{\text{deRham}}(U_i, \mathbb{R}) = 0$ for all $i \in \Lambda$. Then

$$H^1_{\text{deRham}}(M, \mathbb{R}) \cong \text{Cech 1-cohomology of } M \text{ relative to the cover } \{U_i, i \in \Lambda\}.$$

Recall that $H^1_{\text{deR}}(U_i, \mathbb{R}) = 0$ if for example each U_i is simply connected. Thus we have at least some topologically characterized coverings for which the isomorphism holds.

Proof of Theorem: We construct a map $H^1_{\text{deRham}}(M, \mathbb{R})$ into Cech 1-cohomology for the cover as follows:
Given a closed 1-form ω, choose functions $f_i : U_i \to \mathbb{R}$ with $df_i = \omega|_{U_i}$. (This is possible since $H^1_{\text{deR}}(U_i, \mathbb{R}) = 0$ by hypothesis). Then set α_{ij} (if $U_i \cap U_j \neq \phi$) to be the constant value of $f_i - f_j$ on $U_i \cap U_j$. Note that $f_i - f_j$, which is a priori a function, is in fact a constant on $U_i \cap U_j$ since $d(f_i - f_j) \equiv \omega|_{U_i} - \omega|_{U_j} \equiv 0$ on $U_i \cap U_j$

and, because $U_i \cap U_j$ is connected $f_i - f_j$ is thus constant.
So $\alpha_{ij} = f_i - f_j$ really is a number, not a function!

Clearly, if $U_i \cap U_j \cap U_k \neq \phi$, then $\alpha_{ij} + \alpha_{jk} + \alpha_{ki} = f_i - f_j + f_j - f_k + f_k - f_i = 0$. So $\{\alpha_{ij}\}$ is a 1-cocycle. Note, however, that it may not be a coboundary! Even though $\alpha_{ij} = f_i - f_j$, f_i, f_j functions, there need not be numbers β_i such that $\alpha_{ij} = \beta_i - \beta_j$. [Important point to note here!].

So we send $\omega \longrightarrow [\{\alpha_{ij}\}] \in$ Cech 1-cohomology — or at least we would like to! But we have to check that $[\{\alpha_{ij}\}]$ does not depend on our choice of the f_i's satisfying $df_i = \omega|_{U_i}$. Now any other f's satisfying this have the form $f_i + c_i$ $c_i \in \mathbb{R}$ since U_i is connected. Then $\hat{\alpha}_{ij} = (f_i + c_i) - (f_j + c_j) = \alpha_{ij} + c_i - c_j$ where $\hat{\alpha}_{ij}$ denotes what we would get from our construction if we used $f_i + c_i$'s in place of the original f_i's. But note that $\{\hat{\alpha}_{ij}\}$ and $\{\alpha_{ij}\}$ differ by a Cech coboundary! So $[\{\hat{\alpha}_{ij}\}] = [\{\alpha_{ij}\}]$ even though $\{\hat{\alpha}_{ij}\}$ is not the same as $\{\alpha_{ij}\}$. They are different but are the same "mod coboundaries".

We are hoping that $\omega \to [\{\alpha_{ij}\}]$ will induce a map on $H^1_{deR}(M, \mathbb{R})$. For this, we need that $[\hat{\omega}_1] = [\omega] \Rightarrow$ the image of $\hat{\omega}_1$ under our construction is the same Cech 1-cohomology class of the image of ω. Now $\hat{\omega}_1 = \omega + df$. So if we have f_i such that $df_i = \omega|_{U_i}$ then $F_i = f_i + f$ have $dF_i = \hat{\omega}|_{U_i}$. So F_i work for $\hat{\omega}$. But $F_i - F_j = f_i - f_j$!

So we have a well defined map from de Rham 1-cohomology classes to Čech 1-cohomology classes relative to the cover $\{U_i\}$.

We shall show that this map is an isomorphism by constructing an inverse for it. For this, suppose we are given a 1-cocycle α_{ij}. Choose a partition of unity $\{\rho_i\}$ subordinate to the cover $\{U_i\}$ and define $f_i : U_i \to \mathbb{R}$ by

$$f_i = \sum_{\ell \in \Lambda} \rho_\ell \alpha_{\ell i}.$$

Then, on $U_i \cap U_j \neq \emptyset$, $f_i - f_j = \sum \rho_\ell \alpha_{\ell i} - \sum \rho_\ell \alpha_{\ell j}$

$= -\sum \rho_\ell \alpha_{i\ell} - \sum \rho_\ell \alpha_{\ell j} = \sum \rho_\ell (-\alpha_{i\ell} - \alpha_{\ell j})$

$= \sum \rho_\ell \alpha_{ij} = \alpha_{ij}$. In particular, $df_i \equiv df_j$ on $U_i \cap U_j$. So the df_i's together give a global 1-form on M. Clearly this form is closed, since locally it is d of some function. We leave as an exercise that: (a) If $[\cup(df_i)]$ is mapped as earlier to a Čech 1-cocycle, this Čech 1-cocycle is (in the same Čech cohomology as) $\{\alpha_{ij}\}$ and (b) If $[\{\hat{\alpha}_{ij}\}] = [\{\alpha_{ij}\}]$ (Čech cohomology classes) then $[\hat{\omega}] = [\omega]$ (deRham cohomology classes)

[Outline for (b): $d(\sum_\ell \rho_\ell (\beta_\ell - \beta_i)) = d(\sum_\ell \rho_\ell \beta_\ell) - d(\sum_\ell \rho_\ell \beta_i)$
$= d(\sum \rho_\ell \beta_\ell)$ since $\sum \rho_\ell \beta_i = \beta_i$ constant and $\sum \rho_\ell \beta_\ell$ is global func.]

Thus deRham isomorphism (at the $k=1$ level) is established. $\square$

It is not hard to show (we'll do it later) that each differentiable manifold M has a "good" open cover $\{U_i : i \in \Lambda\}$ in the sense that the cover is locally finite, each U_i is connected and simply connected, and each nonempty $U_i \cap U_j$ is connected. Note that if $H: M_1 \to M_2$ is a homeomorphism (but not necessarily a diffeomorphism) of one differentiable manifold to another and is $\{U_i, i \in \Lambda\}$ is such a cover of M_1, then $\{H(U_i) : i \in \Lambda\}$ is such a cover of M_2. Moreover, the Čech 1-cohomology of the two covers is the same, since it depends only on which intersections are nonempty, and this is obviously the same for the two covers. The deRham Isomorphism Theorem thus implies immediately that

$$H^1_{deR}(M_1, \mathbb{R}) \cong H^1_{deR}(M_2, \mathbb{R}).$$

So deRham k cohomology is topological for $k = 1$. This is actually true for all k. Similar methods apply but further work is needed. (For one thing, simple connectivity has to be replaced by something else. Also the Čech cohomology idea has to be extended beyond the 1-level). Later!