Another Example of the Poincare Lemma ~~Proof~~ via the $\mathcal{L}_X = d\, i_X + i_X\, d$ Method

On $\mathbb{R}^2$, let $\omega = P\, dx + Q\, dy$ $\quad P, Q : \mathbb{R}^2 \to \mathbb{R}$

with $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ ($\Leftrightarrow$ ω closed).

Contradiction: $F_t((x,y)) = (tx, ty)$ $\quad F_1 = $ identity

$\underline{\text{notation}}$ $H((x,y),t)$ $\quad F_0 = $ cons. map to $(0,0)$

$i_{\frac{\partial}{\partial t}}(H^* \omega)$ is what we need to compute since we want to reason that $\mathcal{L}_{\frac{\partial}{\partial t}} F_t^* \omega =$ dx, dy terms of $\mathcal{L}_{\frac{\partial}{\partial t}} H^* \omega$

and $\mathcal{L}_{\frac{\partial}{\partial t}} H^* \omega = d\left(i_{\frac{\partial}{\partial t}} H^* \omega\right) + i_{\frac{\partial}{\partial t}}\big(\underbrace{d(H^*\omega)}_{H^* d\omega = 0}\big)$.

Now $H_* \frac{\partial}{\partial t}\Big|_{(x,y,t)} = \frac{d}{dt}(xt, ty) = (x,y)$ or in our usual notation

$$H_* \frac{\partial}{\partial t}\Big|_{(x,y,t)} = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$$

since ω is a 1-form i is just substituting in

$$\begin{aligned} \text{Hence } i_{\frac{\partial}{\partial t}}(H^* \omega)\Big|_{(x,y,t)} &= \omega\left(H_* \frac{\partial}{\partial t}\right)\Big|_{H(x,y,t)} \\ &= (P\,dx + Q\,dy)\left(x\frac{\partial}{\partial x} + y \frac{\partial}{\partial y}\right)\Big|_{H(x,y,t)} \\ &= x P\Big|_{H(x,y,t)} + y Q\Big|_{H(x,y,t)} \\ &= x\, P\Big|_{(tx,ty)} + y Q\Big|_{(tx,ty)} \end{aligned}$$

So we are looking hoping for $\omega|_{(x,y)} = (\omega - 0)|_{(x,y)} = F_1^* \omega|_{(x,y)} - F_0^* \omega|_{(x,y)}$

$$= \int_0^1 \mathcal{L}_{\frac{\partial}{\partial t}} H^* \omega \;(dx, dy \text{ part only}) = \int_0^1 d_{\mathbb{R}^2}\left(i_{\frac{\partial}{\partial t}} H^* \omega\right)$$

$$= d_{\mathbb{R}^2}\left(\int_0^1 (x P|_{(tx,ty)} + y Q|_{(tx,ty)}\, dt\right).$$

The first thing to note is that if we already knew that $\omega = df$ for some f then we would have

$$\int_0^1 x\, P|_{(tx,ty)} + y\, Q|_{(tx,ty)}\, dt = \int_0^1 \left(x \frac{\partial f}{\partial x}\Big|_{(tx,ty)} + y \frac{\partial f}{\partial y}\Big|_{(tx,ty)}\right) dt$$

$$= \int_0^1 \frac{d}{dt}\big(f(tx,ty)\big)\, dt = f(x,y) - f(0,0)$$

and the whole thing would work: $d(f(x,y) - f(0,0)) = df(x,y) = \omega$. So if $\omega = df$, we are getting the (a) right answer here!

But of course we really want to prove that

$$d_{\mathbb{R}^2}\left(\int_0^1 x P|_{(tx,ty)} + y\, Q|_{(tx,ty)}\, dt\right) = \omega$$ without assuming

$\omega = df$ for some f in advance! (We do not want to do circular reasoning though of course it is reassuring that it does work in the case we are given $\omega = df$).

For this, we need to compute that $\frac{\partial}{\partial x}\left(\int_0^1 (x P|_{(tx,ty)} + y\, Q|_{(tx,ty)})\, dt\right) = P|_{(x,y)}$ and $\frac{\partial}{\partial y}(\int\ ) = Q$. Let us try $\frac{\partial}{\partial x}$:

$$\frac{\partial}{\partial x}\left(\int_0^1\ \right) = \int_0^1 \frac{\partial}{\partial x}\left(x P|_{(tx,ty)} + y\, Q|_{(tx,ty)}\right) dt$$

$$= \int_0^1 P|_{(tx,ty)} + \int_0^1 \left(x \frac{\partial}{\partial x}(P|_{(tx,ty)}) + y \frac{\partial}{\partial x}(Q|_{(tx,ty)})\right) dt$$

$$= \int_0^1 P|_{(tx,ty)} + \int_0^1 x\, t \left(\frac{\partial P}{\partial x}\Big|_{(tx,ty)}\right) + y\, t\left(\frac{\partial Q}{\partial x}\Big|_{(tx,ty)}\right) dt$$

(now using $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$)

$$= \int_0^1 P|_{(tx,ty)} + \int_0^1 t \underbrace{\left(x \frac{\partial P}{\partial x}\Big|_{(tx,ty)} + y \frac{\partial P}{\partial y}\Big|_{(tx,ty)}\right)} dt$$

$$= \int_0^1 P|_{(tx,ty)} + \int_0^1 t\, \frac{\partial P}{\partial t}\Big|_{(tx,ty)}$$

Now we integrate by parts

$$\int_0^1 t \frac{\partial P}{\partial t}\Big|_{(tx,ty)} dt = t P|_{(tx,ty)} \Big]_{t=0}^{t=1} - \int P|_{(tx,ty)} dt$$

$$= P(x,y) - \int_0^1 P|_{(tx,ty)} dt$$

Substituting this in, we get

$$\frac{\partial}{\partial x}\left(\int_0^1 (xP|_{(tx,ty)} + yQ|_{(tx,ty)})\, dt\right)$$

$$= \int_0^1 P|_{(tx,ty)} + P(x,y) - \int_0^1 P|_{(tx,ty)} dt$$

$$= P(x,y).$$ It worked!

Things get fairly complicated even in this simple example, which is almost the simplest possible nontrivial example. This shows (I think) why the rather abstract and formal $\mathcal{L}_{\partial/\partial t}$ process is needed. To do these computations concretely in more complicated cases would be an intimidating process. The formalism makes things much simpler, since it isolates in advance the role of the form ω being closed: it makes one term of $i_X d + d i_X = 0$!