Examples of first order linear equations applied

($y' + p(x)y = g(x)$. Solution: $Ce^{-\int p} + e^{-\int p}\int e^{\int p} g$. Last time!)

Falling body with air resistance (assumed proportional to velocity). Equation $x''(t) = -g - C_R x'(t)$, t time* independent variable, $x(t)$ = height at time t, dependent variable

Write $v(t) = x'(t)$. Equation becomes $v'(t) = -g - C_R v$.

$v' + C_R v = -g$

$p = C_R \quad \int p = C_R t$

Solution: $v(t) = c e^{-C_R t} + e^{-C_R t}\int_0^t e^{C_R s}(-g)\,ds$

$$= c e^{-C_R t} + \frac{1}{C_R} e^{C_R t}(-g)e^{-C_R t} + \frac{g}{C_R}e^{-C_R t}$$

Given specified $v(0)$,

solution is $v(t) = -\frac{g}{C_R} + e^{-C_R t}\left(\frac{g}{C_R}\right) + v(0)e^{-C_R t}$

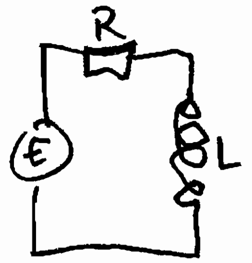
[Note that $(e^{-C_R t})' + C_R e^{-C_R t} = 0$ while $\left(-\frac{g}{C_R}\right)' + C\left(-\frac{g}{C_R}\right) = -g$

so this really does solve the equation $v(t)' + C v(t) = -g$ and it does have value $V(0)$ when $t = 0$!].

As $t \to \infty$, v converges to $\frac{-g}{C_R}$ "terminal velocity" — the minus sign means the fall is down, not up!

* C_R = constant of proportionality for air resistance.

Example 2:



E voltage source $E(t)$ varies with time

resistor R, inductor L

Equation for current $I(t)$

$$L\,I'(t) + R\,I(t) = E(t).$$

Solution: See "Lecture I" at www.math.ucla.edu/~greene under Math 135, Spring 2010.

Involved in this solution is finding (in case $E(t) = \sin \omega t$) the integral $\int_0^t e^{kx} \sin \omega x \, dx$ $\quad k = R/L$

For this, use complex exponentials:

$e^{i\omega x} = \cos \omega x + i \sin \omega x$ so $\sin \omega x = \operatorname{Im} e^{i\omega x}$

so $$\int_0^t e^{kx} \sin \omega x \, dx = \operatorname{Im} \int_0^t e^{kx + i\omega x} dx$$

$$= \operatorname{Im} \int_0^t e^{(k+i\omega)x} dx = \operatorname{Im} \left. \frac{1}{k+i\omega} e^{(k+i\omega)x} \right]_0^t$$

$$\frac{1}{k+i\omega} e^{(k+i\omega)x} = \frac{k - i\omega}{k^2+\omega^2} e^{kx} (\cos \omega x + i \sin \omega x)$$

so Im part $= \dfrac{e^{kx}}{k^2+\omega^2} (k \sin \omega x - \omega \cos \omega x)$.

Note that this really works

$$\left[e^{kx} (k \sin \omega x - \omega \cos \omega x) \right]' =$$

$$k e^{kx}(k \sin \omega x - \omega \cos \omega x)$$

$$+ e^{kx}(\omega k \cos \omega x + \omega^2 \sin \omega x) = e^{kx}(k^2+\omega^2)\sin \omega x$$

(cosine terms cancel!). So

$$\left[\frac{1}{k^2+\omega^2} e^{kx}(k \sin \omega x - \omega \cos \omega x)\right]' = e^{kx} \sin \omega x$$

as required.

Be sure to get used to this complex exponential stuff. It will turn up often!

"Undetermined coefficients" idea: If we wanted to solve $I'(t) + \frac{R}{L} I(t) = \frac{1}{L} E(t)$, with $E(t) = e^{i\omega t}$, we might try to look for (complex) solutions of the form $I(t) = C e^{i\omega t}$, C a complex number. Then $I'(t) = C i \omega e^{i\omega t}$ and we would want $C i \omega e^{i\omega t} + k C e^{i\omega t} = \frac{1}{L} e^{i\omega t}$ (where $k = \frac{R}{L}$ as before). So we would need

$$C(k + i\omega) = 1/L \quad \text{or}$$

$$C = \frac{1}{L} \cdot \frac{1}{k+i\omega} = \frac{1}{L} \frac{k-i\omega}{k^2+\omega^2}$$

To get the case $E(t) = \sin \omega t$, we would just take imaginary parts:

Im (complex solution) with righthand side $= \frac{1}{L} e^{i\omega t}$ will give a real solution of

$$I' + \frac{R}{L} I = \frac{1}{L} \sin \omega t$$

since the equation has real coefficients and $\sin \omega t = \text{Im}\, e^{i\omega t}$

[Think about this point!].

So $E(t) = \sin \omega t$ case has a solution

$$\text{Im}\left(\frac{1}{L} \frac{k - i\omega}{k^2 + \omega^2} e^{i\omega t} \right) = \frac{1}{L(k^2+\omega^2)} \text{Im}\left((k - i\omega)(\cos \omega t + i \sin \omega t \right)$$

$$= \frac{1}{L(k^2+\omega^2)} (k \sin \omega t - \omega \cos \omega t)$$

as before. To get the "general solution" we take this one and add $C \cdot (\text{solution of } I' + \frac{R}{L} I = 0)$

namely $C e^{-kt}$

So general solution is

$$I(t) = C e^{-\frac{R}{L}t} + \frac{1}{L\left(\frac{R^2}{L^2} + \omega^2\right)} (k \sin \omega t - \omega \cos \omega t)$$

$$= C e^{-\frac{R}{L}t} + \frac{1}{\sqrt{R^2 + L^2 \omega^2}} \sin(\omega t - \phi)$$

for suitable choice of ϕ. Choosing C enables one to specific $I(0)$.

Details of trigonometry

$$\frac{1}{L}\frac{1}{k^2+\omega^2}\left(k\sin\omega t - \omega\cos\omega t\right)$$

$$= \frac{1}{L}\frac{1}{\sqrt{k^2+\omega^2}}\left(\frac{k}{\sqrt{k^2+\omega^2}}\sin\omega t - \frac{\omega}{\sqrt{k^2+\omega^2}}\cos\omega t\right)$$

$$= \frac{1}{L}\frac{1}{\sqrt{\frac{R^2}{L^2}+\omega^2}}\left(\sin(\omega t-\phi)\right)$$

$$= \frac{1}{\sqrt{R^2+L^2\omega^2}}\sin(\omega t-\phi)$$

if ϕ is chosen so that

$$\cos\phi = \frac{k}{\sqrt{k^2+\omega^2}} \quad \text{and} \quad \sin\phi = \frac{\omega}{\sqrt{k^2+\omega^2}}.$$

This uses $\sin(\omega t-\phi) = \sin\omega t\cos\phi - \cos\omega t\sin\phi$.

The choice of ϕ is possible because

$$\left(\frac{k}{\sqrt{k^2+\omega^2}}\right)^2 + \left(\frac{\omega}{\sqrt{k^2+\omega^2}}\right)^2 = 1 \quad !$$