Examples of $\oint_\gamma \frac{\partial f}{\partial z}\,dz$ = change in $f \overset{\text{notation}}{=} \Delta f$
= f(end) − f(beginning)

(1) $f(z)=z \qquad \frac{\partial f}{\partial z}=1 \qquad \left(\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)(x+iy) = \frac{1}{2}(1-i^2\cdot 1) + i(0+0) = 1 \right).$

$$\oint 1\,dz = \oint 1\cdot dx + i\,dy = \Delta x + i\,\Delta y = \Delta f$$

(2) $f(z)=z^2 \qquad \frac{\partial f}{\partial z} = 2z$

$$\oint 2z\,dz = \oint 2(x+iy)(dx+i\,dy)$$
$$= \oint 2x\,dx + i^2\,2y\,dy \quad + 2i\oint y\,dx + x\,dy$$
$$= \oint d(x^2-y^2) + 2i\oint d(xy) = \Delta(x^2-y^2) + i\Delta(2xy)$$
$$= \Delta\,(x^2-y^2+2ixy) = \Delta\,(x+iy)^2 = \Delta f$$

(3) $f(z)=\frac{1}{z} \qquad f(x+iy) = \frac{1}{x+iy} = \frac{\overset{u}{x}}{x^2+y^2} - \frac{\overset{v}{iy}}{x^2+y^2}$

$\frac{\partial\left(\frac{1}{z}\right)}{\partial z} = -\frac{1}{z^2} \qquad$ (Real check: $\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \left(\frac{(x^2+y^2)\cdot 1 - x\,(2x)}{(x^2+y^2)^2}\right.$

$\frac{\partial}{\partial x}\left(\frac{-y}{x^2+y^2}\right) = \frac{+y\cdot 2x}{(x^2+y^2)^2} \qquad \frac{\partial f}{\partial z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{2xyi}{(x^2+y^2)^2}$

$$= \frac{-1}{(x^2-y^2)+2ixy} = -1/z^2$$

since $\frac{1}{(x^2-y^2)+2ixy} = \frac{(x^2-y^2)-2xyi}{(x^2-y^2)^2+2^2x^2y^2} = \frac{(x^2-y^2)}{(x^2+y^2)^2} - \frac{2xyi}{(x^2+y^2)^2}$

$\hookrightarrow x^4-2x^2y^2+y^4+4x^2y^2 =$

So $\oint -\frac{1}{z^2}\,dz = \oint\left[\frac{y^2-x^2}{(x^2+y^2)^2} + \frac{2xyi}{(x^2+y^2)^2}\right](dx+i\,dy)$

$$= \oint\left[\frac{y^2-x^2}{(x^2+y^2)^2}dx + \frac{2xy(-1)}{(x^2+y^2)^2}dy\right] + i\oint\left[\frac{2xy}{(x^2+y^2)^2}dx + \frac{y^2-x^2}{(x^2+y^2)^2}dy\right]$$
$$= \oint d\left(\frac{x}{x^2+y^2}\right) + i\oint d\left(\frac{-y}{x^2+y^2}\right) = \Delta\left(\frac{x}{x^2+y^2}\right) - i\,\Delta\left(\frac{y}{x^2+y^2}\right) = \Delta\left(\frac{1}{z}\right)$$