Gauss Bonnet Theorem:

A Worked out example:

Recall from homework that if $(x(s), y(s))$ is an arclength parameter curve then the surface obtained by rotation around the x axis, namely

$$S(s,\theta) = (x(s),\ y(s)\cos\theta,\ y(s)\sin\theta)$$

has Gauss curvature $-y''(s)/y(s)$.

Now

$$\frac{dy}{dx} = \frac{dy}{ds}\cdot\frac{ds}{dx} = \frac{dy}{ds}\sqrt{1+(y')^2} \quad \text{where } y' = \frac{dy}{dx}$$

where we consider y now as a function of x (as well as of s: we are using that $\frac{dx(s)}{ds} > 0$ by assumption). So applying this to dy/ds) we get

$$\frac{d}{ds}\left(\frac{dy}{ds}\right) = \frac{d}{ds}\left(\frac{dy}{dx}\Big/\sqrt{1+(y')^2}\right)$$

$$= \frac{1}{\sqrt{1+(y')^2}}\frac{d}{dx}\left(\frac{dy}{dx}\Big/\sqrt{1+(y')^2}\right)$$

$$= \frac{1}{\sqrt{1+(y')^2}}\left(\frac{1}{\sqrt{1+(y')^2}}\frac{d^2y}{dx^2} - \left(\frac{1}{\sqrt{1+(y')^2}}\right)^3\left(\frac{dy}{dx}\right)^2\frac{d^2y}{dx^2}\right)$$

$$= \frac{1}{(1+(y')^2)^2}\cdot\left((1+(y')^2)\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2\frac{d^2y}{dx^2}\right)$$

$$= \frac{d^2y}{dx^2}\Big/\left(\sqrt{1+\left(\frac{dy}{dx}\right)^2}\right)^2$$

(we are using $\frac{dy}{dx} = y'$: just notation)

So Gauss curvature of $S = -\dfrac{\frac{d^2y}{dx^2}}{\left(1+\left(\frac{dy}{dx}\right)^2\right)^2 y(x)}$

Now recall from calculus that on a surface of revolution

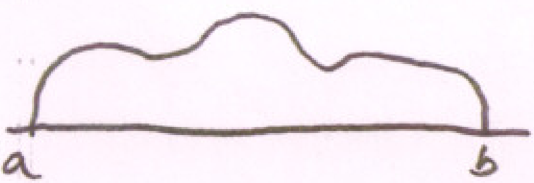
$$d(\text{area}) = y(x)\, ds\, d\theta$$
$$= y(x)\sqrt{1+(y')^2}\, dx\, d\theta$$

Hence

$$\int \text{Gauss curvature} = \int_0^{2\pi} \int_a^b -\frac{y''}{y(1+(y')^2)^2}\, y\sqrt{1+(y')^2}\, dx\, d\theta$$

$$= 2\pi \int_a^b \frac{-y''}{\left(\sqrt{1+(y')^2}\right)^3}\, dx \,,$$

where the a to b interval goes from where $y' = +\infty$ to where $y' = -\infty$ (for a closed, smooth surface)



Now $\displaystyle\int_a^b \frac{y''\,dx}{(1+(y')^2)^{3/2}}$

$$= \int_a^b \frac{d(y')}{(1+(y')^2)^{3/2}} = -\int_{-\infty}^{+\infty} \frac{du}{(1+u^2)^{3/2}} .$$

We do this last integral by substitution:

$u = \tan\theta$ where θ goes from $-\pi/2$ to $+\pi/2$

$du = \sec^2\theta\, d\theta \qquad 1+u^2 = \sec^2\theta$

$$\frac{du}{(1+u^2)^{3/2}} = \frac{\sec^2\theta\, d\theta}{(\sec^2\theta)^{3/2}} = \frac{1}{\sec\theta}\, d\theta = \cos\theta\, d\theta$$

So the integral $\displaystyle\int_{-\infty}^{+\infty} \frac{du}{(1+u^2)^{3/2}} = \int_{-\pi/2}^{+\pi/2} \cos\theta\, d\theta = 2$

Hence $\int$ Gauss curvature $= 2\pi\,(-(-2)) = 4\pi$.
Thus we have check the Gauss-Bonnet Theorem for this case.