More on the Gauss Map in General

As usual, let $\Gamma : S \to S^2$ be the Gauss map of a closed surface in $\mathbb{R}^3$. We have already observed that Γ's Jacobian at $p \in S$ is the Gauss curvature at p. Let us agree to call at point $u \in S^2$ a "critical value" of Γ if and only if there is a point $p \in S$ with $\Gamma(p) = u$ and with Gauss curvature at $p = 0$, or, equivalently, with Jacobian of Γ at $p = 0$. [This definition can be made in general: if $f : M \to N$ is a differentiable mapping of one surface to another, then (by definition) $q \in N$ is a critical value if $\exists\, p \in M$ such that $f(p) = q$ and such that f is singular at p, that is, has 0 Jacobian determinant at p in local coordinates on M and N].

In fact, it always happens that most $u \in S^2$ are <u>not</u> critical values. The critical values have measure 0 (or area 0 in this case) in a sense we shall make precise momentarily.

Now suppose that neither $u \in S^2$ nor $-u$ is critical value for Γ. Define the "height function" H_u on S by $H_u(p) = \langle u, p \rangle$. (If $u = (0,0,1)$, then H_u would be just the z-coordinate, for example). Of course the function H_u has critical points on S, a maximum and minimum at the very least. But when $\pm u$ are both noncritical values of Γ, these critical points are nondegenerate in a certain sense.

For convenience, we look at this in more detail in the case $u = (0,0,1)$ so that $H_u = z$. This is without loss of generality, since any u can be made $= (0,0,1)$ by rotation of coordinates. Thus at each point with normal $= \pm(0,0,1)$, we can consider S as graphed over its horizontal tangent plane, i.e. S locally looks like $(u, v, f(u,v))$ with $f_u = f_v = 0$ at $(u,v) = (0,0)$. Since in this situation, Gauss curv. to $= f_{uu}f_{vv} - f_{uv}^2$ (at $(u,v) = (0,0)$), it follows that (1) f has either a strict local minimum or a strict local maximum, strict in the extended sense that the quadratic part of the Taylor expansion is either positive definite or negative definite. <u>or</u> (2) f has a definite saddle point in the sense that the quadratic part of the Taylor expansion is a nondegenate quadratic form with one positive eigenvalue and one negative eigenvalue. Case (1) corresponds to $K>0$ case (2) to $K<0$. This all just standard two variable calculus, once we have S graphed over its tangent plane.

Now it is not hard to see (pictorially!) that no. of local max + no of local min. − no. of saddlepts = the Euler characteristic of S. Considering how the Euler characteristic changes as one goes up in z level, i.e., how the Euler characteristic of $\{ p \in S : z(p) < \alpha \}$ as α varies over $[\min_S z, \max_S z]$, changes with increasing α.

Each local minimum introduces a new, isolated point which turns into a disc for slightly larger α. This ups Euler characteristic by 1. Each local maximum caps off a circle with a disc, the circle being in slightly lower levels. This also increase Euler characteristic by 1. Cross a sadde point decreases Euler characteristic by 1 (See figures). So the formula

$$\chi(S) = \text{no. of loc. max} - \text{no. of saddle pts.}$$

$$+ \text{no of loc. min.}$$

$$= \text{no of Gauss curv pos. pts.} - \text{no. of Gauss curv neg pts}$$

among pts with exterior normal $= \pm(0,0,1)$.
We can also think of this of course as

(1) $\left(\begin{array}{l} \text{no of Gauss curv.} > 0 \text{ pts with } N = (0,0,1) \\ - \text{no of Gauss curv} < 0 \text{ pts with } N = (0,0,1) \end{array} \right)$

$+$ $+\left(\begin{array}{l} \text{no of Gauss curv} > 0 \text{ pts with } N = -(0,0,1) \\ - \text{no of Gauss curv} < 0 \text{ pts with } N = -(0,0,1) \end{array} \right)$ (2)

$$\text{Let } C = \{ u \in S^2 : \text{either } u \text{ or } -u \text{ is a critical value of } \Gamma \}$$

Since the set of critical values has measure 0, so does C. We now integrate with respect to a unit vector $u \in S^2 - C$ ~~than~~

with (0,0,1) replaced by u!

the formula (with ① & ② as above), $\underbrace{}$

$$\chi(S) = ① + ②$$

over $u \in S^2 \setminus C$. The left hand side gives

$$\int_{S^2 - C} \chi(S)\, d(\text{area}) = \int_{S^2} \chi(S)\, d(\text{area}) = 4\pi\, \chi(S)$$

since $\chi(S)$ is a constant and C has measure 0. By the definition of C, the set $-C = C$. So

$$\int_{S^2 - C} ① = \int_{S^2 - C} ② .$$

Thus we get

$$2 \int_{S^2 - C} ① = 4\pi\, \chi(S)$$

or

$$\int_{S^2 - C} ① = 2\pi\, \chi(S).$$

Finally, we can use the fact that the Jacobian (including orientation!) of Γ is K to get that

$$\int_{S^2 - C} ① = \int_S K\, d(\text{area}).$$

This then yields $\int_S K\, d(\text{area}) = 2\pi\, \chi(S)$.

So we have recovered the Gauss Bonnet Theorem (or surface in $\mathbb{R}^3$ at least) from this viewpoint. To complete the argument, we need to give a few more details of why $\int_{S^2 - C} ① = \int_S K \, d(\text{area})$. Now a certain difficulty arises: there are points $x \in S$ with $K(x) \neq 0$ but with $\Gamma(x) \in C$: this happens when some other point $y \in S$ has $K(y) = 0$ and $N(y) = \pm N(x)$. However, these points x, which we shall call points of the "second kind" turn out to have measure 0 in S. [The other points of S where $K \neq 0$ are called "first kind".] Thus their omission from further discussion will not affect the value of $\int_S K \, d(\text{area})$: in particular,

$$\int_S K \, d(\text{area}) = \int_{\substack{\text{points of } S \\ \text{with } K \neq 0 \text{ but not of the second kind}}} K \, d(\text{area})$$

We can leave out $K=0$ points and also points of the second kind, the $K=0$ pts since integrating 0 gives 0 and the second kind points since they form a set of measure 0 (which we shall prove at the end).

Now suppose x is a point of S with $K \neq 0$ of the first kind. Then $\Gamma^{-1}(\Gamma(x))$ is a finite set consisting of $p_1, \dots, p_k$ points where $K > 0$ and $q_1, \dots, q_\ell$ points where $K < 0$. By the Inverse Function Theorem, $\exists$ open sets $V_1, \dots, V_k$ and $W_1, \dots, W_\ell$ with $p_j \in V_j$, $\forall j$ and $q_m \in W_m$, $\forall m$, with $K > 0$ on the V's, $K < 0$ on the W's and Γ restricted to each V or W a diffeomorphism onto an open set U_x (same open set U_x for all, by taking intersection). From the change of variables formula for double integrals and our interpretation of the Gauss curvature as Jacobian, we get that

$$\int_{U \subset S^2} ① \, d(\mathrm{area}_{S^2}) = \int_U (k-l) \, d(\mathrm{area}_{S^2})$$

$$= \sum_{j=1}^{k} \int_{V_j \subset S} K \, d(\mathrm{area}_S) + \sum_{m=1}^{l} \int_{W_m \subset S} K \, d(\mathrm{area}_S) = \int_{\Gamma^{-1}(U)} K \, d(\mathrm{area})$$

Note that the "orientation reversal" on Ws of Γ is incorporated in the negativity of K on the W_s which in turn corresponds to the negatively counted points in the counting for ①.

Since this formula

$$\int_{U \subset S^2 - C} ① = \int_{\Gamma^{-1}(U)} K \, d(\mathrm{area}_S)$$

holds in a neighborhood of each point in $S^2 - C$, the additivity of the integral gives

$$\int_{S^2 - C} ① = \int_{\Gamma^{-1}(S^2 - C)} K \, d(\mathrm{area}_S) = \int_{\{p \in S : K(p) \neq 0\}} K \, d(\mathrm{area}_S)$$

as we wanted.

The one remaining point is to see why the set of points with $K \neq 0$ of the second kind has measure 0. For this, note that if x has $K(x) \neq 0$ but $\Gamma(x) \in \dot{C}$ (i.e., $\exists y \ni N(y) = \pm N(x)$ but $K(y) = 0$), then $\Gamma |$ some open neighborhood V_x of x is a diffeomorphism. Hence $\Gamma^{-1}(C) \cap V_x$ has measure 0 since: C has measure 0 so $\Gamma(V_x) \cap C$ has measure 0 so $\Gamma^{-1}(C) \cap V_x = \Gamma^{-1}(\Gamma(V_x) \cap C)$ has measure 0 because the diffeomorphism Γ^{-1} on $\Gamma(V_x)$ preserves measure 0.

So the points of the second kind in $\{x : K(x) \neq 0\}$ have measure 0 in a neighborhood of each point in the open set $\{x : K(x) \neq 0\}$. (Note that points of the second kind lie in $\Gamma^{-1}(C)$!). Thus the second kind points form a set of measure zero in S as required.