Mathematics 120B, Spring 2008, Homework I

Linear algebra set-up: A <u>symmetric</u> <u>bilinear</u> <u>form</u> is a function $B: V \times V \to \mathbb{R}$ (V a vector space) that is linear in each variable and that satisfies $B(\vec{v},\vec{w}) = B(\vec{w},\vec{v})$, all $\vec{v},\vec{w} \in V$. Associated to such a B, is a "<u>quadratic</u> <u>form</u>" $Q: V \to \mathbb{R}$ defined by $Q(\vec{v}) = B(\vec{v},\vec{v})$.

1. Suppose B is a symmetric bilinear form on a vector space V of dimension 2 with a (positive definite) inner product $\langle\ ,\ \rangle$. [A positive definite inner product is by definition a symmetric bilinear form which is nonnegative on pairs $(\vec{v},\vec{v})$ and $= 0$ on $(\vec{v},\vec{v})$ only if $\vec{v} = \vec{0}$]. Let $\vec{u}_1, \vec{v}_1$ and $\vec{u}_2, \vec{v}_2$ be orthonormal bases for V (orthonormal relative to $\langle\ ,\ \rangle$).

Prove:

(1) $B(\vec{u}_1,\vec{u}_1) + B(\vec{v}_1,\vec{v}_1) = B(\vec{u}_2,\vec{u}_2) + B(\vec{v}_2,\vec{v}_2)$.

(2) $B(\vec{u}_1,\vec{u}_1)B(\vec{v}_1,\vec{v}_1) - B^2(\vec{u}_1,\vec{v}_1) = B(\vec{u}_2,\vec{u}_2)B(\vec{v}_2,\vec{v}_2) - B^2(\vec{u}_2,\vec{v}_2)$

[Suggestion: Write $\vec{u}_2, \vec{v}_2$ in terms of a rotation of $\vec{u}_1, \vec{v}_1$, with a possible sign change of one vector].

Terminology: $B(\vec{u}_1,\vec{u}_1) + B(\vec{v}_1,\vec{v}_1)$ is the <u>trace</u> of B (relative to $\langle\,\rangle$). $B(\vec{u}_1,\vec{u}_1)B(\vec{v}_1,\vec{v}_1) - B^2(\vec{u}_1,\vec{v}_1)$ is the <u>determinant</u> of B (relative to $\langle\ ,\ \rangle$). These are, as indicated, not dependent on basis choice.

2. Let $V = \mathrm{span}(S_u, S_v)$, S a surface patch ($V =$ tangent space). Let (in prob. 1's notation) B be defined by

$$B(a_1 S_u + b_1 S_v,\ a_2 S_u + b_2 S_v)$$
$$= L_{11}(a_1 a_2) + L_{12}(a_1 b_2 + b_1 a_2) + L_{22}(b_1 b_2)$$

and $\langle a_1 S_u + b_1 S_v, a_2 S_u + b_2 S_v \rangle$

$$= E(a_1 a_2) + (a_1 b_2 + b_1 a_2) F + G(b_1 b_2).$$

Show that (e.g., by finding an orthonormal basis rel. to $\langle\ \rangle$):

(a) trace B (relative to $\langle , \rangle$) $= (EN + GL - 2FM)/(EG - F^2)$

(b) det B (relative to $\langle , \rangle$) $= (L_{11} L_{22} - L_{12}^2)/(EG - F^2)$

3. Suppose γ is an arclength-parameter curve in a surface $S(u,v)$, i.e. $\gamma(s) = S(u(s), v(s))$ and

$$E\left(\frac{du}{ds}\right)^2 + 2F\left(\frac{du}{ds}\right)\left(\frac{dv}{ds}\right) + G\left(\frac{dv}{ds}\right)^2 = 1.$$

(a) Prove that the normal component of the acceleration $\frac{d^2\gamma}{ds^2}$ is uniquely determined by $\frac{du}{ds}$ and $\frac{dv}{ds}$.

(b) In the notation of prob. 2, prove that

trace B = sum of the maximum and minimum values of $\langle N$, normal component of acc.$\rangle$

where N = a fixed choice ($S_u \times S_v / \| S_u \times S_v \|$) of unit normal and the maximum and minimum are taken over all such γ (through the given point of S)

(c) As in part (b), prove

det B = product of the maximum and minimum values of $\langle N$, normal component of acceleration$\rangle$.

(d) Interpret this as Gauss curvature = product of max and min curvatures of normal plane sections (sections by planes containing the normal N) and similarly for trace B (= "mean curvature" by definition). Watch out for $\pm$ signs.

4. Integration with respect to area on a surface patch S is by definition

$$\int f(u,v)\ \|S_u \times S_v\|\ du\,dv$$

where f is a function on $\mathcal{U}$, $S: \mathcal{U} \to \mathbb{R}^3$ being the patch. (Alternatively, one could think of f as a function on $S(\mathcal{U})$, since we suppose always that S is injective (= one-to-one)).

(a) Discuss carefully why this is a good definition in the following sense: that if one "reparameterizes" by looking at $\hat{S}(\hat{u},\hat{v}) = S(F(\hat{u},\hat{v}))$ where $F: \hat{\mathcal{U}} \to \mathcal{U}$ is a 1-1 onto nonsingular smooth mapping with $F(\hat{u},\hat{v}) = (u(\hat{u},\hat{v}), v(\hat{u},\hat{v}))$ then

$$\int f(u(\hat{u},\hat{v}), v(\hat{u},\hat{v}))\ \|\hat{S}_{\hat{u}} \times \hat{S}_{\hat{v}}\|\ d\hat{u}\,d\hat{v}$$

$$= \int f(u,v)\ \|S_u \times S_v\|\ du\,dv$$

[You will need to recall how change of variables in double integrals works].

(b) Prove that $\int f(u,v)\,\|S_u \times S_v\|\ du\,dv$

$$= \int f(u,v)\sqrt{EG - F^2}\ du\,dv.$$

5. Let $S : U \to \mathbb{R}^3$ be a surface path and $F : U \to \mathbb{R}$ be a smooth function. Define a family $S_t : U \to \mathbb{R}^3$ of surface patches by

$$S_t(u,v) = S(u,v) + t\, F(u,v)\, N$$

where N is the usual unit normal $\left((S_u \times S_v)/\|S_u \times S_v\| \right)$.

Let

$$A_t = \text{area of } S_t \stackrel{\text{def.}}{=} \int_1 \|(S_t)_u \times (S_t)_v\| \, du\, dv$$

Show that

$$\left.\frac{dA_t}{dt}\right|_{t=0} =$$

$$= \int H\Big|_{S(u,v)} F(u,v)\, \|S_u \times S_v\| \, du\, dv.$$

where H = the mean curvature (defined in the previous problem).

[Note: You will need to use $\|V \times W\|^2 = \|V\|^2 \|W\|^2 - \langle V, W\rangle^2$]

* 6. Let S be a closed surface in $\mathbb{R}^3$. Suppose that S encloses maximum volume for given area. Prove that S has constant mean curvature. [You may assume that "pushing in or out" S by $t(\text{function})\vec{N}$ giving volume V_t has the property that $\left.\frac{dV_t}{dt}\right|_{t=0} = \int_S \begin{pmatrix}\text{given}\\ \text{function}\end{pmatrix}$

where $\int_S$ is in the sense of the area integral in problem 4). [Suggestion: Look at how similar problem was done for area inside a curve]

7. Find the (intrinsic) Gauss curvature of $E = \lambda^2(x,y)$, $G = \lambda^2(x,y)$ $F = 0$.

8. Recall that the Gauss curvature of the surface of revolution

$$S(s,\theta) = (x(s), y(s)\cos\theta, y(s)\sin\theta)$$

where $x'(s) > 0$, $x'(s)^2 + y'(s)^2 = 1$ is

$$-y''(s)/y(s).$$

Use this to find surfaces of revolution of Gauss curvature $\equiv 1$ that are not (part of) a sphere.

9. Use the idea of problem 8 to find surfaces of revolution with Gauss curvature $\equiv -1$.

10. Use problem 7 to show that the Gauss curvature of the metric

$E = 4/(1-x^2-y^2)^2$

$G = 4/(1-x^2-y^2)^2$

$F = 0$

on $\{(x,y) : x^2+y^2 < 1\}$ has Gauss curvature $\equiv -1$.

11. What is the mean curvature of the surface of problem 8 (surface of revolution)?