Sample Problems for Midterm II

Note correction: $T(S_{uu})$ not just S_{uu}

1. Explain how to compute $\langle S_{uu}, S_v \rangle$ and $\langle S_{uu}, S_u \rangle$ from E, F, and G (and their derivatives).

→ 2. Writing $T(S_{uu}) = a S_u + b S_v$, explain how to find a and b from given E, F, G (and derivatives).

3. Explain how to use $(S_{uv})_u = (S_{uu})_v$ to show that $L_{11}L_{22} - L_{12}^2$ is intrinsic.

4. Prove that if $(f(s), g(s))$ is an arclength parameter curve and $S(s,\theta)$ $(= S(u,v))$ $= (f(s), g(s)\cos\theta, g(s)\sin\theta)$ is the associated surface of revolution, then the Gauss curvature at (s,θ) is

Note correction: g goes here, not f! → $-\frac{d^2}{ds^2}g(s)/g(s)$. Illustrate for the unit sphere.

5. What is the Gauss curvature of $S(u,v) = (u, v, f(u,v))$ in terms of the derivatives of f at a point where $f_u = f_v = 0$?

6. What about at a point when f_u, f_v are not necessarily $=0$? (Hint: Find N and compute via $L_{11} = -\langle N, S_{uu} \rangle$ etc.)

7. Define the eigenvalues of a quadratic form $Q(x,y) = ax^2 + 2bxy + cy^2$ associated to the matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ and show they are = the max & min values of Q on $\{(x,y) : x^2+y^2=1\}$

Show that the product of the eigenvalues $= ac - b^2$.

8. Use problem 7 to show that if $Q_1(x,y) \geq Q_2(x,y) \geq 0$ for all (x,y) with $x^2+y^2=1$, then the product of the eigenvalues for $Q_1 \geq$ the product of the eigenvalues for Q_2.

9. Using problems 8 and 5 show that a compact surface (with no boundary) in $\mathbb{R}^3$ has a point of positive curvature.

10. Verify the Gauss Bonnet Theorem for a sphere of arbitrary radius in $\mathbb{R}^3$ by computing its area and (constant) Gauss curvature.

11. Explain why the Gauss curvature of $\lambda S(u,v)$, $\lambda \in \mathbb{R}$, $\lambda > 0$, is $(1/\lambda^2)$ Gauss curvature of $S(u,v)$ for any surface patch S.

12. Use the Inverse Function Theorem to explain why, if $S(u,v)$ is a nonsingular surface patch and $(u_0,v_0) \in U$ $(S: U \to \mathbb{R}^3)$ then in some neighborhood of (u_0,v_0), S is a graph over the (xy) plane or over the (yz) plane or over the (xz) plane, thus deducing that Monge patch arguments are general proofs.

13. What is the normal component of the acceleration of the curve $S(u(t), v(t))$? Show it is determined by the tangent vector $(u'(t), v'(t))$.

14. Compute the Gauss curvature of the unit sphere by writing it as a Monge patch $(u, v, f(u,v))$ with $f_u = f_v = 0$ at the given point where you are computing curvature.

15. Does the paraboloid $z = x^2 + y^2$ have positive Gauss curvature everywhere? Prove your answer.

16. In problem 15, is there an $\varepsilon > 0$ such that the Gauss curvature $\geq \varepsilon$ everywhere? Prove your answer.

* 17. Prove that the sum of the angles of a triangle is $180°$. Think about why the sum is $> 180°$ on a sphere.

* = "bonus" — not really going to be on midterm!