Appendix on Details of Inoue Surfaces

In the Kodaira classification of compact complex surfaces already discussed, a special position is occupied by what are called the Class VII_0 surfaces. They are (minimal) surfaces with first Betti number $b_1 = 1$.

These in turn are subdivided. First, there are surfaces with $b_2=0$ also (along with $b_1 = 1$) and containing at least one curve. This class includes Hopf surfaces (obtained as quotients of two – dimensional complex Euclidean space with the origin removed under the cyclic group generated by multiplication by a complex number of absolute value not equal to 1) and some elliptic surfaces. Kodaira proved that only those examples occur in this class.

Thus in the class VII_0 , it remains only to consider the two classes:

1. second Betti number nonzero

and

2. second Betti number 0 and containing no curves.

(In both cases, the space of meromorphic functions consists of constants only, so algebraic dimension is 0). Inoue constructed families of surfaces in case 2. He also showed that any example in case 2 that admitted a line bundle L such that the space of holomorphic (1,0) forms with values in L was nontrivial must belong to one of the families he constructed. Later Bogomolov, and by a different and simpler argument Li, Yau , and Zheng showed that such a line bundle always exists. Thus Inoue's families are the only examples of surfaces for case 2. The basic method of Li, Yau, and Zheng is to note that if no such bundle L exists, then the holomorphic tangent bundle of the surface is stable. Under that hypothesis, they then show that the tangent bundle has a Hermitian Yang Mills connection and deduce from that the surface admits a Kahler structure, contradicting that the first Betti number is 1.

Inoue's examples are quotients of HxC, the product of the upper half plane and the complex numbers. The transformation groups by which the quotienting occurs are properly discontinuous, fixed point-free groups of affine transformations. The exact descriptions of these groups are given at the end of this appendix, extracted from Inoue's paper "On Surfaces of Class VII_0 " ,Inventiones Math. 24(1974) , 269-310).

Topologically, these surfaces are:
Type S^M : a 3-torus bundle over a circle
Type S^+ : a bundle over a circle with fibre itself a bundle, namely a circle bundle over a 2-torus
Type S^- : a quotient of an S^+ by a Z_2 action, that is , some S^+ is unramified double cover of a given S^-

Note that these topological types are different from the Hopf surface situation, where,e.g., the well known complex structure on S^3xS^1 is obtained by regarding the surface as a fibration over S^2 with a torus fibre (the famous "Hopf fibration" of S^3 over S^1, producted with S^1). A similar idea was used in the construction of the well-known Calabi-Eckmann complex structures on the product of any two odd-dimensional spheres, the product in that case being regarded as a fibration over the product of two complex projective spaces, with a torus fibre.

Inoue's description of the construction of his families of surfaces:

Surfaces S_M

Let $M=(m_{ij})\in SL(3,\mathbb{Z})$ be a unimodular matrix with eigen-values $\alpha, \beta, \bar\beta$ such that $\alpha>1$ and $\beta\neq\bar\beta$. We choose a *real* eigen-vector (a_1, a_2, a_3) and an eigen-vector (b_1, b_2, b_3) of M corresponding to α and β, respectively. Clearly α is an irrational number and

(14) $(a_1, b_1), (a_2, b_2), (a_3, b_3)$ are linearly independent over $\mathbb{R}$,

(15) $$(\alpha a_j, \beta b_j)=\sum_{k=1}^{3} m_{jk}(a_k, b_k) \quad \text{for } j=1,2,3.$$

By $\mathbb{H}$ we denote the upper half of the complex plane. Let G_M be the group of analytic automorphisms of $\mathbb{H}\times\mathbb{C}$ generated by

(16) $$\begin{aligned} &g_0: (w,z)\to(\alpha w, \beta z),\\ &g_i: (w,z)\to(w+a_i, z+b_i) \quad \text{for } i=1,2,3. \end{aligned}$$

(14) and (15) imply that the action of G_M on $\mathbb{H}\times\mathbb{C}$ is properly discontinuous and has no fixed points. We define S_M to be the quotient surface $\mathbb{H}\times\mathbb{C}/G_M$.

Surfaces $S^{(+)}_{N,p,q,r;t}$

Let $N=(n_{ij})\in SL(2,\mathbb{Z})$ be a unimodular matrix with two real eigen-values α, $1/\alpha$, $\alpha>1$. We choose *real* eigenvectors (a_1, a_2) and (b_1, b_2) of N corresponding to α and $1/\alpha$, respectively, and we fix integers p, q, r $(r\neq 0)$ and a complex number t. We define (c_1, c_2) to be the solution of the following equation:

(17) $$(c_1, c_2)=(c_1, c_2)\cdot {}^tN+(e_1, e_2)+\frac{b_1 a_2-b_2 a_1}{r}(p,q)$$

where $e_i=\frac{1}{2}n_{i1}(n_{i1}-1)a_1 b_1+\frac{1}{2}n_{i2}(n_{i2}-1)a_2 b_2+n_{i1}n_{i2}b_1 a_2$ $(i=1,2)$. Let $G^{(+)}_{N,p,q,r;t}$ be the group of analytic automorphisms of $\mathbb{H}\times\mathbb{C}$ generated by

(18) $$\begin{aligned} &g_0: (w,z)\to(\alpha w, z+t)\\ &g_i: (w,z)\to(w+a_i, z+b_i w+c_i) \quad i=1,2\\ &g_3: (w,z)\to\left(w, z+\frac{b_1 a_2-b_2 a_1}{r}\right). \end{aligned}$$

Let Γ be the subgroup $\langle g_1, g_2, g_3\rangle$ of $G^{(+)}_{N,p,q,r;t}$. For each fixed $y=\operatorname{Im} w$, Γ defines an automorphism group Γ_y of $\mathbb{R}\times\mathbb{C}\simeq\mathbb{R}^3$ in an obvious manner. Then g_3 commutes with every element of $G^{(+)}_{N,p,q,r;t}$, $g_1^{-1}g_2^{-1}g_1 g_2=g_3^r$ and g_0 normalizes Γ. Moreover,

(19) the action of Γ_y on $\mathbb{R}^3$ is properly discontinuous and has no fixed points.

This follows from the fact that (a_1, b_1) and (a_2, b_2) are linearly independent over $\mathbb{R}$. (17), (18) and (19) imply that the action of $G^{(+)}_{N,p,q,r;t}$ on $\mathbb{H}\times\mathbb{C}$ is properly discontinuous and has no fixed points. We define $S^{(+)}_{N,p,q,r;t}$ to be the quotient surface $\mathbb{H}\times\mathbb{C}/G^{(+)}_{N,p,q,r;t}$.

Surfaces $S^{(-)}_{N,p,q,r}$

Let $N=(n_{ij})\in GL(2,\mathbb{Z})$ be a 2×2 matrix with $\det N=-1$ having two real eigen-values α, $-1/\alpha$ such that $\alpha>1$. We choose *real* eigen-vectors (a_1,a_2) and (b_1,b_2) of N corresponding to α and $-1/\alpha$, respectively, and we fix integers p, q, r $(r\neq 0)$. We define (c_1,c_2) to be the solution of the following equation:

$$-(c_1,c_2)=(c_1,c_2)\,{}^tN+(e_1,e_2)+\frac{b_1a_2-b_2a_1}{r}(p,q), \tag{20}$$

where $e_i=\frac{1}{2}n_{i1}(n_{i1}-1)a_1b_1+\frac{1}{2}n_{i2}(n_{i2}-1)a_2b_2+n_{i1}n_{i2}b_1a_2$ $(i=1,2)$. Let $G^{(-)}_{N,p,q,r}$ be the group of analytic automorphisms of $\mathbb{H}\times\mathbb{C}$ generated by

$$\begin{aligned}
&g_0\colon (w,z)\to(\alpha w,\,-z),\\
&g_i\colon (w,z)\to(w+a_i,\,z+b_iw+c_i) \qquad \text{for } i=1,2,\\
&g_3\colon (w,z)\to\left(w,\,z+\frac{b_1a_2-b_2a_1}{r}\right).
\end{aligned} \tag{21}$$

Then the subgroup $\langle g_0^2,g_1,g_2,g_3\rangle$ coincides with $G^{(+)}_{N^2,p_1,q_1,r;0}$ for certain $(p_1,q_1)\in\mathbb{Z}^2$ of which index in $G^{(-)}_{N,p,q,r}$ equals 2. g_0 defines an involution of $S^{(+)}_{N^2,p_1,q_1,r;0}$ free from fixed points. Thus the action of $G^{(-)}_{N,p,q,r}$ on $\mathbb{H}\times\mathbb{C}$ is properly discontinuous and has no fixed points. We define a surface $S^{(-)}_{N,p,q,r}$ to be the quotient surface $\mathbb{H}\times\mathbb{C}/G^{(-)}_{N,p,q,r}$.

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