COMPLEX DIFFERENTIAL GEOMETRY

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1. Complex manifolds

Let $\mathbf{C}$ = the complex numbers and $\mathbf{C}^n = \mathbf{C} \times \cdots \times \mathbf{C}$ (n factors). $\mathbf{C}^n$ will be referred to as n-dimensional complex euclidean space. Of course, $\mathbf{C}^n$ is topologically 2n -dimensional; more specifically, $\mathbf{C}^n$ can and will be identified homeomorphically with $\mathbf{R}^{2n}$ as follows: If $(z_1,\ldots,z_n) \in \mathbf{C}_n$ and if $z_j = x_j + iy_j, x_j, y_j \in \mathbf{R}$, $j = 1,\ldots n$ then $(z_1,\ldots,z_n)$ is to be identified with $(x_1,y_1,\ldots,x_n,y_n) \in \mathbf{R}^{2n}$. That is, $(z_1,\ldots,z_n)$ is identified with $(\text{Re } z_1, \text{Im } z_1,\ldots,\text{Re } z_n, \text{Im } z_n)$, where Re and Im denote real and imaginary parts, respectively.

DEFINITION. A function $f : D \to \mathbf{C}$ defined on an open subset D of $\mathbf{C}^n$ is *holomorphic* if f considered as a function from $D \subset \mathbf{R}^{2n}$ to $\mathbf{R}^2$ is C^∞ and satisfies the Cauchy-Riemann equations:

$$\frac{\partial(\text{Re } f)}{\partial x_j} = \frac{\partial(\text{Im } f)}{\partial y_j} \quad \text{and} \quad \frac{\partial(\text{Re } f)}{\partial y_j} = -\frac{\partial(\text{Im } f)}{\partial x_j}$$

for all $j = 1,\ldots,n$. A mapping $f : D \to \mathbf{C}^m$ is *holomorphic* if each component of f is holomorphic; that is, if $f_1,\ldots,f_m$ are holomorphic, where $f_j : D \to \mathbf{C}$, $j = 1,\ldots,m$ are defined by $f(z) = (f_1(z),\ldots,f_m(z)) \in \mathbf{C}^m$ for $z \in D$.

Note that if $D = \cup_{\lambda \in \Lambda} D_\lambda$ with each D_λ open, then $f : D \to \mathbf{C}^m$ is holomorphic if and only if $f \mid D_\lambda$ is holomorphic for every λ.

Multiplication by i, $(z_1,\ldots,z_n) \to (iz_1,\ldots,iz_n)$, defines a mapping from $\mathbf{C}^n$ to $\mathbf{C}^n$, whose square is multiplication by -1. The induced mapping from $\mathbf{R}^{2n}$ to $\mathbf{R}^{2n}$, which will be denoted by J, is a real linear endomorphism, whose composition with itself is again multiplication by -1. The following lemma shows that the endomorphism J can be used to characterize holomorphic mappings:

LEMMA 1. *A mapping* $f : D \to \mathbf{C}^m$, $D^{\text{open}} \subset \mathbf{C}^n$, *which is* C^∞ *when considered as a function from* $D \subset \mathbf{R}^{2n}$ *to* R^{2m} *is holomorphic if and only if* $f_* J_{\mathbf{R}^{2n}} = J_{\mathbf{R}^{2m}} f_*$. *Here* f_* *is the real Jacobian of* f.

Proof. Write $f(z) = (f_1(z),\ldots,f_m(z))$, $f_j(z) \in \mathbf{C}$ for $j = 1,\ldots,m$. In matrix form relative to the standard bases of $\mathbf{R}^{2n}$ and $\mathbf{R}^{2m}$,

$$f_* = \begin{pmatrix} \frac{\partial(\mathrm{Re} f_1)}{\partial x_1} & \frac{\partial(\mathrm{Re} f_2)}{\partial y_1} & \cdots & \frac{\partial(\mathrm{Re} f_m)}{\partial x_n} & \frac{\partial(\mathrm{Re} f_m)}{\partial y_n} \\ \frac{\partial(\mathrm{Im} f_1)}{\partial x_1} & \frac{\partial(\mathrm{Im} f_1)}{\partial y_1} & \cdots & \frac{\partial(\mathrm{Im} f_m)}{\partial x_n} & \frac{\partial(\mathrm{Im} f_m)}{\partial y_n} \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \frac{\partial(\mathrm{Re} f_m)}{\partial x_1} & \frac{\partial(\mathrm{Re} f_m)}{\partial y_1} & \cdots & \frac{\partial(\mathrm{Re} f_m)}{\partial x_n} & \frac{\partial(\mathrm{Re} f_m)}{\partial y_n} \\ \frac{\partial(\mathrm{Im} f_m)}{\partial x_1} & \frac{\partial(\mathrm{Im} f_m)}{\partial y_1} & \cdots & \frac{\partial(\mathrm{Im} f_m)}{\partial x_n} & \frac{\partial(\mathrm{Im} f_m)}{\partial y_n} \end{pmatrix}$$

$$J_{\mathbf{R}^{2m}} = \begin{pmatrix} \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} & & & 0 \\ & \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} & & \\ & & \ddots & \\ 0 & & & \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} \end{pmatrix}$$

$$J_{\mathbf{R}^{2n}} = \begin{pmatrix} 0 & -1 & & & & & & \\ 1 & 0 & & & & & 0 & \\ & & 0 & -1 & & & & \\ & & 1 & 0 & & & & \\ & & & & \cdot & & & \\ & & & & & \cdot & & \\ & & & & & & \cdot & \\ & 0 & & & & & 0 & -1 \\ & & & & & & 1 & 0 \end{pmatrix}$$

Straightforward matrix multiplication shows that $f_*J = Jf_*$ if and only if

$$\frac{\partial(\mathrm{Re}\, f_k)}{\partial x_j} = \frac{\partial(\mathrm{Im}\, f_k)}{\partial y_j} \quad \text{and} \quad \frac{\partial(\mathrm{Im}\, f_k)}{\partial x_j} = - \frac{\partial(\mathrm{Re}\, f_k)}{\partial y_j}$$

for all $k = 1,\ldots,m$ and $j = 1,\ldots,n$.

Lemma 1 implies immediately the following corollaries:

COROLLARY. *If* $f : D \to \mathbf{C}^m$ *and* $g : D_1 \to \mathbf{C}^k$ *are holomorphic mappings and if* $f(D) \subset D_1^{\text{open}} \subset \mathbf{C}^m$, *then* $g \circ f : D \to \mathbf{C}^k$ *is holomorphic.*

COROLLARY. *If* $f : D \to \mathbf{C}^n$, $D^{\text{open}} \subset \mathbf{C}^n$, *is holomorphic and is a diffeomorphism onto its image when considered as a mapping from* $D \subset \mathbf{R}^{2n}$ *to* $\mathbf{R}^{2n}$ *then* $f^{-1} : f(D) \to \mathbf{C}^n$ *is holomorphic. (Note that in this case* $f(D)$ *is necessarily open in* $\mathbf{C}^n$.*)*

DEFINITION. A *complex structure* on a C^∞ manifold M of even dimension $2n$ is a maximal collection of C^∞ charts indexed by a set Λ:

$$\{(\psi_\lambda, U_\lambda) : \lambda \in \Lambda, \psi_\lambda : U_\lambda \to R^{2n}\}$$

having $\cup_{\lambda \in \Lambda} U_\lambda = M$ and satisfying the following condition (holomorphic transition) for all $\lambda, \mu \in \Lambda$,

$$\psi_\lambda \psi_\mu^{-1} : \psi_\mu(U_\lambda \cap U_\mu) \to \psi_\lambda(U_\lambda \cap U_\mu) \subset \mathbf{R}^{2n}$$

is a holomorphic function considered as a function from an open subset of $\mathbf{C}^n$ to $\mathbf{C}^n$.

A *complex manifold* is a paracompact C^∞ manifold M together with a complex structure. The number n is the *complex dimension* of M. The coordinate charts of the complex structure of a complex manifold are called *holomorphic coordinate charts* or *holomorphic coordinate systems*.

Note that an open subset of a complex manifold is itself a complex manifold in a standard fashion.

LEMMA 2. *Any collection of C^∞ charts on a paracompact C^∞ manifold M which cover M and which satisfy the holomorphic transition condition determine one and only one complex structure on M; i.e. any such collection is contained in a unique maximal such collection.*

The proof of this lemma follows exactly the pattern (using the corollaries of Lemma 1) of the standard proof of the corresponding result for real manifolds.

Let M be a complex manifold and p be a point of M. Denote the (real) tangent space of M at p by M_p. An endomorphism $J_p : M_p \to M_p$ of the real vector space M_p with $J_p^2 =$ multiplication by -1 can be defined as follows: Choose a holomorphic coordinate system $\psi : U \to \mathbf{R}^{2n}$ defined in a neighborhood U of p. Then $\psi_{*p} : M_p \to \mathbf{R}^{2n}$ is an isomorphism (the tangent space to $\mathbf{R}^{2n}$ at $\psi(p)$ being identified with $\mathbf{R}^{2n}$). J_p is then defined to be $\psi_{*p}^{-1} J_{\mathbf{R}^{2n}} \psi_{*p}$. Lemma 2 implies immediately that J_p thus defined does not depend on the choice of ψ : if ϕ is a second holomorphic coordinate system around p then at p

$$\phi_*^{-1} J_{\mathbf{R}^{2n}} \phi_* = \phi_*^{-1} (\psi\phi^{-1})_*^{-1} J_{\mathbf{R}^{2n}} (\psi\phi^{-1})_* \phi_*$$

because $(\psi\phi^{-1})_*$ commutes with $J_{\mathbf{R}^{2n}}$ by Lemma 2 and the holomorpiicity of $\psi\phi^{-1}$. Hence

$$\phi_*^{-1} J_{\mathbf{R}^{2n}} \phi_* = \phi_*^{-1} \phi_* \psi_*^{-1} J_{\mathbf{R}^{2n}} \psi_* \phi_*^{-1} \phi_* = \psi_*^{-1} J_{\mathbf{R}^{2n}} \psi_* \,.$$

Thus J_p is independent of holomorphic coordinate choice; that $J_p^2 =$ multiplication by

-1 is apparent from the fact that $J^2_{\mathbf{R}^{2n}} =$ multiplication by -1.

Since holomorphic coordinate systems are always C^∞, $p \to J_p$ considered as a (1,1) tensor on M is C^∞. Note that this C^∞ tensor completely determines the complex structure on M since a C^∞ coordinate system $\psi : U \to \mathbf{R}^{2n}$ is a holomorphic coordinate system if and only if $\psi_{*p}J_p = J_{\mathbf{R}^{2n}}\psi_{*p}$ for all $p \in U$.

DEFINITION. A C^∞ mapping $f : M \to M'$ from one complex manifold M to another M' is *holomorphic* if $f_*J_M = J_{M'}f_*$.

Lemma 2 shows that this definition is equivalent to the requirement that f be holomorphic when expressed (locally) in holomorphic coordinate systems on M and M'. The definition as given has the advantage of avoiding local coordinate expressions.

Note that if M' is a complex manifold and M a C^∞ real submanifold then M has at most one complex structure such that the injection $i : M \to M'$ is holomorphic since at most one J_M can satisfy $i_*J_M = J_{M'}i_*$, i_* being injective. Of course such a complex structure on M may fail to exist. If one (and necessarily only one) such does exist, then M is said to be a *complex submanifold* of M'.

Since the (1,1) tensor J with $J^2 =$ multiplication by -1 on a complex manifold determines the complex structure, it is reasonable to consider such tensors independent of any complex structure:

DEFINITION. Let M be a C^∞ manifold. An *almost complex structure* on M *is a* C^∞ (1,1) tensor J *on* M *such that, for every* $p \in M$, $J_p^2 : M_p \to M_p$ *is multiplication by* -1.

LEMMA 3. A *real finite dimensional vector space which admits an endomorphism* $J : V \to V$ *satisfying* $J^2 =$ *multiplication by* -1 *is necessarily even dimensional.*

Proof. Let $X,Y \to g(X,Y)$ be any positive definite inner product on V. Then

$(X,Y) \to H(X,Y)$ defined by $H(X,Y) = g(X,Y) + g(JX,JY)$ is a positive definite inner product on V relative to which J acts as an isometry. In particular it follows that any J-invariant subspace has a J-invariant complementary subspace.

Now note that if $X_1 \in V$ and $X_1 \neq 0$ then X_1 and JX_1 are real linearly independent since if $JX_1 = \alpha X_1, \alpha \in \mathbf{R}$, then $X_1 = -J^2X_1 = -\alpha JX_1 = -\alpha^2 X_1$ or $\alpha^2 = -1$, which is impossible for $\alpha \in \mathbf{R}$. Hence X_1 and JX_1 span a 2-dimensional subspace, which is obviously J-invariant. This subspace has then a J-invariant complement, say V_1, of dimension equal to two less than the dimension of V. The lemma now follows by an obvious induction on dimension argument.

REMARK. The existence of a J-invariant complement for every J-invariant subspace is of course a special case of the complete reducibility of representations of finite groups.

Lemma 3 shows that any manifold admitting an almost complex structure must be even dimensional.

LEMMA 4. *Let M be a paracompact (even dimensional) C^∞ manifold with an almost complex structure J_M. In order that J be the almost complex structure on M associated to complex structure on M, it is necessary and sufficient that for each point p of M there exists a C^∞ coordinate chart $\psi : U \to \mathbf{R}^{2n}$, $p \in U$, such that $\psi_* J_M = J_{\mathbf{R}^{2n}} \psi_*$ everywhere on U.*

Proof. The necessity of the condition is immediate. To prove the sufficiency, note that if $\psi : U_1 \to \mathbf{R}^{2n}$ and $\phi : U_2 \to \mathbf{R}^{2n}$ satisfy $\psi_* J_M = J_{\mathbf{R}^{2n}} \psi_*$ everywhere on U_1 and $\phi_* J_M = J_{\mathbf{R}^{2n}} \phi_*$ everywhere on U_2 then $\psi\phi^{-1} : \phi(U_1 \cap U_2) \to \psi(U_1 \cap U_2)$ satisfies

$$(\psi\phi^{-1})_* J_{\mathbf{R}^{2n}} = \psi_* \phi_*^{-1} J_{\mathbf{R}^{2n}} = \psi_* J_M \phi_*^{-1} = J_{\mathbf{R}^{2n}} \psi_* \phi_*^{-1} = J_{\mathbf{R}^{2n}} (\psi\phi^{-1})_*$$

everywhere on $\phi(U_1 \cap U_2)$. By Lemma 1, $\psi\phi^{-1}$ is holomorphic on the open set

$\phi(U_1 \cap U_2)$, and Lemma 2 now implies the desired conclusion.

It is possible to find conditions expressed directly in terms of the almost complex structure tensor J which are necessary and sufficient for J to arise from a complex structure. These conditions are discussed in the Appendix Z.

2. Basic Examples

Two complex manifolds M and N are *biholomorphic* if there is a (real) diffeomorphism $F : M \to N$ that is holomorphic. It follows from the results in §1 that F^{-1} is then also holomorphic. From this, it then follows that the relation of being biholomorphic is an equivalence relation. Two complex manifolds that are biholomorphic are the same object as far as the purposes of complex manifold theory go.

The most fundamental examples for the theory are the following:

(i) $\mathbf{C}^n$. This is $\mathbf{R}^{2n}$ with the complex structure J determined by

$$\mathbf{R}^{2n} = \{(x_1,y_1,\ldots,x_n,y_n)\}$$

$$J\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i}\,,\ J\left(\frac{\partial}{\partial y_i}\right) = -\frac{\partial}{\partial x_i} \qquad i = 1,2,\ldots,n\,.$$

The holomorphic coordinate system $(x_1 + \sqrt{-1}\,y_1,\ldots,x_n + \sqrt{-1}\,y_n)$ determines a complex manifold structure.

(ii) $B^n = \left\{(z_1,\ldots,z_n) \in \mathbf{C}^n \middle| \sum_{j=1}^{n} |z_j|^2 < 1\right\}$,

the unit ball in $\mathbf{C}^n$. Since this is an open subset of $\mathbf{C}^n$, it inherits a complex manifold structure automatically. It is important to note that $\mathbf{C}^n$ and B^n are *not* biholomorphic even though they are real diffeomorphic. To see this, note that a holomorphic mapping $F : \mathbf{C}^n \to B^n$, $F = (F_1,\ldots,F_n)$, $F_j : \mathbf{C}^k \to \mathbf{C}$ is necessarily constant because Liouville's Theorem immediately implies that each F_j is constant. (In detail: $z \to F_j(a + bz)$, $a,b \in \mathbf{C}$, $z \in \mathbf{C}$ is constant so F_j is itself constant.)

(iii) Variations of B^n. If S is a C^∞ submanifold that is C^1 close to $\{(x_1,y_1,\ldots,x_n,y_n) \mid \Sigma x_i^2 + \Sigma y_i^2 = 1\}$, then the interior U_s of S is an open subset of $\mathbf{C}^n$ that is real diffeomorphic to B^n. If $n = 1$, then U_s is biholomorphic to B^1, by the Riemann Mapping Theorem. But if $n \geq 2$, U_s is generically *not* biholomorphic to B^n. The U_s so obtained in fact form a collection of complex manifolds that in a suitable sense give an infinite dimensional family of biholomorphic equivalence classes.

(iv) $P_n\mathbf{C}$, complex projective space of (complex) dimension n. The notation $\mathbf{C}P^n$ is also used. To define $P_n\mathbf{C}$, define an equivalence relation on $\mathbf{C}^{n+1} - \{(0,\ldots,0)\}$ as follows:

$$(z_1,\ldots,z_{n+1}) \sim (w_1,\ldots,w_{n+1})$$

if and only if $\exists\ \lambda \in \mathbf{C}$ such that $z_j = \lambda w_j$ for each $j = 1,2,\ldots,n+1$. (Exercise: Check that this is an equivalence relation.) As a set, $P_n\mathbf{C}$ is the set of all equivalence classes of $\mathbf{C}^{n+1} - \{(0,\ldots,0)\}/\sim$. Define a topology on $P_n\mathbf{C}$ by declaring a set $U \subset P_n\mathbf{C}$ to be open if and only if

$$\{(z_1,\ldots,z_{n+1}) \in \mathbf{C}^{n+1} - \{(0,\ldots,0)\} \mid [(z_1,\ldots,z_{n+1})] \in U\}$$

is open in $\mathbf{C}^{n+1}$, where [g] denotes the equivalence class of $(z_1,\ldots,z_{n+1})$. (This is the topology usually called the quotient topology relative to the equivalence relation.) $P_n\mathbf{C}$ is compact because the map

$$\{(z_1,\ldots,z_{n+1}) \, | \Sigma |z_j|^2 = 1\} \to P_n\mathbf{C}$$

obtained by taking equivalence classes is surjective (and continuous by definition).

To define a complex structure on $P_n\mathbf{C}$, we exhibit coordinate systems as follows: For each j, set $U_j = \{[(z_1,\ldots,z_{n+1})] \mid z_j \neq 0\}$. Clearly $\cup\, U_j = P_n\mathbf{C}$. Define $F_j : U_j \to \mathbf{C}^n$ by

$$[(z_1,\ldots,z_{n+1})] \to (z_1/z_j,\ldots,z_1/z_j,\ldots,z_n/z_j), \quad \ell \neq j\ .$$

The transition map from $F_j(U_j)$ to $F_k(U_k)$ defined on $F_j(U_j \cap U_k)$ is easily seen to be holomorphic: it is essentially multiplication by z_j/z_k. (Exercise: Check details of this.) Thus the $F_j : U_j \to \mathbf{C}^n$ maps define a complex manifold structure on $P_n\mathbf{C}$.

(v) Submanifolds (of $P_n\mathbf{C}$): The definition of (complex) submanifold of a complex manifold runs parallel with the real theory: A subset N of a complex manifold M is an (embedded) submanifold if (1) N is a C^∞ real embedded submanifold of M and (2) T_pN, all $p \in N$, is a J invariant subspace of T_pM ($T_pN =$ real tangent space) of N at p.

This definition condition (2) can be checked to be equivalent to the idea that in a neighborhood in M of each point $q \in N$, $N \cap$ neighborhood is a slice in some holomorphic coordinate system, i.e.,

$$N \cap \text{nbhd} = \{(z_1,\dots,z_n) \mid z_j = z_{j+1} = \cdots = z_n = 0\}$$

in some (holomorphic) coordinates $(z_1,\dots,z_n)$.

A compact submanifold of $P_n\mathbf{C}$ is called a (compact) algebraic variety. The fact that a compact complex submanifold of $P_n\mathbf{C}$ is an algebraic variety in the usual sense (of being the common zeroes of a set of homogeneous polynomials) is true, but hard to prove. A converse statement is, however, relatively easy.

If $P(z_1,\dots,z_{n+1})$ is a homogeneous polynomial, then, for $\lambda \neq 0$, $P(\lambda z_1,\dots,\lambda z_{n+1}) = 0$ if and only if $P(z_1,\dots,z_{n+1}) = 0$ since

$$P(\lambda z_1,\dots,\lambda z_{n+1}) = \lambda^d P(z_1,\dots,z_n)$$

where $d =$ degree (of homogeneity) of P. Thus it makes sense to refer to P vanishing (or not) at a point of $P_n\mathbf{C}$. If $P_1,\dots,P_k$ is a (finite) set of polynomials that are homogeneous, set $V(P_1,\dots,P_k) =$ the points of $P_n\mathbf{C}$ at which all the $P_1,\dots,P_k$ vanish. The set $V(P_1,\dots,P_k)$ need not be a real submanifold: it could have singularities. But if it is a real submanifold, then it is in fact necessarily a complex submanifold. Rather than considering this fact in full generality, we illustrate with a concrete example.

Set

$$V = V(z_1^2 + z_2^2 + z_3^2) \subset P_2\mathbf{C} .$$

Consider $V \cap U_j$, $j \in \{1,2,3\}$ where $U_j = \{[(z_1,z_2,z_3)] | z_j \neq 0\}$ as before, with $j = 3$, say. The coordinates on U_3 are (z_1,z_2) with $(z_1,z_2) \leftrightarrow [(z_1,z_2,1)] \in P_2\mathbf{C}$. The polynomial vanishes at $[(z_1,z_2,1)]$ if and only if $z_1^2 + z_2^2 + 1 = 0$. So $V \cap U_3$ corresponds in (z_1,z_2) coordinates to

$$\{(z_1,z_2) \mid z_1^2 + z_2^2 + 1 = 0\} .$$

This is a complex submanifold in $\mathbf{C}^2$ because, near (w_1,w_2), $w_2 \neq 0$, with $w_1^2 + w_2^2 + 1 = 0$, the functions

$$(z_1, z_1^2 + z_2^2 + 1)$$

form a holomorphic coordinate system and $\{(z_1,z_2) \mid z_1^2 + z_2^2 + 1 = 0\}$ is obviously a coordinate slice in this coordinate system (what if $w_2 = 0$? Exercise). So, checking similarly on $V \cap U_1$ and $V \cap U_2$, we see that V is a complex submanifold of $P_2\mathbf{C}$.

It is important to realize that the analogue of Whitney's Embedding Theorem does not hold in the complex case. First of all, a connected, compact complex submanifold of $\mathbf{C}^n$ must be a point. (Proof outline: $z_j \mid N$ is holomorphic if N is a complex submanifold of $\mathbf{C}^n$. If N is compact and connected, the Maximum Modulus Principle implies z_j is constant. Think for yourself about why the Maximum Modulus Principle applies in several variables!) So $P_n\mathbf{C}$ is *not* realizable as a submanifold of $\mathbf{C}^n$.

You might be inclined to hope that $P_n\mathbf{C}$ would play some sort of universal embedding theorem role. But this does not work, either. There are compact complex manifolds that are not submanifolds (in the complex sense) of any $P_n\mathbf{C}$. It is not trivial to see why, however. See §3.

3. Hermitian and Kähler Metrics

DEFINITION. A C^∞ Riemannian metric g on a complex manifold M is an

Hermitian metric if for each $p \in M$ and each pair of tangent vectors $X,Y \in M_p$, the tangent space of M at p,

$$g(X,Y) = g(JX,JY) .$$

Every complex manifold admits an Hermitian metric. To verify this fact, note that, since a complex manifold M is paracompact by definition, it necessarily admits a C^∞ Riemannian metric g_1. If g is defined by $g(X,Y) = g_1(X,Y) + g_1(JX,JY)$, then g is clearly an Hermitian metric.

The following lemma describes the pointwise structure of Hermitian metrics:

LEMMA 1. *Let* V *be a real vector space and* J *an endomorphism of* V *satisfying* $J^2 =$ *multiplication by* -1. *Suppose that* g *is a positive definite inner product on* V *satisfying* $g(X,Y) = g(JX,JY)$ *for all* $X,Y \in V$. *Then*

(a) $g(X,JY) = -g(JX,Y)$.

(b) there exists an orthonormal basis for V of the form $X_1,JX_1,X_2,JX_2,\ldots,X_n,JX_n$.

Proof. To prove (a), note that $g(X,JY) = g(JX,J(JY))$ while $g(JX,J(JY)) = g(JX,-Y) = -g(JX,Y)$.

To prove (b), one follows the method used to prove Lemma 3 of §1: Let X_1 be a unit vector in V. Then JX_1 is a unit vector since $g(JX_1,JX_1) = g(X_1,X_1)$. Also $g(X_1,JX) = -g(JX_1,X_1)$ by part (a) so $g(X_1,JX_1) = 0$. The subspace generated by X_1 and JX_1 is J-invariant, and hence its orthogonal complement is also J-invariant. The desired conclusion now follows by induction on the dimension of V.

If g is an Hermitian metric, then the 2-tensor ω defined by $\omega(X,Y) = g(JX,Y)$ is antisymmetric by part (a) of Lemma 1.

DEFINITION. If g is an Hermitian metric, the 2-form ω defined by $\omega(X,Y) = g(JX,Y)$ is the *Kähler form* of g.

LEMMA 2. *If g is an Hermitian metric, then $\omega \wedge \cdots \wedge \omega$ (n factors, $n =$ the complex dimension of M) is nowhere zero.*

Proof. Given $p \in M$, let $X_1, JX_1, \ldots, X_n, JX_n$ be an orthonormal basis relative to g for M_p. Such a basis exists by Lemma 1b). Then

$$\omega(X_i, JX_i) = g(JX_i, JX_i) = 1$$

while

$$\omega(X_j, JX_i) = g(JX_j, JX_i) = 0 \text{ if } i \neq j .$$

Also

$$\omega(X_i, X_j) = \omega(JX_i, JX_j) = 0 \text{ for all } i,j .$$

Then

$$(\omega \wedge \cdots \wedge \omega)\,(X_1, JX_1, \ldots, X_n, JX_n) = n!\,\omega(X_1, JX_1) \cdot \cdots \cdot \omega(X_n, JX_n) = n!$$

In particular, $(\omega \wedge \cdots \wedge \omega) \neq 0$ at p.

PROPOSITION 1. *A complex manifold is orientable.*

Proof. Since any complex manifold admits an Hermitian metric, Lemma 2 shows that there exists a nowhere vanishing $2n$ form on any complex manifold of real dimension $2n$.

Actually, it can be shown that a complex manifold is orientable without introducing any metric concepts. In fact, a holomorphic mapping from a domain in $\mathbf{C}^n$ to $\mathbf{C}^n$ which is a real diffeomorphism onto its image is necessarily orientation preserving on the underlying $\mathbf{R}^{2n}$ since its (real) Jacobian determinant is positive (by a calculation with determinants using the Cauchy-Riemann equations). Thus a covering by coordinate systems all of whose transition mappings have positive Jacobian determinant is given directly by the complex structure.

DEFINITION. An Hermitian metric g on a complex manifold is a *Kähler metric* if the Kähler form ω associated to g is closed, i.e. $d\omega = 0$.

The condition $d\omega = 0$ implies a surprisingly close relationship between the metric g and the complex structure of M. One aspect of this relationship is expressed in the following proposition.

PROPOSITION 2. *Let M be a complex manifold and g an Hermitian metric on M. Then the following conditions on g are equivalent:*

(a) *g is a Kähler metric*

(b) *If D is the Riemannian covariant differentiation associated to g, then $DJ = 0$.*

Proof that (b) implies (a): Since $Dg = 0$ by definition of g and $\omega(X,Y) = g(JX,Y)$ for all X, Y, the vanishing of DJ implies that of $D\omega$:

$$D_Z\omega(X,Y) = Z\,\omega(X,Y) - \omega(D_ZX,Y) - \omega(X,D_ZY)$$

$$= Zg(JX,Y) - g(J(D_ZX),Y) - g(JX,D_ZY)$$

$$= (D_Zg)\,(JX,Y) + g(D_Z(JX),Y) + g(JX,D_ZY) - g(J(D_ZX),Y) - g(JX,D_ZY)$$

$$= 0 + g((D_ZJ)X,Y) + g(J(D_ZX),Y) + g(JX,D_ZY) - g(J(D_ZX),Y) - g(JX,D_ZY)$$

$$= g(D_ZJ)X,Y) = 0 \text{ if } DJ = 0\,.$$

Now for any form ω, the formula

$$d\omega = \sum_i du^i \wedge D_{\partial/\partial u^i}\,\omega$$

holds, where (u^i) is a real C^∞ local coordinate system. In particular, if $D\omega = 0$ then $d\omega = 0$.

Proof that (a) implies (b): It suffices to establish the following formula for an arbitrary Hermitian metric g and its Kähler form ω:

$$g((D_ZJ)X,Y) = \frac{1}{2}\,d\omega(X,Y,Z) - \frac{1}{2}\,d\omega(JX,JY,Z)\,.$$

For then the vanishing of $d\omega$ implies that $g(D_Z J)X,Y) = 0$ for all X and Y and hence that $D_Z J = 0$. To establish the formula, note first that, all terms being tensors, the formula need be verified only in the case that X, Y, and Z are linear combinations with constant coefficients of the standard vector fields associated to a C^∞ local coordinate system. In fact, this coordinate system may be taken to be the real and imaginary parts of a holomorphic one, say $(x_1,y_1,x_2,y_2,\ldots,x_n,y_n)$. Note that if X, Y, and Z are (locally) such constant coefficient linear combinations of $\partial/\partial x_i$ and $\partial/\partial y_i$ then so are JX, JY, and JZ so that the Lie brackets of any two of the six vector fields X, Y, Z, JX, JY, JZ all vanish.

Then

$$g((D_Z J)X,Y) = g(D_Z(JX) - J(D_Z X),Y)$$

$$= g(D_Z(JX),Y) - g(J(D_Z X),Y) = g(D_Z(JX),Y) + g(D_Z X,JY)$$

$$= \frac{1}{2}\,[(JX)g(Y,Z) + Zg(JX,Y) - Yg(JX,Z)]$$

$$+ \frac{1}{2}\,[Zg(X,JY) + Xg(JY,Z) - (JY)g(X,Z)]\ ;$$

and on the other hand

$$d\omega(X,Y,Z) = X\omega(Y,Z) + Z\omega(X,Y) - Y\omega(X,Z)$$

$$= Xg(JY,Z) + Zg(JX,T) - Yg(JX,Z)$$

and

$$d\omega(JX,JY,Z) = (JX)\omega(JY,Z) + Z\omega(JX,JY) - (JY)\omega(JX,Z)$$

$$= -\ (JX)g(Y,Z) - Zg(X,JY) + (JY)g(X,Z)\ .$$

The desired formula thus follows.

Proposition 2 implies that the tensor J is invariant under parallel translation relative to a Kähler metric; in particular, the value J_p of J at a single point $p \in M$ together with the Kähler metric on M determines J everywhere on M (assuming that M is connected) and thus the complex structure of M.

If g is a Kähler metric, the Kähler form ω, being closed, determines a class in deRham 2-cohomology. Because $\omega\wedge\cdots\wedge\omega$ (n times) is nowhere vanishing, and hence a nonvanishing volume form multiple, ω cannot be exact. (Detail: If $\omega = d\theta$, then $\omega\wedge\cdots\wedge\omega$ (n times) $= d(\theta\wedge\omega\cdots\wedge\omega)$, $n-1$ ω's. So

$$0 = \int d(\theta\wedge\omega\wedge\cdots\wedge\omega) = \int \omega\wedge\cdots\wedge\omega \neq 0\ ,$$

a contradiction.) Thus the cohomology class of ω is nonzero. Hence any Kähler manifold has nonvanishing 2-cohomology in the deRham sense. It can be seen that $M = S^{2p+1} \times S^{2q+1}$ admits a complex structure, all $p,q \geq 1$ (cf. [2]). But then M has zero 2-cohomology and hence does not admit Kähler metrics (see §10), so such an M cannot be realized as a submanifold of $P_n\mathbf{C}$.

4. Complexification of the Tangent and Cotangent Spaces

DEFINITION. Let V be a real vector space. The *complexification* of V, to be denoted $V^{\mathbf{C}}$, is the complex vector space consisting of all ordered pairs $(v,w), v, w \in V$ with operations defined by

$$(v_1,w_1) + (v_2,w_2) = (v_1 + w_1, v_2 + w_2)\ .$$

$$(\alpha + i\beta)\,(v,w) = (\alpha v - \beta w, \alpha w + \beta v)\ , \quad \alpha,\beta \in \mathbf{R}\ .$$

LEMMA 1. *If* V *is a real vector space of dimension* n *and* $v_1,\ldots,v_n$ *is a basis for* V, *then*

(a) $V^{\mathbf{C}}$ *has complex dimension* n *and* $(v_1,0)\cdots(v_n,0)$ *is a (complex) basis for* $V^{\mathbf{C}}$

(b) $V^{\mathbf{C}}$ *considered as a real vector space has dimension* $2n$ *and* $(v_1,0),\ldots,(v_n,0),$ $(0,v_1),\ldots,(0,v_n)$ is a (real) basis for $V^{\mathbf{C}}$.

Proof. As an example, the complex linear independence of $(v_1,0),\ldots,(v_n,0)$ will be verified, the other verifications being left to the reader: If for $\alpha_\ell, \beta_\ell \in R$, $\ell = 1,\ldots,n$,

$$\sum_{\ell=1}^{n} (\alpha_\ell + i\beta_\ell)(v_\ell,0) = (0,0)$$

then

$$(0,0) = \sum_{\ell=1}^{n} (\alpha_\ell v_\ell, \beta_\ell v_\ell) = \left(\sum_{\ell}^{n} \alpha_\ell v_\ell, \sum_{\ell=1}^{n}, \beta_\ell v_\ell\right)$$

or $\sum_{\ell=1}^{n} \alpha_\ell v_\ell = 0$ and $\sum_{\ell=1}^{n} \beta_\ell v_\ell = 0$. Hence $\alpha_\ell = 0$ for all ℓ and $\beta_\ell = 0$ for all ℓ by the real linear independence of the $v_1,\ldots,v_n$.

Since in $V^{\mathbf{C}}$ $i(w,0) = (0,w)$ for any $w \in V$, it is reasonable to introduce the notation $v + iw$ for the element (v,w) of $V^{\mathbf{C}}$.

DEFINITION. Let M be a complex manifold of complex dimension n, p be a point of M, and $(z_1,\ldots,z_n) = (x_1,y_1,\ldots,x_n,y_n)$ be a holomorphic coordinate system defined in a neighborhood of p. Then $\partial/\partial z_i|_p$ and $\partial/\partial \bar{z}_i|_p$, $i = 1,\ldots,n$, are the elements of $M_p^{\mathbf{C}}$ given by

$$\left.\frac{\partial}{\partial z_i}\right|_p = \frac{1}{2}\left(\left.\frac{\partial}{\partial x_i}\right|_p - i\left.\frac{\partial}{\partial y_i}\right|_p\right)$$

$$\left.\frac{\partial}{\partial \bar{z}_i}\right|_p = \frac{1}{2}\left(\left.\frac{\partial}{\partial x_i}\right|_p + i\left.\frac{\partial}{\partial y_i}\right|_p\right).$$

When the point p is clear from the context the abbreviated notations $\partial/\partial z_i$, $\partial/\partial \bar{z}_i$ will be used.

Since

$$\frac{\partial}{\partial x_i} = \frac{\partial}{\partial z_i} + \frac{\partial}{\partial \bar{z}_i}$$

and

$$\frac{\partial}{\partial y_1} = -i\left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial \bar{z}_i}\right)$$

it follows from Lemma 1 that $\partial/\partial z_i,\ldots,\partial/\partial z_n$, $\partial/\partial \bar{z}_1,\ldots,\partial/\partial \bar{z}_1$, being thus a spanning set of $2n$ elements, are a complex basis for $M_p^{\mathbf{C}}$.

Lemma 2. *The complex subspace of $M_p^{\mathbf{C}}$ spanned by $\partial/\partial z_1,\ldots,\partial/\partial z_n$ is independent of the choice of holomorphic coordinate system $(z_1,\ldots,z_n)$. The same is true of the subspace spanned by $\partial/\partial \bar{z}_1,\ldots,\partial/\partial \bar{z}_n$.*

Proof. Let $(z'_1,\ldots,z'_n) = (x'_1,y'_1,\ldots,x'_n,y'_n)$ be another holomorphic coordinate system in a neighborhood of p. Then at p for any $\ell = 1,\ldots,n$

$$\frac{\partial}{\partial z_\ell} = \frac{1}{2}\left(\frac{\partial}{\partial x'_\ell} - i\,\frac{\partial}{\partial x'_\ell}\right) = \frac{1}{2}\left(\sum_{j=1}^{n} \frac{\partial x_j}{\partial x'_j}\frac{\partial}{\partial x_j} + \sum_{j=1}^{n} \frac{\partial y_j}{\partial x'_\ell}\frac{\partial}{\partial y_j}\right)$$

$$-\frac{i}{2}\left(\sum_{j=1}^{n} \frac{\partial x_j}{\partial y'_\ell}\frac{\partial}{\partial x_j} + \sum_{j=1}^{n} \frac{\partial y_j}{\partial y'_\ell}\frac{\partial}{\partial y_j}\right)$$

$$= \frac{1}{2}\sum_{j=1}^{n}\left(\frac{\partial x_j}{\partial x'_\ell} - i\,\frac{\partial x_j}{\partial y'_\ell}\right)\frac{\partial}{\partial x_j} - \frac{1}{2}\left(\sum_{j=1}^{n}\left(\frac{\partial y_i}{\partial y'_\ell} + i\,\frac{\partial y_i}{\partial x'_\ell}\right)\right.$$

$$= \sum_{j=1}^{n}\left(\frac{\partial x_j}{\partial x'_\ell} - i\,\frac{\partial x_j}{\partial y'_\ell}\right)\left[\frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\,\frac{\partial}{\partial y_\ell}\right)\right]$$

since

$$\frac{\partial x_j}{\partial x'_\ell} - i\,\frac{\partial x_j}{\partial y'_\ell} = \frac{\partial y_j}{\partial y'_j} + i\,\frac{\partial y_j}{\partial x'_\ell}$$

by the Cauchy-Riemann equations. Thus

$$\frac{\partial}{\partial z'_\ell} = \sum_{j=1}^{n}\left(\frac{\partial x_j}{\partial x'_\ell} - i\,\frac{\partial x_j}{\partial y'_\ell}\right)\frac{\partial}{\partial z_j}$$

so $\partial/\partial z'_\ell$ belongs to the complex subspace spanned by $\partial/\partial z_1,\ldots,\partial/\partial z_n$.

The proof for the subspace spanned by $\partial/\partial\bar{z}_1,\ldots,\partial/\partial/\bar{z}_n$ is similar and will be omitted.

If f is a real C^∞ function defined in a neighborhood of p, then each element of $M_p^{\mathbf{C}}$ acts on f by complex linear extension of the action of the elements of M_p on f by differention:

$$(v + iw)f = (vf) + i(wf) \quad v,w \in M_p\,.$$

A second complex linear extension definition gives an action of each element of $M_p^{\mathbf{C}}$ on complex valued functions defined in a neighborhood of p which are C^∞ considered as functions into R^2:

$$(v + iw)\,(f) = (v + iw)\mathrm{Ref} + i(v + iw)\mathrm{Imf}$$

$$= [v(\mathrm{Ref}\,) - w(\mathrm{Imf}\,)] + i[w(\mathrm{Ref}\,) + v(\mathrm{Imf}\,)]$$

for $v,w \in M_p$, Ref , Imf as usual. Note that in this sense the last equation in the proof of Lemma 2 can be rewritten

$$\frac{\partial}{\partial z'_\ell} = \sum_{j=1}^{n}\left(\frac{\partial}{\partial z'_\ell}\,z_j\right)\frac{\partial}{\partial z_j}$$

since

$$\frac{\partial}{\partial z'_\ell}\,z_j = \frac{1}{2}\left(\frac{\partial}{\partial x'_\ell} - i\,\frac{\partial}{\partial y'_\ell}\right)(x_j + iy_j)$$

$$= \frac{1}{2}\left(\frac{\partial x_j}{\partial x'_\ell} + \frac{\partial y_j}{\partial y'_\ell} - i\,\frac{\partial x_j}{\partial y'_\ell} + i\,\frac{\partial y_j}{\partial x'_\ell}\right) = \left(\frac{\partial x_j}{\partial x'_\ell} - i\,\frac{\partial x_j}{\partial y'_\ell}\right),$$

the last equality following from again applying the Cauchy-Riemann equations. Similar computations can be used to show that

$$\frac{\partial}{\partial \bar{z}'_\ell} = \sum_{j=1}^{n}\left(\frac{\partial}{\partial \bar{z}'_\ell}\,\bar{z}_j\right)\frac{\partial}{\partial \bar{z}_j}$$

where the $\bar{z}_j$ in the parenthetical expression denotes the function $x_j - iy_j$.

LEMMA 3. *Let f be a complex valued function defined in a neighborhood of p in M which is C^∞ as a function into R^2 and $(z_1,\ldots,z_n)$ be a holomorphic coordinate system in a neighborhood of p. Then*

(a) *for all $\ell = 1,\ldots,n$*

$$\left(\frac{\partial}{\partial z_\ell}\overline{f}\right) = \overline{\frac{\partial}{\partial \overline{z}_\ell} f}$$

(where $\overline{}$ denotes complex conjugation).

(b) *f is holomorphic on the domain of definition of the coordinate system $(z_1,\ldots,z_n)$ if and only if everywhere on the domain*

$$\frac{\partial}{\partial \overline{z}_\ell} f = 0 \text{ for all } \ell = 1,\ldots,n\ .$$

Proof. The assertion (a) follows immediately from the definitions. To prove (b), recall that f is holomorphic if and only if $f_*J_M = J_{\mathbf{R}^2}f_*$ everywhere on the domain. Since

$$J\left(\frac{\partial}{\partial x_\ell}\right) = \frac{\partial}{\partial y_\ell} \text{ and } J\left(\frac{\partial}{\partial y_\ell}\right) = -\frac{\partial}{\partial x_\ell}\ ,$$

f is holomorphic if and only if

$$f_*\left(\frac{\partial}{\partial y_\ell}\right) = J_{\mathbf{R}^2}f_*\left(\frac{\partial}{\partial x_\ell}\right)$$

and

$$f_*\left(-\frac{\partial}{\partial x_\ell}\right) = J_{\mathbf{R}^2}f_*\left(\frac{\partial}{\partial y_\ell}\right).$$

Now

$$f_*\left(\frac{\partial}{\partial x_\ell}\right) = \left(\frac{\partial}{\partial x_\ell}(\text{Ref}\,)\ ,\ \frac{\partial}{\partial x_\ell}(\text{Imf}\,)\right)$$

$$f_*\left(\frac{\partial}{\partial y_\ell}\right) = \left(\frac{\partial}{\partial y_\ell}(\text{Re}f), \frac{\partial}{\partial y_\ell}(\text{Im}f)\right)$$

so

$$J_{\mathbf{R}^2}f_*\left(\frac{\partial}{\partial x_\ell}\right) = \left(-\frac{\partial}{\partial x_\ell}(\text{Im}f), \frac{\partial}{\partial x_\ell}(\text{Re}f)\right)$$

$$J_{\mathbf{R}^2}f_*\left(\frac{\partial}{\partial y_\ell}\right) = \left(-\frac{\partial}{\partial y_\ell}(\text{Im}f), \frac{\partial}{\partial y_\ell}(\text{Re}f)\right).$$

Thus the necessary and sufficient condition for f to be holomorphic is that

$$\frac{\partial}{\partial x_\ell}(\text{Re}f) = \frac{\partial}{\partial y_\ell}(\text{Im}f) \quad \text{and} \quad \frac{\partial}{\partial x_\ell}(\text{Im}f) = -\frac{\partial}{\partial y_\ell}(\text{Re}f).$$

Since

$$\frac{\partial}{\partial \bar{z}_\ell}f = \frac{1}{2}\left(\frac{\partial}{\partial x_\ell}(\text{Re}f) - \frac{\partial}{\partial y_\ell}(\text{Im}f)\right) + \frac{i}{2}\left(\frac{\partial}{\partial x_\ell}(\text{Im}f) + \frac{\partial}{\partial y_\ell}(\text{Re}f)\right)$$

the conclusion now follows.

The process of complexifying the tangent space and relating this complexification to holomorphic coordinate systems has an analogue in the case of the cotangent space:

DEFINITION. Let $(z_1,\ldots,z_n)$ be a holomorphic coordinate system defined in a neighborhood of a point $p \in M$. Then the elements dz_i, $d\bar{z}_i \in (M^*_p)^{\mathbf{C}}$ are given by

$$dz_i \overset{\text{Def}}{=} dx_i + idy$$
$$d\bar{z}_i \overset{\text{Def}}{=} dx_i - idy.$$

Here M^*_p is the real cotangent space of M at p.

Since $dx_i = (dz_i + d\bar{z}_i)/2$ and $dy_i = (d\bar{z}_i - dz_i)/2$, it follows from Lemma 1 that $dz_i,\ldots,dz_n$, $d\bar{z}_1,\ldots,d\bar{z}_n$ form a complex basis for $(M^*_p)^{\mathbf{C}}$.

The previous definition is a special case of the following definition:

DEFINITION. Let f be a complex-valued function defined in a neighborhood of $p \in M$ which is C^∞ considered as a function into R^2. Then $df \in (M^*_p)^C$ is given by

$$df \overset{\text{Def}}{=} d(\text{Re} f) + id(\text{Im} f)\ .$$

LEMMA 4. *For any f as in the previous definition, at the point $p \in M$:*

$$df = \sum_{i=1}^{n} \left(\frac{\partial}{\partial z_i} f\right) dz_i + \sum_{i=1}^{n} \left(\frac{\partial}{\partial \bar{z}_i} f\right) d\bar{z}_i$$

for any holomorphic coordinate system $(z_1,\ldots,z_n)$ defined in a neighborhood of p.

Proof.

$$\frac{\partial}{\partial z_i} f = \frac{1}{2} \left(\frac{\partial}{\partial x_i} - i \frac{\partial}{\partial x_i}\right) f$$

$$\left(\frac{\partial}{\partial z_i} f\right) dz_i = \left(\frac{\partial}{\partial z_i} f\right) \left(dx_i + idy_i\right)$$

$$= \frac{1}{2} \left[\left(\frac{\partial}{\partial x_i} f\right) dx_i + \left(\frac{\partial}{\partial y_i} f\right) dy_i\right] + \frac{i}{2} \left[\left(\frac{\partial}{\partial x_i} f\right) dy_i - \left(\frac{\partial}{\partial y_i} f\right) dx_i\right] .$$

Similarly

$$\left(\frac{\partial}{\partial \bar{z}_i} f\right) d\bar{z}_i = \frac{1}{2} \left[\left(\frac{\partial}{\partial x_i} f\right) dx_i + \left(\frac{\partial}{\partial y_i} f\right) dy_i\right]$$

$$- \frac{i}{2} \left[\left(\frac{\partial}{\partial x_i} f\right) dy_i - \left(\frac{\partial}{\partial y_i} f\right) dx_i\right] .$$

Thus

$$\left(\frac{\partial}{\partial z_i} f\right) dz_i + \left(\frac{\partial}{\partial \bar{z}_i} f\right) d\bar{z}_i = \left(\frac{\partial}{\partial x_i} f\right) dx_i + \left(\frac{\partial}{\partial y_i} f\right) dy_i$$

from which the formula of the lemma follows.

LEMMA 5. *The subspace of* $(M^*_p)^{\mathbf{C}}$ *spanned by* $dz_1,\ldots,dz_n$ *is independent of the holomorphic coordinate system defined in a neighborhood of* p. *The same is true of the subspace of* $(M^*_p)^{\mathbf{C}}$ *spanned by* $d\bar{z}_1,\ldots,d\bar{z}_n$.

Proof. Let $(z'_1,\ldots,z'_n)$, $(z_1,\ldots,z_n)$ be two holomorphic coordinate systems each defined in a neighborhood of $p \in M$. Then by Lemma 4

$$dz'_j = \sum_{j=1}^{n} \left(\frac{\partial}{\partial z_j} z'_i\right) dz_j + \sum_{j=1}^{n} \left(\frac{\partial}{\partial \bar{z}_j} z'_i\right) d\bar{z}_j .$$

Lemma 3 implies that $(\partial/\partial\bar{z}_j)\, z^*_i = \overline{(\partial/\partial z_j)\bar{z}_i} = 0$, so that dz'_i is a linear combination of the dz_j's. Similarly, $d\bar{z}_i$ is a linear combination of $d\bar{z}_j$'s.

An element $\omega_1 + i\omega_2 \in (M^*_p)^{\mathbf{C}}$, $\omega_1,\omega_2 \in M^*_p$, gives rise to a complex linear functional on $(M^*_p)^{\mathbf{C}}$ by taking

$$(\omega_1 + i\omega_2)\,(v + iw) = \omega_1(v) - \omega_2(w) + i\omega_2(v) + i\omega_1(w)$$

for $v,w \in M_p$. $(M^*_p)^{\mathbf{C}}$ can be thus identified with a subspace of the complex dual of $M_p^{\mathbf{C}}$, and by dimensionality considerations this subspace is in fact the whole of the complex dual of $M_p^{\mathbf{C}}$. Note that using this identification procedure, $dz_1,\ldots,dz_n$, $d\bar{z}_1,\ldots,d\bar{z}_n$ is the complex basis of $(M^{*p})^{\mathbf{C}}$ which is dual to the basis $\partial/\partial z_1,\ldots,\partial/\partial z_n$, $\partial/\partial\bar{z}_1,\ldots,\partial/\partial\bar{z}_n$ of $(M_p)^{\mathbf{C}}$ since

$$dz_i\left(\frac{\partial}{\partial \bar{z}_j}\right) = \frac{1}{2}\,(dx_i + i dy_i)\left(\frac{\partial}{\partial x_j} + i\,\frac{\partial}{\partial y_j}\right)$$

$$= \frac{1}{2}\,(\delta_{ij} - \delta_{ij}) + i0 = 0$$

while

$$dz_i\left(\frac{\partial}{\partial z_j}\right) = \frac{1}{2}\,(dx_i + i dy_i)\left(\frac{\partial}{\partial x_j} - i\,\frac{\partial}{\partial y_j}\right)$$

$$= \frac{1}{2}\,(\delta_{ij} + \delta_{ij}) + i(0) = \delta_{ij}$$

and similarly

$$d\bar{z}_i\left(\frac{\partial}{\partial z_j}\right) = 0$$

while

$$d\bar{z}_i\left(\frac{\partial}{\partial \bar{z}_j}\right) = \delta_{ij} .$$

(Here $\delta_{ij} = 0$ if $i \neq j = 1$ if $i = j$.)

The definition of $df \in (M^*_p)^{\mathbf{C}}$, where f is complex valued, combined with the definition of the action of the elements of $(M^*_p)^{\mathbf{C}}$ on $M_p^{\mathbf{C}}$ yields immediately that for all $V \in M_p^{\mathbf{C}}$

$$df(V) = Vf$$

where Vf is defined as previously. The duality of $dz_1,\ldots,dz_n$, $d\bar{z}_1,\ldots,d\bar{z}_n$ and $\partial/\partial z_1,\ldots,\partial/\partial z_1,\ldots,\partial/\partial z_n$, $\partial/\partial\bar{z}_1,\ldots,\partial/\partial z_n$ can be interpreted from this viewpoint also, since $(\partial/\partial z_i)\bar{z}_j = 0$, $(\partial/\partial z_i)z_j = \delta_{ij}$, etc.

5. Complex-Valued Differential Forms

DEFINITION. A *complex-valued* r-form (or *complex* r-form) on a real vector space V is an element of $(\Lambda^r V^*)^C$, where $\Lambda^r V^* =$ the real vector space of real r-forms on V. The (complex) *wedge product* $\wedge : (\Lambda^r V^*)^{\mathbf{C}} \times (\Lambda^s V^*)^{\mathbf{C}} \to (\Lambda^{r+s} V^*)^{\mathbf{C}}$ is the complex linear extension of the real wedge product $\wedge : (\Lambda^r V^*) \times (\Lambda^s V^*) \to \Lambda^{r+s} V^*$.

The complex wedge product has the associativity and skewcommutativity properties of the real wedge product.

LEMMA 1. If V *is a real vector space of dimension* N, *then* $(\Lambda^r V^*)^{\mathbf{C}}$ *has complex dimension (the binomial coefficient)* $\binom{N}{r}$; *and if* $\{\omega_1,\ldots,\omega_N\}$ *is any (complex) basis for* $(V^*)^{\mathbf{C}}$ *then the set of forms*

$$\{\omega_{i_1} \wedge \cdots \wedge \omega_{i_r} \mid 1 \le i_1 < i_2 < \cdots < i_r \le N\}$$

is a basis for $(\Lambda^r V^*)^{\mathbf{C}}$.

Proof. let $\theta_1,\ldots,\theta_N$ be a (real) basis for V^*. Then the set $\{\theta_{i_1} \wedge \cdots \wedge \theta_{i_r} \mid 1 \le i_1 < \cdots < i_r \le N\}$ is a basis for $\Lambda^r V^*$, which thus has dimension $\binom{N}{r}$. Hence $(\Lambda^r V^*)^{\mathbf{C}}$ has complex dimension $\binom{N}{r}$ by Lemma 1 of §4. Since the elements $\omega_i \wedge \cdots \wedge \omega_{i_r}$, $1 \le i_1 < \cdots < i_r \le N$, are $\binom{N}{r}$ in number, they necessarily form a complex basis of $(\Lambda_r V^*)^{\mathbf{C}}$ if they generate $(\Lambda_r V^*)^{\mathbf{C}}$. Now, for each i, $\theta_i = \sum_{j=i}^N \alpha_{ij}\omega_j$ for some (uniquely determined) complex numbers α_{ij}, because each $\theta_i \in (V^*)^{\mathbf{C}}$ and the ω_j's are a basis for $(V^*)^{\mathbf{C}}$. Hence

$$\theta_{i_1} \wedge \cdots \wedge \theta_{i_r} = \left(\sum_{j=1}^N \alpha_{i_1 j}\omega_j\right) \wedge \cdots \wedge \left(\sum_{j=1}^N \alpha_{i_r j}\omega_j\right),$$

from which equation it follows that $\theta_{i_1} \wedge \cdots \wedge \theta_{i_r}$ is a linear combination with complex coefficients of wedge products of ω_j's. Since any such wedge product is $\pm$ a wedge product of the form $\omega_{j_1} \wedge \cdots \wedge \omega_{j_r}$, $1 \le j_1 < \cdots < j_r \le N$, each $\theta_{i_1} \wedge \cdots \wedge \theta_{i_r}$ belongs to the complex subspace of $(\Lambda^r V^*)^{\mathbf{C}}$ generated by $\{\omega_{j_1} \wedge \cdots \wedge \omega_{j_r} \mid 1 \le j_1 < j_2 < \cdots < j_r \le N\}$. Hence this subspace is in fact all of $(\Lambda^r V^*)^{\mathbf{C}}$.

If M is a complex manifold of complex dimension n and $(z_1,\ldots,z_n)$ a holomorphic coordinate system in a neighborhood of a point $p \in M$, then $\{dz_1,\ldots,dz_n, d\bar{z}_1,\ldots, d\bar{z}_n\}$ is a basis for $(M^*_p)^{\mathbf{C}}$. Thus it follows from Lemma 1 that

$$\{dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q} \mid 1 \le i_1 < i_2 \cdots < i_p \le n,$$
$$1 \le j_i < j_2 < \cdots < j_q \le n, p + q = r\}$$

is a basis for $(\Lambda^r M^*_p)\mathbf{C}$.

Note: We have to rely on context to distinguish the integer p from the point $p \in M$ in what follows. Twenty-six letters just are not enough! Apologies.

LEMMA 2. *If* $(z_1,\ldots,z_n)$ *and* $(w_1,\ldots,w_n)$ *are two holomorphic coordinate systems both defined in a neighborhood of* $p \in M$, *then* $dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$ *in* $(\Lambda^{p+q} M^*{}_p)^{\mathbf{C}}$ *is a linear combination with complex coefficients of forms of the type* $dw_{k_1} \wedge \cdots \wedge dw_{k_p} \wedge d\bar{w}_{\ell_1} \wedge \cdots \wedge d\bar{w}_{\ell_q}$ *(p and q are here fixed throughout).*

Proof. Each dz_i at p is a linear combination of dw_j's and each dz_i a linear combination of $d\bar{w}_j$'s (Lemma 5, §4); the result follows directly.

DEFINITION. An element ω of $(\Lambda^r M^*{}_p)^{\mathbf{C}}$ is said to be of *type* (p,q), $p + q = r$, if for some (and hence by Lemma 2 any) holomorphic coordinate system $(z_1,\ldots,z_n)$ defined in a neighborhood of p, Ω belongs to the subspace of $(\Lambda^r M^*{}_p)^{\mathbf{C}}$ spanned by the set

$$\{dz_1 \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} < \cdots \wedge d\bar{z}_{j_q} \mid 1 \le i_1 < i_2 < \cdots < i_p \le n,$$
$$1 \le j_1 < j_2 < \cdots < j_q \le n\}.$$

It follows from Lemmas 1 and 2 that any element Ω of $(\Lambda^r M^*{}_p)^{\mathbf{C}}$ can be expressed as a sum of elements $\Omega^{(p,q)}$, $p + q = r$, $p \ge 0$, $q \ge 0$, of type (p,q); moreover, the elements $\Omega^{(p,q)}$ are uniquely determined. Thus well-defined complex linear maps $\pi_{(p,q)} : (\Lambda^r M^*{}_p)^{\mathbf{C}} \to (\Lambda^r M^*{}_p)\mathbf{C}$ can be obtained by setting $\pi_{(p,q)} = \Omega^{(p,q)}$. Then for any Ω, $\Omega = \sum_{p+q=r,p\ge 0,q\ge 0} \pi_{(p,q)}\,\Omega$.

The elements of $(\Lambda^r M^*{}_p)^{\mathbf{C}}$ act on r-tuples $v_1,\ldots,v_r$ of elements of M_p as complex-valued real multilinear alternating mappings. By complex multilinear extension, the elements of $(\Lambda^r M^*{}_p)^{\mathbf{C}}$ can be taken to act on r-tuples of elements of $M_p^{\mathbf{C}}$ now as complex-valued complex multilinear alternating mappings.

A real linear conjugation operation on $(\Lambda^r M^*{}_p)^{\mathbf{C}}$ can be defined as a special case of a general conjugation on $V^{\mathbf{C}}$, V any real vector space $\overline{v + iw} = v - iw$, $v,w \in V$.

This operation is conjugate linear relative to the complex vector space structure on $V^{\mathbb{C}}$. An element of $V^{\mathbb{C}}$ is an element of $V \subset V^{\mathbb{C}}$ if and only if it is invariant under the conjugation operation $\overline{}$. Note that if an element of $(\Lambda^r M^*_p)^{\mathbb{C}}$ is of type (p,q) then its conjugate is of type (q,p) since $\overline{dz_i} = d\overline{z}_i$ and $\overline{\Omega_1 \wedge \Omega_2} = \overline{\Omega}_1 \wedge \overline{\Omega}_2$ for any elements Ω_1, Ω_2 of $(\Lambda^r M^*_p)^{\mathbb{C}}$.

DEFINITION. A (C^∞) complex-value r-form ω on M is a mapping $\omega : M \to \cup_{p \in M} (\Lambda^r M^{*p})^{\mathbb{C}}$ such that

(a) $\omega(p) \in (\Lambda^r M^*_p)^{\mathbb{C}}$ for all $p \in M$

(b) ω is C^∞ in the sense that when ω is expressed in terms of a

$$\{dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q} \mid 1 \le i_1 < \cdots < i_p \le n, \quad 1 \le j_1 < \cdots < j_q \le n,\ p + q = r\}$$

basis, the coefficients are C^∞ (complex-valued) functions of $x_1, y_1, \ldots, x_n, y_n$ on the domain of holomorphic coordinate system $(z_1, \ldots, z_n)$. The complex vector space of all complex-valued r forms on M will be denoted $\Omega^r(M)$. An r-form $\omega \in \Omega^r(M)$ is *of type* (p,q) if $\omega(p)$ if of type (p,q) in $(\Lambda^r M^*_p)^{\mathbb{C}}$ for every $p \in M$. The vector subspace of $\Omega^r(M)$ consisting of all r-forms of type (p,q) will be denoted $\Omega^{p,q}(M)$.

It is easy to verify that a complex-valued form ω which has the property that $\omega(p) \in \Lambda^r M^*_p$ for every $p \in M$ is a (real) C^∞ form in the usual sense and that, on the other hand, any real C^∞ form ω on M can be considered to be a complex-valued form. Moreover, since $dz_1, \ldots, dz_n$, $d\overline{z}_1, \ldots, d\overline{z}_n$ is dual to $\partial/\partial z_1, \ldots, \partial/\partial z_n$, $\partial/\partial \overline{z}_1, \ldots, \partial/\partial \overline{z}_1$, the condition b) in the definition of a complex-valued r-form can be reformulated as the requirement that

$$\omega(p) \left(\left.\frac{\partial}{\partial z_{i_1}}\right|_p, \ldots, \left.\frac{\partial}{\partial z_{i_p}}\right|_p, \left.\frac{\partial}{\partial \overline{z}_{j_1}}\right|_p, \ldots, \left.\frac{\partial}{\partial \overline{z}_{j_q}}\right|_p \right), p + q = r,$$

be a C^∞ (complex-valued) function of $p \in M$ on the domain of $(z_1, \ldots, z_n)$ for any set of indices $i_1,\ldots,i_p, j_1,\ldots,j_q$. It follows that if $v_1,\ldots,v_r$ are any r complex linear combinations of $\partial/\partial z_1,\ldots,\partial/\partial z_n,\partial/\partial\bar{z}_1,\ldots,\partial/\partial\bar{z}_n$ with C^∞ coefficients then $\omega(p)\,(v_1|_p,\ldots,v_r|_p)$ is a C^∞ function of p. In particular,

$$\omega(p)\left(\frac{\partial}{\partial x_1}\bigg|_p, \frac{\partial}{\partial y_1}\bigg|_p,\ldots,\frac{\partial}{\partial x_n}\bigg|_p, \frac{\partial}{\partial y_n}\bigg|_p\right)$$

is a C^∞ function. An extended (to arbitrary C^∞ real coordinate systems) converse of this last result also holds: $\omega : M \to \cup_{p\in M} (\Lambda^r M^*{}_p)^{\mathbb{C}}$ with $\omega(p) \in (\Lambda^r M^*{}_p)\mathbb{C}$ is C^∞ if and only if it is C^∞ in real C^∞ coordinate expressions.

By complex linearity, the operator d can be extended to $\Omega^r(M)$: Explicitly, let $(u_1,\ldots,u_{2n})$ be a real coordinate system in a neighborhood of $p \in M$ and write (uniquely)

$$\omega = \sum_{1\le i_1<\cdots<i_r\le 2n} f_{i_1,\ldots,i_r}\, du_{i_1} \wedge \cdots \wedge du_{i_r}\,,$$

where the f's are C^∞ complex-valued. Then

$$d\omega = \sum_{1\le i_1<\cdots<i_r\le 2n} df_{i_1,\ldots,i_r} \wedge du_{i_1} \wedge \cdots \wedge du_{i_r}$$

where $df_{i_1,\ldots,i_r}$ is defined as in §4. The usual argument from the real case shows that $d\omega \in \Omega^{r+1}(M)$ thus defined is independent of the choice of the real coordinate system $(u_1,\ldots,u_{2n})$. Also from the real case it follows easily that d thus extended to complex-valued forms satisfies

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^r\, \omega_1 \wedge d\omega_2$$

for $\omega \in \Omega^r(M)$.

LEMMA 3.

$$d(f\, dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q})$$

$$= \sum_{i=1}^{n} \left(\frac{\partial f}{\partial z_i}\, dz_i \wedge dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}\right)$$

$$+ \sum_{i=1}^{n} \left((-1)^p \frac{\partial f}{\partial \bar{z}_i} dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_i \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q} \right) .$$

Proof. Since $d(dz_\ell) = d(d\bar{z}_\ell) = 0$ for all $\ell = 1,\ldots,n$, the formula given follows by repeated application of $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^r \omega_1 \wedge d\omega_2$, $\omega_1, \omega_2 \in \Omega^r$, together with Lemma 4 of §4.

Lemma 3 implies immediately that, for $\omega \in \Omega^{p,q}(M)$,

$$d\omega \in \Omega^{p+1,q}(M) + \Omega^{p,q+1}(M) .$$

DEFINITION. $\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M)$ is by definition $\pi_{p+1,q} \circ d : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M)$. $\bar{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)$ is $\pi_{p,q+1} \circ d : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)$. More generally,

$$\partial : \Omega^r(M) \to \Omega^{r+1}(M) \text{ is the operator } \textstyle\sum_{p+q=r} \pi_{p+1,q}\, d\, \pi_{p,q}$$

and

$$\bar{\partial} : \Omega^r(M) \to \Omega^{r+1}(M) \text{ is } \textstyle\sum_{p+q=r} \pi_{p,q+1}\, d\, \pi_{p,q} .$$

In case the meaning is clear, the symbols ∂ and $\bar{\partial}$ will be used without explicit notation of their domain, as is usually done with the operator d.

LEMMA 4. $d = \partial + \bar{\partial}$, $\partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0$. *If* $\omega \in \Omega^{p,q}(M)$ *then* $\bar{\omega} \in \Omega^{q,p}(M)$ *and* $\bar{\partial}\bar{\omega} = \overline{\partial\omega}$.

Proof. All the assertions follow easily from the definitions of ∂ and $\bar{\partial}$, the fact that $d^2 = 0$, the direct sum decomposition of $\Omega^r(M) = \sum_{p+q=r} \Omega^{p,q}(M)$, and the behavior of types under conjugation. The details are left to the reader.

Lemma 3 provides a computational description of ∂ and $\bar{\partial}$:

$$\partial(f\, dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge dz_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q})$$

$$= \sum_{i=1}^{n} \frac{\partial f}{\partial z_i} dz_i \wedge dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$$

$$\bar\partial(f\, dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar z_{j_1} \wedge \cdots \wedge d\bar z_{j_q})$$

$$= \sum_{i=1}^{n} (-1)^p \frac{\partial f}{\partial \bar z_i}\, dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar z_i \wedge d\bar z_{j_1} \wedge \cdots \wedge d\bar z_{j_q} .$$

These formulae can be used to provide computational proof of Lemma 4.

6. Hermitian and Kähler Metrics in Complex Notation

The subspace of $M_p^{\mathbb{C}}$ spanned by $\partial/\partial z_1|_p, \ldots, \partial/\partial z_n|_p$ (where $(z_1, \ldots, z_n)$ is a holomorphic coordinate system in a neighborhood of $p \in M$) is called the *holomorphic tangent space* at p. *It is a complex vector space of dimension* n. By Lemma 2, §4, it is well defined independently of the choice of $(z_1, \ldots, z_n)$ coordinates. Notation: M_p^h.

If g is a Hermitian metric on M (i.e., a J-invariant Riemannian metric), then g determines a Hermitian metric on each holomorphic tangent space M_p^h in the conventional sense of complex linear algebra as follows: First note that g extends by complex linearity to be a complex bilinear form $\hat g$ on $M_p^{\mathbb{C}}$. (This form cannot be positive definite!) Define for $V, W \in M_p^h$

$$g(V,W) = \hat g(V, \overline{W}) .$$

Then g on M_p^h so defined is a positive definite Hermitian metric on M_p^h. (Note: The use— for both M_p and M_p^h— of the same symbol g can lead to no confusion, especially since, for real vectors, the conjugation in the definition of g on M_p^h would be of no effect.)

Define, given $(z_1, \ldots, z_n)$ holomorphic coordinates,

$$g_{i\bar j} = g\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right) = \hat g\left(\frac{\partial}{\partial z_i}, \overline{\frac{\partial}{\partial z_j}}\right).$$

Then

$$g_{i\bar j} = \frac{1}{4} \hat g\left(\frac{\partial}{\partial x_i} - \sqrt{-1}\,\frac{\partial}{\partial y_i}, \frac{\partial}{\partial x_j} + \sqrt{-1}\,\frac{\partial}{\partial y_j}\right)$$

$$= \frac{1}{4}\left[g\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right)+g\left(\frac{\partial}{\partial y_i},\frac{\partial}{\partial y_j}\right)\right]+\frac{\sqrt{-1}}{4}\left[g\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial y_j}\right)-g\left(\frac{\partial}{\partial y_i},\frac{\partial}{\partial x_j}\right)\right]$$

$$= \frac{1}{2}\,g\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right)+\frac{\sqrt{-1}}{2}\,g\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial y_j}\right)$$

because

$$g\left(\frac{\partial}{\partial y_i},\frac{\partial}{\partial y_j}\right)=g\left(J\,\frac{\partial}{\partial x_i},J\,\frac{\partial}{\partial x_j}\right)=g\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right)$$

and

$$g\left(\frac{\partial}{\partial y_i},\frac{\partial}{\partial x_j}\right)=g\left(J\,\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right)=-\,g\left(\frac{\partial}{\partial x_i},J\,\frac{\partial}{\partial x_j}\right)=-\,g\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial y_i}\right).$$

These formulae express g on M_p^h in terms of g on M_p. They also show (when run the other way) the following: Every Hermitian metric on M_p^h arises from exactly one J-invariant (Riemannian) metric on M_p.

Of course, since all these considerations happen at one point p at a time, they are really facts about complex linear algebra!

As a tensor, the metric on $M_{\mathbb{C}}^{h}$ is given by

$$G=\sum_{i,j} g_{i\bar{j}}\,dz_i\otimes d\bar{z}_j$$

*in the sense that**

$$g(V,W)=G(V,\overline{W})\qquad V,W\in M_p^h\,.$$

Associated to the tensor G $(=\hat{g})$ is a (real) 2-form

$$-\sqrt{-1}\sum_{i,j} g_{i\bar{j}}\,dz_i\wedge d\bar{z}_j\,.$$

*No complex tensor could operate on V,W to give $g(V,W)$: there must be a complex conjugation applied to W first since $g(V,+\sqrt{-1}\,W)=-\sqrt{-1}\,g(V,W)$!

Exercise. Check that this association is well defined independently of coordinate choice. This form is just the Kähler form as defined on page 3.2 (page 2 of §3).

Philosophical remark. This is one of many places where it is possible to get $\pm$ signs wrong, or to gain or lose factors of 1/2 or 1/4. Usually it does not matter. But in cases where it does (later: positive vs. negative curvature), it is safest to check all computations by simple examples. There are many papers that are supposed to be about positive bundle (definition later) that are actually about negative ones, etc.

The metric is Kähler if and only if

$$0 = d\left(\sum_{i,j} g_{i\bar{j}}\, dz_i \wedge d\bar{z}_j\right) \quad \Leftrightarrow \quad 0 = \sum_{i,j,\ell} \left(\frac{\partial g_{i\bar{j}}}{\partial z_\ell}\, dz_\ell + \frac{\partial g_{i\bar{j}}}{\partial \bar{z}_\ell}\, d\bar{z}_\ell\right) dz_i \wedge d\bar{z}_j\,.$$

Since the terms from the dz_ℓ have type (2,1) while those from the $d\bar{z}_\ell$ have type (1,2) each sum

$$\sum \frac{\partial g_{i\bar{j}}}{\partial z_\ell}\, dz_\ell \wedge dz_i \wedge d\bar{z}_j \quad \text{and} \quad \sum \frac{\partial g_{i\bar{j}}}{\partial \bar{z}_\ell}\, d\bar{z}_\ell \wedge dz_i \wedge d\bar{z}_j$$

must vanish separately if (and only if) g is a Kähler metric. It follows that: g is a Kähler metric if and only if

$$\frac{\partial g_{i\bar{j}}}{\partial z_\ell} = \frac{\partial g_{\ell\bar{j}}}{\partial z_i} \quad \text{and} \quad \frac{\partial g_{i\bar{j}}}{\partial \bar{z}_\ell} = \frac{\partial g_{i\bar{\ell}}}{\partial \bar{z}_j}$$

for all i,j,ℓ. We shall use this soon, in relation to holomorphic normal coordinates.

7. Holomorphic Normal Coordinates

In this section, we want to prove the following computationally useful proposition:

PROPOSITION. *Let M be a complex manifold with Hermitian metric g, and let ω be the Kähler form of g. Let $p \in M$. Then $d\omega|_p = 0$ if and only if $\exists$ a holomorphic coordinate system $(z_1,\ldots,z_n)$ defined in a neighborhood of p with*

$$(1)\qquad g\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right)\bigg|_p = 0 \quad i \neq j$$
$$= 1 \quad i = j$$

and

$$(2)\qquad d\left\{g\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right)\right\}\bigg|_p = 0 \ \ all \ \ i,j .$$

Proof. Conditions [(1) and (2)] imply that $d\omega|_p = 0$ by the formulae of §6. The converse is more complicated. We do it as follows:

Choose coordinates (holomorphic) $w_1,\dots,w_n$ with $p = (0,\dots,0)$. Changing the w coordinates by a constant coefficient linear transformation, we can assume condition (1) to hold. To arrange for condition (2) to hold, we introduce $z_1,\dots,z_n$ defined by

$$z_i + \sum_{j,k} \alpha^i_{jk}\, z_j z_k = w_i\ ,$$

where α^i_{jk} are complex constants to be chosen later and which are to satisfy $\alpha^i_{jk} = \alpha^i_{kj}$. Note that for each choice of α's, the equations define a holomorphic coordinate system $(z_1,\dots,z_n)$ in some (smaller) neighborhood of p: this follows from the holomorphic implicit function theorem (actually, inverse function theorem). By the (holomorphic) chain rule:

$$\frac{\partial}{\partial z_i} = \sum_\ell \frac{\partial w_\ell}{\partial z_i}\frac{\partial}{\partial w_\ell} = \sum_\ell \left(\delta_{i\ell} + \sum_j (\alpha^\ell_{ij} + \alpha^\ell_{ji}) z_j\right)\frac{\partial}{\partial w_\ell}$$

and

$$\frac{\partial}{\partial \bar z_i} = \sum_\ell \left(\delta_{ij} + \sum_j (\bar\alpha^\ell_{ij} + \bar\alpha^\ell_{ji}) \bar z_j\right)\frac{\partial}{\partial \bar w_\ell} .$$

Hence, $g_{i\bar j}$ in z coordinates, $h_{i\bar j}$ for w coordinates,

$$g_{i\bar j} = \hat g\left(\sum_\ell \left(\delta_{i\ell} + 2\sum_q \alpha^\ell_{iq}\, z_q\right)\frac{\partial}{\partial w_\ell}, \sum_k \left(\delta_{jk} + 2\sum_m \bar\alpha^k_{jm}\, \bar z_m\right)\frac{\partial}{\partial \bar w_k}\right)$$

$$= h_{i\bar{j}} + 2\hat{g}\left(\sum_q \alpha^{\ell}_{iq}\, z_q\, \frac{\partial}{\partial w_\ell}, \frac{\partial}{\partial \bar{w}_j}\right) + 2\hat{g}\left(\frac{\partial}{\partial w_i}, \sum_k \bar{\alpha}^k_{jm}\, \bar{z}_m\, \frac{\partial}{\partial \bar{w}_k}\right)$$

+ terms quadratic in z's .

At p, $g_{i\bar{j}} = h_{i\bar{j}}$ so condition (1) remains satisfied. For condition (2), we need

$$\left.\frac{\partial g_{i\bar{j}}}{\partial z_r}\right|_p = 0 \quad \text{and} \quad \left.\frac{\partial g_{i\bar{j}}}{\partial \bar{z}_r}\right|_p = 0\,.$$

Again at p (where z's = w's = 0)

(*) $$\frac{\partial g_{i\bar{j}}}{\partial z_r} = \frac{\partial h_{i\bar{j}}}{\partial z_r} + 2\,\bar{\alpha}^j_{ir}$$

because $\hat{g}(\partial/\partial w_i, \partial/\partial \bar{w}_j) = \delta_{ij}$ at p. Also at p

(**) $$\frac{\partial g_{i\bar{j}}}{\partial \bar{z}_r} = \frac{\partial h_{i\bar{j}}}{\partial \bar{z}_r} = \frac{\partial h_{i\bar{j}}}{\partial \bar{z}_r} + 2\bar{\alpha}^i_{jr}\,.$$

So from (*) and (**) condition (2) holds if

(†) $$\alpha^j_{ir} = -\frac{1}{2}\,\frac{\partial h_{i\bar{j}}}{\partial z_r}$$

and

(††) $$\bar{\alpha}^i_{jr} = -\frac{1}{2}\,\frac{\partial h_{i\bar{j}}}{\partial \bar{z}_r}\,.$$

If (†) holds (for all i, j, r) then so does (††) because

$$\overline{\frac{\partial h_{i\bar{j}}}{\partial \bar{z}_r}} = \frac{\overline{\partial h_{i\bar{j}}}}{\partial z_r} = \frac{\partial h_{j\bar{i}}}{\partial z_r}\,.$$

So we need only arrange for (†) to hold for all i, j, r *at once*. This will be possible if and only if the righthand side of (†) is symmetric in i and r. (Remember: We assumed α^j_{ir} was symmetric in i and r, without loss of generality.) In other words, we must have

$$(\dagger\dagger\dagger) \qquad \frac{\partial h_{i\bar{j}}}{\partial z_r} = \frac{\partial h_{r\bar{j}}}{\partial z_i}$$

and, if we do have this, then we can solve (†). But (†††) is true if g is Kähler (end of §6). So the proof is complete. ■

While this proof is clear computationally, it is worthwhile to look at the situation philosophically, too. In general, Riemann normal coordinates (i.e., $\exp^{-1}$) will not be real and imaginary part of a holomorphic coordinate system, i.e., J on M_p will not be taken, in general, to J on M by $\exp_p$. Of course, since $d(\exp_p)|_p$ = identity, J at p is taken to itself. Also, if the metric is Kähler so that $DJ|_p = 0$, then [because Riemann normal coordinates are parallel-at-p (i.e. D(coord field)$|_p = 0$] the exponential map is J-preserving at p to one higher order than just the 0-order, $d\exp_p|_p$ = identity. So it makes sense that there is some holomorphic coordinate system with real and imaginary parts $x_1, y_1, \ldots, x_n, y_n$ matching Riemann normal coordinates to an extra order, i.e., having covariant derivative 0 at p. This is just what condition (2) of the Proposition gives.

Exercise. Show that with $(z_1, \ldots, z_n)$ as in the Proposition and $z_j = x_j + \sqrt{-1}\, y_j$, it holds that

$$D_{\partial/\partial x_j} \frac{\partial}{\partial y_\ell} = 0 \qquad D_{\partial/\partial x_j} \frac{\partial}{\partial y_\ell} = 0$$
$$D_{\partial/\partial x_j} \frac{\partial}{\partial x_\ell} = 0 \qquad D_{\partial/\partial y_j} \frac{\partial}{\partial x_\ell} = 0$$

all at p, all j, ℓ.

8. Basic Examples of Kähler Manifolds

(1) $\mathbf{C}^n$ with its standard metric (as $\mathbf{R}^{2n}$). This is easily seen to be Kähler because in standard coordinates its metric coefficients are constant.

(2) $P_n\mathbf{C}$: Denote by $P : S^{2n+1} \to \mathbf{C}P^n$ the projection (to equivalence classes) of $S^{2n+1} = \{(z_1,\ldots,z_{n+1}) \in \mathbf{C}^{n+1} \mid \Sigma|z_j|^2 = 1\}$ onto $\mathbf{C}P^n$. If

$$(z_1,\ldots,z_{n+1}) = (x_1,y_1,\ldots,x_{n+1},y_{n+1}) ,$$

then $(x_1,\ldots,y_{n+1})$ and $(-y_1,x_1,\ldots,-y_{n+1},x_{n+1})$ together span a J-invariant subspace S of $(\mathbf{C}^{n+1})_{(x_1\cdots y_{n+1})}$ = the real tangent space of $\mathbf{C}^{n+1} = \mathbf{R}^{2n+2}$ at $(x_1,\ldots,y_{n+1})$. The orthogonal complement of this subspace S projects J-equivariantly under dP one-to-one onto the tangent space of $\mathbf{C}P^n$ at $P(z_1,\ldots,z_{n+1})$. To see this note that the two have the same real dimension $2n$. Also the vector $(-y_1,x_1,\ldots,-y_{n+1},x_{n+1})$ generates the (one dimensional) tangent space to the fibre of P through $(x_1,\ldots,y_{n+1})$ because it is the tangent to the curve

$$t \to (\cos t + i\sin t) \cdot (z_1,\ldots,z_{n+1})$$

which is the fibre. So $dP|$ orthogonal complement S is injective.

We now define a metric on $\mathbf{C}P^n$ by declaring $dP|$ orthogonal complement of S to be isometric. It is algebra to check that the resulting metric is C^∞ Hermitian. To see that it is Kähler is more tedious if one attempts direct calculation. But the situation can be simplified as follows:

First note that the unitary group acting on $\mathbf{C}^{n+1}$ induces an action $\mathbf{C}P^n$; and, as is easy to see, this action is isometric relative to the metric we have defined. It is also biholomorphic (i.e., each induced mapping is a biholomorphic map of $\mathbf{C}P^n$ to itself). Thus we need only check the vanishing of DJ (or $d\omega$) at one point of $\mathbf{C}P^n$ – because the group action is transitive and DJ (or $d\omega$) is an invariant under biholomorphic isometries. So let us check what happens at $[(1,0,\ldots,0)]$. Writing for convenience $(z_0,\ldots,z_n)$ as coordinates on $\mathbf{C}^{n+1}$ we have the coordinates on $\mathbf{C}P^n$

$$(z_1,\ldots,z_n) \to [(1,z_1,\ldots,z_n)]$$

as a holomorphic coordinate system near $[(1,0,\ldots,0)]$. If this coordinate system has $dg_{i\bar{j}}|_{(0,\ldots,0)} = 0$, then the formulae in §6 show that $DJ = 0$ (or $d\omega = 0$) at $(0,\ldots,0)$.

Now computing the length of $\partial/\partial z_i, i \geq 1$, (or the inner product $\hat{g}(\partial/\partial z_i, \partial/\partial \bar{z}_j)$) will involve computing the dP preimage in the orthogonal complement of

$$\left(\frac{1}{\sqrt{1+z_1^2+\cdots z_n^2}}, \ldots, \frac{z_n}{\sqrt{1+\cdots z_n^2}}\right)$$

and J of that vector (considered as a real vector). More precisely

$$g\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_i}\right) = H\left(Q\left(\frac{\partial}{\partial z_i}\right), Q\left(\frac{\partial}{\partial z_i}\right)\right)$$

where $H =$ the Euclidean metric on $\mathbf{C}^{n+1}$ and $Q =$ projection on the orthogonal complement of $\{(1/\sqrt{\ },\ldots,z_n/\sqrt{\ })$ and J of that vector$\}$. A straightforward calculation involving order of magnitude argument only* shows that

$$g\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_i}\right)$$

is constant to second order (i.e., has 0 first derivatives), and similarly for

$$g\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right).$$

(Note: This argument also incidentally shows in detail that the metric on $\mathbf{C}P^n$ is J-invariant).

(3) B^n and its Poincare metric $B^n = \{(z_1,\ldots,z_n) \mid \Sigma |z_j|^2 < 1\}$).

There is a large group of biholomorphic mappings acting on B^n. First of all, the unitary group acts fixing the origin. It is well known (and fairly easy to see) that these are the only biholomorphic mappings that fix the origin. In addition, the maps (for each a, $a \in \mathbf{C}$, $|a| < 1$)

$$(z_1,\ldots,z_n) \to \left(\frac{z_1 - a}{1 - \bar{a}z_1}, \frac{z_2(1-|a|^2)^{1/2}}{1-\bar{a}z_1}, \ldots, \frac{z_n(1-|a|^2)^{1/2}}{1-\bar{a}z_1}\right)$$

*$Q(\partial/\partial z_i) = \partial/\partial z_i$ - term vanishing to 1st order that is perpendicular to $(1/\sqrt{\ },\ldots,z_n/\sqrt{\ })$ and J of that.

are easily checked to be biholomorphic. Combining these two types of maps, we see that biholomorphic maps act transitively on B^n. We define a metric on B^n by setting the metric at $p \in B^n = \alpha^*_p\, g_0$, where $g_0 =$ Euclidean metric at origin and $\alpha_p : B^n \to B^n$ is a biholomorphic map taking p to the origin. (Any two such differ by a unitary group element, so the metric is well defined.)

The metric we have defined is obviously J-invariant. (Exercise: Show it is C^∞). To see that it is Kähler, it is enough to see that $d\omega = 0$ (or $DJ = 0$) at $(0,\ldots,0)$ since all points are biholomorphically isometrically equivalent to $(0,\ldots,0)$. Also, since the unitary group acts transitively on directions at $(0,\ldots,0)$, it is enough to check that $D_XJ = 0$ where $X = \partial/\partial x_1|_{(0,\ldots,0)}$. For this, consider the (standard) holomorphic coordinate system $(z_1,\ldots,z_n)$. If

$$0 = D_X \frac{\partial}{\partial X_i} = D_X \frac{\partial}{\partial y_j} \quad \text{all} \quad i, j$$

then $D_XJ = 0$ since $J(\partial/\partial x_i) = \partial/\partial y_i$.

But straightforward calculation shows that $g(\partial/\partial z_i,\, \partial/\partial \bar{z}_j)$ is 2nd order constant at $(0,\ldots,0)$ so $D\, \partial/\partial x_1 = D\, \partial/\partial y_i = 0$ at $(0,\ldots,0)$. ∎

9. Curvature Properties of Kähler Manifolds

Recall the Riemann curvature 3-tensor

$$R(X,Y)Z \overset{\text{def}}{=} (D_XD_Y - D_YD_X - D_{[X,Y]})Z$$

and 4-tensor

$$R(X,Y,Z,W) \overset{\text{def}}{=} -\, g(R(X,Y)Z,W)\ .$$

On a Kähler manifold, these tensors have special properties relative to J: Since J is parallel, it follows that

$$R(X,Y)\,(JZ) = JR(X,Y)Z\ .$$

From this follows

$$- R(X,Y,JZ,JW) = g(R(X,Y)JZ,JW) = g(JR(X,Y)Z,JW)$$

$$= g(R(X,Y)Z,W) = - R(X,Y,Z,W) .$$

Also since $R(X,Y,Z,W) = R(Z,W,X,Y) = R(Z,W,X,Y)$ on any Riemannian manifold, we have

$$R(JX,JY,Z,W) = R(Z,W,JX,JY) = R(Z,W,X,Y) = R(X,Y,Z,W) .$$

So $R(,,,)$ is J-invariant in the first two slots and in the last two.

Let X be a unit vector. The number $R(X,JX,X,JX)$ is called the *holomorphic sectional curvature* of the 2-plane spanned by X, JX. If Y is any other unit vector in this 2-plane, then $R(X,JX,X,JX) = R(Y,JY,Y,JY)$ (Exercise: Check this.) Thus holomorphic sectional curvature is a well-defined function on the J-invariant 2-planes in M_p.

The holomorphic sectional curvatures at p determine the whole curvature tensor at p. In fact, a purely algebraic kind of determination holds:

Proposition: Suppose R and T are 4-tensors on a vector space V with endomorphism $J : V \to V \ni J^2 = -1$. Suppose R and T have the symmetries of a Riemman curvature tensor (antisymmetric in 1st two, last two; symmetric under $1234 \to 3412$; 1st Bianchi) and that both are J invariant in first two and in last two shots. Suppose also that for all $X \in V$

$$R(X,JX,X,JX) = T(X,JX,X,JX) .$$

Then

$$R = T .$$

Proof: It suffices to consider the case $T = 0$. Look at the map

$$(X,Y,U,V) \overset{\hat{R}}{\to} R(X,JY,U,JV) + R(X,JU,Y,JV) + R(X,JV,Y,JU) .$$

This is symmetric in X, Y, U and V (check this). Also it $= 0$ for $X = Y = U = V$ since $T = R = 0$ for X, JX, X, JX etc. By polarization, $\hat{R} \equiv 0$. With $X = U$, $Y = V$, one gets

$$(^*) \qquad 2R(X,JY,X,JY) + R(X,JX,Y,JY) = 0 \,.$$

From the Bianchi identity

$$R(X,JX,Y,JY) + R(X,Y,JY,JX) + R(X,JY,JX,Y) = 0$$

so

$$R(X,JX,Y,JY) - R(X,Y,X,Y) - R(X,JY,X,JY) = 0 \,.$$

Adding minus this to (*) gives

$$3R(X,JY,X,JY) + R(X,Y,X,Y) = 0$$

Replace Y by JY:

$$3R(X,Y,X,Y) + R(X,JY,X,JY) = 0 \,.$$

These last two imply $R(X,Y,X,Y) = 0$. Since sec. curv $\equiv 0 \Rightarrow R \equiv 0$ (in a purely algebraic way), it follows that

$$R = 0. \qquad \blacksquare$$

The proposition we have just proved shows that there could be at most one tensor R (with all the symmetries and J invariances) $\ni R(X,JX,X,JX) = +1$ for $X \ni \|X\| = 1$. We can in fact exhibit one such. (Note: We shall do so by just writing one down. But in principle we could *compute* one, by noting that $\mathbf{C}P^n$ has constant holomorphic sectional curvature, since the biholomorphic isometries act transitively on the J-invariant 2-planes. So we could just compute the curvature tensor of $\mathbf{C}P^n$, and even at one point, in fact. Then we would have at most to multiply it by a constant). With g as the real J-invariant inner product set

$$\begin{aligned} R_0(X,Y,U,V) = \frac{1}{4}\{&g(X,U)g(Y,V) - g(X,V)g(Y,U) \\ &+ g(X,JU)g(Y,JV) - g(X,JV)g(Y,JU) \end{aligned}$$

$$+ 2g(X,JY)g(U,JV)\} .$$

A tedious but easy check shows R_0 has the required symmetries and satisfies

$$R_0(X,JX,X,JX) = g(X,X)^2 .$$

Also,

$$R_0(X,Y,X,Y) = \frac{1}{4}\{g(X,X)g(Y,Y) - g(X,Y)^2 + 3g(X,JY)^2\} .$$

This last formula has a pleasant geometric meaning.

Suppose P is a 2-plane. Then we define the angle α_P between P and JP by $\alpha_p =$ minimum angle between $X \in P$ and $Z \in JP$. Then

$$\cos(\alpha(p)) = |g(X,JY)|$$

where X, Y is an orthonormal basis for P. (Exercise: Prove this.) Thus the sectional curvature (for R_0 as curvature tensor)

$$K(P) = \frac{1}{4}(1 + 3\cos^2(\alpha_p)) .$$

We can summarize all this as follows: If at a point $p \in M$, M a Kähler manifold, all J-invariant 2-planes have the same sectional curvature c then the curvature tensor at p is cR_0 and for any 2-plane P in M_p

$$K(P) = (1/4)\,(1 + 3\cos^2(\alpha_1))c .$$

The next result shows that a Kähler manifold of $\mathbb{C}$-dimension ≥ 2 can satisfy the previous condition at each point only in a special way:

PROPOSITION. *If M is a (connected) Kähler manifold with $\dim_{\mathbb{R}} M = 2\dim_{\mathbb{C}} 2M \geq 4$ and if, at each point $p \in M$, $R = c_p R_0$ for some constant c, then in fact c is independent of p!*

Proof. Let Ric denote the Ricci tensor of M. Compute for an arbitrary $X, X \in M_p$, $\|X\| = 1$

$$\mathrm{Ric}(X,X) = R(X,JX,X,JX) + \sum_{j=2}^{n} [R(X,Y_j,X,Y_j) + R(X,JY_j,X,JY_j)]$$

where X, JX, Y_2, $JY_2, \ldots, Y_n$, JY_n is an orthonormal basis of M_p. Now $R(X,JX,X,JX) = c_p$ while

$$R(X,Y_j,X,Y_j) = R(X,JY_j,X,JY_j) = 1/4\, c_p\ .$$

So

$$\mathrm{Ric}(X,X) = c_p + 2[(n-1)/4]c_p = \left(\frac{n+1}{2}\right)c_p\ .$$

In particular, $\mathrm{Ric}(X,X)$ is independent of $X \in M_p$, $\|X\| = 1$. So M is Einstein (at p). By a classical theorem the quotient $c_p = \{1/(n + 1/2)\}\ [\mathrm{Ric}/g]$ is independent of p.

■

Of course, none of this discussion about independence of point has any relevance in case $n = 1$ (a 2-dimensional manifold in the sense of real dimension).

Our three basic Kähler manifolds $-$ $\mathbf{C}^n$, $\mathbf{C}P^n$, B^n $-$ all have constant holomorphic sectional curvature. $\mathbf{C}^n$ is of course flat: $R \equiv 0$. $\mathbf{C}P^n$ has constant holomorphic sectional curvature (because the biholomorphic isometries act transitively on the J-invariant 2-planes). $\mathbf{C}P^n$ is simply connected and compact so it cannot admit a (necessarily complete) Riemannian metric with sectional curvature ≤ 0. Hence its holomorphic sectional curvature (which is constant) is > 0. Similarly, the complete metric on B^n cannot have positive curvature, since it would be positive bounded away from zero, contradicting the noncompactness of B^n. So $R_{B^n} = cR_0$, $c \leq 0$. It cannot be the case that $c = 0$ because then B^n would be biholomorphically isometric to $\mathbf{C}^n$: namely, the exponential map would be such a biholomorphic isometry. So $c < 0$ in this case.

These three examples in fact are all the possibilities for constant holomorphic sectional curvature, up to coverings and constant factors. Specifically, the following result holds:

PROPOSITION. *Any two complete n-dimensional* **C** *simply connected Kähler manifolds with the same constant positive holomorphic sectional curvature are biholomorphically isometric to each other.*

Proof. Let M_1 and M_2 be two such manifolds. Choose $p_1 \in M_i$ and $p_2 \in M_2$ and let $T : M_{p_1} \to M_{p_2}$ be an isometric linear transformation that commutes with J. (Such a transformation can be obtained by choosing orthonormal bases $X_1, JX_1, \ldots, X_n, JX_n$ and $Y_1, JY_1, \ldots, Y_n, JY_n$ and defining $T(X_j) = Y_j, T(JX_j) = JY_j$ for all $j = 1, \ldots, n$.) The transformation T also takes R_{p_1} to R_{p_2}, by our previous results. Moreover, since J_{M_1} and J_{M_2} are both parallel and since the curvature of M_1 (and M_2) is determined by the metric, J and the equal constant holomorphic sectional curvatures, it follows that both M_1 and M_2 have parallel curvature tensors. By simple connectivity of M_1, and standard considerations*, T must be the differential of a locally isometric surjective covering map $T : M_1 \to M_2$. T must be holomorphic (Exercise: Prove this by using parallelism of J to show $dT \circ J_{M_1} = J_{M_2} \circ dT$ at each point of M_1, since the equation holds at p). Since M_2 is simply connected, T must be injective. ■

10. Kähler Submanifolds

Let N be a complex submanifold of a complex manifold M. Since by definition $J_N|_q =$ the restriction to N_q of $J_M|_q$ (N_q being J-invariant), it follows that the restriction to N of a Hermitian metric on M is a Hermitian metric on N. Also, the Kähler form of the metric on N is the restriction to N of the Kähler form of the

*of Riemannian geometry.

Hermitian metric on M. Since $d_N = d_M|N$, it follows that if the metric on M is Kähler then so is the metric on N. The relationship between the curvature of the metric on M and that of N is much closer than in the Riemannian case, where, in the case of large codimension, at least, there is no relationship. The relationship in the Kähler situation comes from special properties of the second fundamental form.

Let $S(X,Y) = D_X^M Y - D_X^N Y$ be the second fundamental of N in M. Then

$$S(X,JY) = D_X^M(JY) - D_X^N(JY) = J_M D_X^M Y - J_N D_X^N Y$$

$$= J_M(D_X^M Y - D_X^N Y) = JS(X,Y) .$$

By symmetry,

$$S(JX,Y) = S(Y,JX) = JS(Y,X) = JS(X,Y) .$$

For any submanifold, the sectional curvatures satisfy (X,Y orthonormal, P = span of X,Y):

$$K_N(P) = R^M(X,Y,X,Y) + g_M(S(X,X),S(Y,Y)) - g_M(S(X,Y),S(X,Y))$$

$$= K_M(P) + g_M(S(X,X),S(Y,Y)) - g_M(S(X,Y),S(X,Y)) .$$

In our case, if $Y = JX$ then

$$K_N(P) = K_M(P) + g_M(S(X,X),S(JX,JX)) - g_M(S(X,JX),S(X,JX))$$

$$= K_M(P) - 2g(S(X,X),S(X,X))$$

because

$$S(JX,JX) = -S(X,X)$$

and

$$g_M(S(X,JX),S(X,JX)) = g_M(JS(X,X),JS(X,X)) = g_M(S(X,X),S(X,X)) .$$

In particular, the holomorphic sectional curvature of N is always $\leq$ the corresponding sectional curvature of M. Let us compute the trace of S: we can of course compute relative to any basis so we use a basis of the form $X_1,JX_1,\ldots X_m,JX_m$, $m = \dim_C N$.

Then

$$\text{Trace } S = S(X_1,X_1) + S(JX_1,JX_1) + \cdots + S(X_n,X_n) + S(JX_n,JX_n) = 0$$

because $S(JX_1,JX_1) = -S(X_1,X_1)$. So N is a minimal submanifold of M. (In case N is compact and so is M, it can be shown that N is absolutely area minimizing in the homology class it represents in M, this homology class being necessarily nontrivial if N is not zero dimensional.)

11. Holomorphic Vector Bundles and Hermitian Metrics and Connections

A holomorphic vector bundle is defined just as is a topological vector bundle with fibres $\mathbf{C}$-vector spaces with an additional restriction, that the transition functions be holomorphic functions of the point in the manifold. More explicitly, with $B \xrightarrow{\pi} M$, M a complex manifold bundle, we suppose given a trivilizing cover U_λ and maps:

$$\phi_\lambda : \pi^{-1}(U_\lambda) \to U_\lambda \times \mathbf{C}^k$$

with $\pi_1 \circ \phi_\lambda = \pi/_{\pi^{-1}(U_\lambda)}$ when $\pi_1 =$ first factor projection and the linear maps, defined for $p \in U_{\lambda_1} \cap U_{\lambda_2}$ of $\mathbf{C}^k \to \mathbf{C}^k$ by $\pi_2 \circ \phi_{\lambda_2} \circ (\phi_{\lambda_1}^{-1} \mid p \times \mathbf{C}^k)$ to depend holomorphically on p. (Notes: $\pi_2 =$ projection on the second factor; since linear maps $\mathbf{C}^k \to \mathbf{C}^k$ are uniquely associated to $k \times k$ $\mathbf{C}$-valued matrices, it makes sense to speak of such maps being holomorphic: it just means that each matrix element is a holomorphic function.)

Notation: $\pi_2 \circ \phi_{\lambda_2} \circ (\phi_{\lambda_1}^{-1} \mid p \times \mathbf{C}^k) = f_{\lambda_1\lambda_2} \in \mathbf{C}^{k^2}$.

A *Hermitian metric* on a holomorphic vector bundle $B \xrightarrow{\pi} M$ is a C^∞ family of Hermitian metric (in the standard linear algebra sense) on each fibre $\pi^{-1}(p)$, $p \in M$.

A $\mathbf{R}$-vector bundle with Riemannian metric in general admits a wide variety of metric-preserving connections, i.e., connections for which parallel translation preserves inner product. (Recall: The unique Riemannian connection on TM, M a Riemannian manifold is made unique only by imposing the additional condition of torsion 0.) Similarly, a complex Hermitian vector bundle admits many Hermitian-metric preserving connections. In the case of a holomorphic Hermitian vector bundle, however, there is a natural way to select a unique metric-preserving connection from among the many possible metric-preserving connections.

DEFINITION. A connection on a holomorphic vector bundle $B \xrightarrow{\pi} M$ is type $(1,0)$ if its connection forms relative to a local holomorphic frame in B are type $(1,0)$.

It is easy to check that the definition does not depend on which holomorphic local frame is used.

The basic uniqueness and existence result is the following:

THEOREM. *If $B \xrightarrow{\pi} M$ is a holomorphic vector bundle and if h is a Hermitian (fibre) metric on B, then $\exists$ a unique type $(1,0)$ connection on B that is metric preserving.*

Proof. Choose a local holomorphic frame, i.e., a trivialization $\phi : \pi^{-1}(U) \to U \times \mathbf{C}^k$ and set $\sigma_j = \phi^{-1}((0 \cdots 1 \cdots 0)$ so that $\sigma_1,\ldots,\sigma_k$ are holomorphic and span $\pi^{-1}(p)$ at each $p \in U$. Set

$$h_{\alpha\beta} = h(\sigma_\alpha,\sigma_\beta) \qquad 1 \le \alpha \le k, 1 \le \beta \le k .$$

Define the connection forms of a covariant differentiation (connection) D to be those determined by

$$D\sigma_\alpha = \sum_{\beta=1}^{k} \omega_\alpha^\beta \, \sigma_\beta , \qquad \alpha = 1,\ldots,k .$$

(i.e.

$$D_X\sigma_\alpha = \sum_{\beta=1}^{k} \omega_\alpha^\beta \,(X)\, \sigma_\beta) .$$

Then D is metric-preserving if and only if

$$dh_{\alpha\beta} = h(D\sigma_\alpha,\sigma_\beta) + h(\sigma_\alpha,D\sigma_\beta)$$

or

$$dh_{\alpha\beta} = h\left(\sum_{\gamma=1}^{k} \omega_\alpha^\gamma\, \sigma_\gamma,\sigma_\beta\right) + h\left(\sigma_\alpha, \sum_{\delta=1}^{k} \omega_\beta^\delta\, \sigma_\delta\right)$$

$$= \sum_{\gamma=1}^{k} \omega_\alpha^\gamma\, h(\sigma_\gamma,\sigma_\beta) + \sum_{\delta=1}^{k} \overline{\omega}_\beta^\delta\, h(\sigma_\alpha,\sigma_\delta)\ .$$

(Note that the $\overline{\omega}$ is conjugate in the second sum because h is conjugate linear in the second variable.)

If ω's are type $(1,0)$ so that $\overline{\omega}$'s are type $(0,1)$ then we must have

$$(\dagger) \qquad \partial h_{\alpha\beta} = \sum_{\gamma=1}^{k} \omega_\alpha^\gamma\, h_{\gamma\beta}$$

and

$$\overline{\partial} h_{\alpha\beta} = \sum_{\delta=1}^{k} \overline{\omega}_\beta^\delta\, h_{\alpha\delta}\ .$$

If the first set of these equations hold, then so do the second because

$$\overline{\partial} h_{\alpha\beta} = \overline{\partial h_{\overline{\alpha\beta}}} = \overline{\partial h_{\beta\alpha}} = \left(\sum_{\delta=1}^{k} \omega_\beta^\delta\, h_{\delta\alpha}\right)^-$$

$$= \sum_{\delta=1}^{k} \overline{\omega}_\beta^\delta\, \overline{h}_{\delta\alpha} = \sum_{\delta=1}^{k} \overline{\omega}_\beta^\delta\, h_{\alpha\delta}\ , \text{ as required}\ .$$

On the other hand, the fact that the matrix $(h_{\gamma\beta})$ is invertible (being positive definite) means that we can choose the ω_α^γ in one and only one way so as to make the first equations $(\dagger)$ work. So we have local existence, and uniqueness for the desired connection. Global existence (and uniqueness) follows as usual. ∎

In a holomorphic frame, we have from the previous

$$\omega_\alpha^\delta = \sum_\beta h^{\beta\delta}\, \partial h_{\alpha\beta}\ .$$

where $h^{\beta\delta}$ is the inverse of h (so $\sum_\lambda h_{\rho\lambda} h^{\lambda\mu} = \delta_\rho^\mu$).

An interesting special case is that of holomorphic line bundles. (Note: Line bundles in the **C** sense are more interesting than those in the **R**-sense. The latter have discrete (± 1) structure group, i.e., are reducible to that group. But in general **C** bundles are reducible only as far as $S^1 \subset \mathbf{C}^*$, not to locally constant transition functions.)

In the line bundle case, h is a 1×1 matrix. Also

$$\omega_1^1 = 1/h_{11}\, \partial h_{11} = \partial(\log h_{11})$$

(for log as in the real sense: $h_{11} > 0$). In a different frame, h_{11} changes to $f\bar{f}\, h_{11}$, f a (nonvanishing) holomorphic function. It follows that

$$\bar{\partial}\partial(\log h_{11}\, f\bar{f}) = \bar{\partial}\partial(\log h_{11} + \log f + \log \bar{f}) = \bar{\partial}\partial \log h_{11} ,$$

where $\log f$ and $\log \bar{f}$ can be any fixed local branch of (holomorphic) "log". So

$$\bar{\partial}\partial(\log h_{11})$$

is in fact a globally defined type (1,1) form.

Associated algebraically to the well-defined form $\bar{\partial}\partial \log h_{11}$ is the Hermitian form

$$(\dagger\dagger) \qquad -\sum_{i,j=1}^{n} \left[\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log h_{11}\right] dz_i \otimes d\bar{z}_j \, .$$

We shall call this form the Hermitian curvature form. The association is essentially that of metric to Kähler form. But to avoid algebraic detail, it is more convenient simply to compute directly that the Hermitian form is independent of local trivialization. This is easy following the line of reasoning used to show that $\bar{\partial}\partial \log h_{11}$ is well defined, and it is left to the reader. (This approach also avoids the possibility of sign errors, which have plagued the transition from type (1,1) to Hermitian forms in the literature.)

Special interest is attached to Hermitian holomorphic line bundles for which the Hermitian curvature form indicated is definite, either positive everywhere or negative everywhere. We make a formal definition:

DEFINITION: A holomorphic line bundle B is *positive* (respectively, *nonnegative*) if for some Hermitian metric on B the Hermitian form (††) is positive definite (respectively, nonnegative definite). The bundle B is *negative* (respectively, *nonpositive*) if for some Hermitian metric on B the Hermitian curvature form is everywhere negative definite (respectively, nonpositive definite).

The logic of the negative sign in the definition of the Hermitian curvature form is as follows. Conventionally, it has been the practice to regard bundles with global holomorphic sections positively, having sections being a good property. Now if a Hermitian line bundle over a compact complex manifold has a nontrivial holomorphic section s then for a local frame field σ_1 we can write $s=f\sigma_1, f$ holomorphic, and then, where $s\neq 0$:

$$\sum_{i,j=1}^{n}\left[\frac{\partial^2}{\partial z_i\partial\bar{z}_j}\log h_{11}\right]dz_i\otimes d\bar{z}_j=\sum_{i,j=1}^{n}\left[\frac{\partial^2}{\partial z_i\partial\bar{z}_j}\log(h_{11}f\bar{f})\right]dz_i\otimes d\bar{z}_j$$

$$=\sum_{i,j=1}^{n}\left[\frac{\partial^2}{\partial z_i\partial\bar{z}_j}\log\|s\|^2\right]dz_i\otimes d\bar{z}_j\,.$$

(Here $\|s\|^2$ is the Hermitian norm squared of the section s.) The last-written form must be nonpositive definite at the point(s) where the global function $\|s\|^2$ attains its maximum. (Exercise in calculus: Prove this.) In particular, there must be points of M where the Hermitian form

$$\sum_{i,j=1}^{n}\left[\frac{\partial^2}{\partial z_i\partial\bar{z}_j}\log h_{11}\right]dz_i\otimes d\bar{z}_j$$

must be nonpositive definite or equivalently points at which

$$-\sum_{i,j=1}^{n}\left[\frac{\partial^2}{\partial z_i\partial\bar{z}_j}\log h_{11}\right]dz_i\otimes d\bar{z}_j$$

is nonnegative definite.

Note that no such reasoning occurs at the minimum of $\|s\|^2$ in general because the minimum may well be zero, in which case $\log \|s\|^2$ is not defined at the minimum. What we do obtain by following the above pattern is that if s is a holomorphic section which nowhere vanishes (so that minimum $\|s\|^2>0$ and $\log \|s\|^2$ is defined and C^∞ globally) then the Hermitian curvature form must be nonpositive definite somewhere. Of course, such a section exists if and only if B is the trivial line bundle. Thus we see that a line bundle on a compact complex manifold that is positive or negative cannot be trivial.

Every complex manifold M has a naturally arising holomorphic line bundle on it. This is the bundle of forms of type $(n,0)$; this bundle is called the *canonical bundle* of M and denoted by K or, when the manifold needs specification, K_M.

If M has a Hermitian metric g then K_M can be given an associated Hermitian metric as follows: Let $(z_1,\ldots,z_n)$ be a local coordinate system on M and set $g_{i\bar{j}}=g(\partial/\partial z_i,\partial/\partial \bar{z}_j)$ as usual. Then put

$$\|dz_1\wedge\cdots\wedge dz_n\|^2 = 1/\det(g_{i\bar{j}}) ,$$

where $\det(g_{i\bar{j}})=$ the determinant of the matrix $(g_{i\bar{j}}), 1\le i,j\le n$. A slightly tedious but routine calculation shows that this definition yields a well defined C^∞ Hermitian metric on K_M. We shall call this the canonical metric on K_M (associated to the metric g on M).

The Hermitian curvature form of the canonical metric on K_M is closely related to the Riemannian curvature tensor for a Kähler manifold M. To make this relationship explicit, let R be the Riemannian curvature (4-) tensor of the Kähler manifold M. Define, as usual, the Ricci tensor Ric of M by (at each point of M)

$$\mathrm{Ric}(X,Y) = \sum_{i=1}^{2n} R(X,e_i,Y,e_i) ,$$

where $\{e_i|i=1,\ldots 2n\}$ is an orthonormal (real) basis for the real tangent space M_p of M at p and $X,Y\in M_p$. It is easy to check that $\mathrm{Ric}(X,Y)$ is independent of the choice of orthonormal basis. It is also easy to see, using the J-symmetries of the curvature tensor, that Ric is an Hermitian form, i.e.,

$$\mathrm{Ric}(JX,JY)=\mathrm{Ric}(X,Y) .$$

The Hermitian real form Ric extends by complex linearity to $M_p\otimes\mathbf{C}$. On the holomorphic tangent space of M at p we define a C-Hermitian form by a now familiar process:

$$(Z,W)\to\mathrm{Ric}(Z,\overline{W}) .$$

The promised relationship between the Hermitian curvature of K_M and the curvature of M is given now by the formula (with $G=\det(g_{i\bar{j}})$, $1\le i,j\le n$):

$$\left[\sum_{i,j=1}^{n}\frac{\partial^2}{\partial z_i\partial\bar{z}_j}(\log G)\ dz_i\otimes d\bar{z}_j\right](Z,\overline{W})=-\mathrm{Ric}(Z,\overline{W})$$

for all Z,W in the holomorphic tangent space of M. Here the left hand side is exactly the Hermitian curvature form of K_M evaluated on $Z,\overline{W}$ because the metric of K_M is (locally) $1/G$ and $\log(1/G)=-\log G$. Thus, for example, if M has positive sectional curvature and hence positive definite (real) Ricci form, it follows that K_M is a negative bundle. Specific examples of this: $P_n\mathbf{C}$, all n. The formula just displayed can be proven by a dedious but straightforward calculation, using the standard rules for differentiation of determinants.

Note: In case M has real dimension 2, then the formula reduces to the familiar expression for the Gauss curvature for a metric of the form $G(dx^2+dy^2)$, namely $k=-1/2\Delta(\log G)$.

We can see that the signs check out correctly in our relationship between Hermitian curvature of the canonical bundle and the Ricci curvature of the manifold by considering complex projective space. It has positive sectional curvature and hence positive Ricci curvature, in its usual Fubini-Study Kähler metric. Thus, according to our calculation, its canonical bundle K is negative (in its canonical metric). Negative bundles cannot have nontrivial holomorphic sections, and sure enough there are no nontrivial holomorphic $(n,0)$ forms on complex projective n-space $P_n\mathbf{C}$.

Looking at this same example the other way around, let us consider K_M^*, the dual bundle of K_M. We can put on K_M^* the dual of the canonical metric, namely, if σ_1 is a local nonvanishing section of K_M^* and σ_2 is a local nonvanishing section of K_M we set

$$\|\sigma_1\|^2 = |\sigma_1(\sigma_2)| / \|\sigma_2\|^2 .$$

It is easy to see that the Hermitian curvature of K_M^* in this metric is just $-1\times$ the Hermitian curvature of K_M. (This hold thing works for any line bundle and its dual: exercise.) Thus K_M^* is a positive bundle. This is as it should be since K_M^* has nontrivial global holomorphic sections (obtained by wedging together holomorphic vector fields). Exercise: Figure out one such section on $P_1\mathbf{C}\cong$ the Riemann sphere.

We have already noted that line bundles with a nontrivial holomorphic section must have positive Hermitian curvature form somewhere (in any Hermitian metric). There is a profound and important converse of this, first established by K. Kodaira.

By definition, the tensor product $B_1 \otimes B_2$ of two line bundles is the line bundle whose transition functions are the products of the transistion functions of B_1 and B_2. (Exercise: $\otimes$ on bundles corresponds to addition of Hermitian curvature forms. Formulate this precisely and prove it.) Along these same lines, it can be shown that sufficiently high $\otimes$-powers actually give an embedding into projective space in the following sense: Let $B \to M$ be a line bundle over a compact complex manifold M. Let $H^\circ(B)=$ the $\mathbf{C}$-vector space of holomorphic sections of B. This is actually a finite-dimensional vector space. (It is easy to see, for instance, that its unit ball is compact in the norm $\|s\|_M = (\int_M \|s\|^2)^{1/2}$ where $\|s\|$ is the pointwise norm in an arbitrary Hermitian metric on B and the integral over M is relative to the volume form of an arbitrary Hermitian metric on M).

Choose a basis $s_1,\ldots,s_k$ for $H^\circ(B)$ and suppose (additionally! - this doesn't have to happen) that there is no point $x\in M$ with all $s_j(x)=0$, $j=1,\ldots,k$. Then we can *define a map* $\mathcal{E}_B:M\to P_{k-1}\mathbf{C}$ as follows. For $x\in M$, choose a $j\in\{1,\ldots,n\}$ with $s_j(x)\neq 0$. Then set the image $\mathcal{E}_B(x)$ of $x=$ the point with homogeneous coordinates $s_1(x)/s_j(x),\ldots,1,\ldots,s_k(x)/s_j(x)$, where the 1 is in the jth slot. That is, the image of x is the image of $(s_1(x)/s_j(x),\ldots,1,\ldots,s_k(x)/s_j(x))\in\mathbf{C}^k-\{(0,\ldots 0\}$, under the usual quotient map $\mathbf{C}^k-\{(0,\ldots,0)\}\to P_{k-1}\mathbf{C}$. The map $\mathcal{E}_B$ is holomorphic. A different choice of the basis s_j changes $\mathcal{E}_B$ only by a linear isomorphism $P_n\mathbf{C}$.

Then the following important theorem holds, the famous Kodaira Embedding Theorem:

If M is a compact complex manifold and if $B\to M$ is a positive line bundle, then there is a positive integer ℓ such that $\mathcal{E}_{B^\ell}$ is defined and is a holomorphic embedding of M into a complex projective space.

As noted, $P_n\mathbf{C}$ has a positive holomorphic line bundle on it, namely K^*. If M is a compact complex submanifold of $P_n\mathbf{C}$ then it is easy to see that $K^*|M$ is positive. (In fact, the restriction of a positive bundle is positive: prove this as an exercise.) Thus we see from the Kodaira Embedding Theorem that a compact complex manifold is biholomorphic to a submanifold of some complex manifold if and only if M has on it a positive holomorphic line bundle.

The proof of the Kodaira Theorem is highly nontrivial and will not be given here.

In case that the canonical bundle is positive or negative or a trivial bundle, it is natural to ask whether it is possible to make the Hermitian curvature form of the canonical bundle equal to a scalar multiple of the metric itself. In particular, one could ask whether there is a Kähler metric g on M such that, for some $c\in\mathbf{R}$,

$$\mathrm{Ric}(Z,\overline{W})=cg(Z,\overline{W})$$

for all (complex) vectors $W\in M_p\otimes\mathbf{C}$, all $p\in M$. In this set up, $c<0$ corresponds to K_M positive, $c = 0$ to K_M trivial, and $c > 0$ to K_M negative. This is known not to be possible in general if K_M is negative. But if K_M is trivial or positive then in fact such Kähler metrics always exist. This important and deep result was shown by S. T Yau after an earlier conjecture by E. Calabi. (The case $c < 0$ was established independently of Yau's work by T. Aubin.) These results are discussed in detail in reference [1].

Another particulary interesting holomorphic vector bundle is the *holomorphic tangent bundle* of a complex manifold M, i.e., the bundle with fibre at $p =$ by definition to the holomorphic tangent space at $p =$ by definition the $\mathbf{C}$-linear span of $\partial/\partial z_1, \ldots, \partial/\partial z_n$, $(z_1, \ldots, z_n)$ a holomorphic coordinate system near p. Suppose M is given a Hermitian metric g, i.e., a J-invariant Riemannian metric. Then $T^h M$, The holomorphic tangent bundle of M, becomes a Hermitian holomorphic vector bundle, with the Hermitian metric determined by g (as in an earlier section). The Kähler case of this situation is especially nice, as we shall see after we prove the following lemma.

LEMMA. *Suppose $B \xrightarrow{\pi} M$ is a Hermitian holomorphic vector bundle and $\Sigma_1, \ldots, \Sigma_k$ is a basis for the fibre B_p at a point $p \in M$. Then $\exists$ a local holomorphic frame $\sigma_1, \ldots, \sigma_k \ni$*

(1) $$\sigma_j|_p = \Sigma_j, \quad \text{all } j = 1, \ldots, k$$

and

(2) $$D\sigma_j|_p = 0, \quad \text{all } j = 1, \ldots, k .$$

Proof. Choose a local holomorphic frame $\gamma_1, \ldots, \gamma_k$ with $\gamma_j|_p = \Sigma_j$, all $j = 1, \ldots, k$. Now compute

$$D(\sum_\beta f^{\alpha\beta} \gamma_\beta) = \sum_\beta df^{\alpha\beta} \gamma_\beta + \sum_{\beta,\delta} f^{\alpha\beta} \omega_\beta^\delta \gamma_\delta .$$

$$df^{\alpha\beta} = - \text{ the } \gamma_\beta \text{ component of } \sum_{\beta,\delta} f^{\alpha\beta}\,\omega_\beta^\delta\,\gamma_\delta$$

$$= - \sum_\rho f^{\alpha\rho}\,\omega_\rho^\beta\,.$$

Since this last form is type (1,0) we can choose a set of holomorphic $f^{\alpha\beta}$ with $f^{\alpha\beta}|_p =$ and $df^{\alpha\beta}$ to satisfy the equations just given. Then $D(\sum_\beta f^{\alpha\beta}\gamma_\beta)|_p = 0$ all $\alpha = 1,\ldots,k$. Since f is then invertible near p,

$$\sigma_\alpha = \sum_\beta f^{\alpha\beta}\,\gamma_\beta$$

fits the requirements. ■

If M is a Hermitian a manifold (i.e. M has a J-invariant Riemannian metric attached) then the Riemannian connection on M induces by complex linear extension a connection on $TM \otimes \mathbf{C}$, the complexified tangent bundle of M. If M is Kähler, then T^hM is a parallel subbundle of $TM \otimes \mathbf{C}$. (Here a subbundle is *parallel* by definition if parallel translation of a vector in the subbundle remains in the subbundle.) This follows because J is parallel so its eigenspaces are preserved by parallel translation.

Moreover, again if M is Kähler, the Riemannian connection on T^hM determined by considering T^hM as a Hermitian vector bundle and finding its unique type (1,0) connection. The proof of this fact is easy: Choose a holomorphic normal coordinate system at a point p, say $(z_1,\ldots,z_n)$. Then for the Riemannian connection D

$$D\left.\frac{\partial}{\partial z_j}\right|_p = 0 \qquad j = 1,\ldots,n$$

by definition. Let $\nabla =$ the Hermitian connection for T^hM. Then also

$$\nabla\left.\frac{\partial}{\partial z_j}\right|_p = 0 \qquad j = 1,\ldots,n\,.$$

because

$$dg\left.\left(\frac{\partial}{\partial z_i}, \frac{\bar\partial}{\partial z_j}\right)\right|_p = 0\,.$$

So $D = \nabla$.

The converse of this is also true. For this, let M be a Hermitian manifold – with Riemannian metric (J-invariant) = g and Riemannian metric (J-invariant) = g and Riemannian connection D. Then D induces a connection $\hat{D}$ on T^hM by setting

$$\hat{D}_zW = \text{ the projection of } D_zW \text{ on } T^hM ,$$

where projection is relative to the *canonical* splitting $TM_p \otimes \mathbf{C} = T^hM_p \oplus T^hM_p$. (This also happens to be orthogonal projection relative to g extended to $TM \otimes \mathbf{C}$ as a Hermitian metric.) In this setting, we can see that if $\hat{D} = \nabla$ ($\nabla \overset{\text{def}}{=}$ the Hermitian connection) then M is Kähler. [Note: On the previous pages, we showed that if M is Kähler, then $\hat{D} = D = \nabla$.]

To prove this, choose a local holomorphic frame σ_j in T^hM which is orthonormal at p and has $\nabla = 0$ at p (by the Lemma). Then, since $\nabla = \hat{D}$ by hypothesis, we have

$$\hat{D}\sigma_j|_p = 0 .$$

But $\hat{D}$ is a length compatible connection [i.e. $g(\hat{D}_XZ,\overline{W}) + g(Z,\hat{D}_XW) = Xg(Z,\overline{W})$] because D is and the projection involved in $\hat{D}$ is orthogonal. So

$$dg(\sigma_j,\overline{\sigma}_k)|_p = 0 \quad j,k = 1,\ldots,n .$$

Moreover, $\hat{D}$ has torsion 0 because D does and Lie brackets of vector fields preserves types, i.e., $[Z,W] \in T^hM$ if $Z,W \in T^hM$. Since $\hat{D}\sigma_j|_p = 0$, it follows that $[\sigma_j,\sigma_k] = 0$ at p, all j, k. Now given a holomorphic local frame σ_j with $[\sigma_j,\sigma_k]|_p = 0$, there is a holomorphic coordinate system $(z_1,\ldots,z_n)$ such that $\sigma_j|_p = \partial/\partial z_j|_p$ and moreover $\sigma_j - \partial/\partial z_j$ vanishes to order two at p. Then (z_j) will be (in our case) a holomorphic normal coordinate system at p. So M is Kähler. ■

Appendix: Integrable and Nonintegrable Almost Complex Structures

For convenience, we based our decomposition of differential forms into types (p,q) on the use of complex (holomorphic) coordinates $(z_1,\ldots,z_n)$: i.e. $\{(p,q) \text{ forms}\}$ = span of $dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$. However, it is important to realize that this was only for convenience. The whole question of types of forms is after all a strictly point-wise item, and one should expect to be able to consider it algebraically. Specifically, we can consider it as follows:

Let V be a real $2n$-dimensional vector space with an endomorphism $J : V \to V \ni J^2 = -1$. Then $V^* \otimes \mathbf{C}$ = direct sum of two subspaces determined as follows: Let $X_1,JX_1,\ldots X_n,JX_n$ be a basis for V and $\omega_1,\omega_2 \cdots \omega_{2n-1},\omega_{2n}$ be the dual basis. [So if $X_1 = \partial/\partial x_1,\ldots,\omega_1 = dx_1$, $\omega_2 = dy_1$ etc.]. Then

$$V^* \otimes \mathbf{C} = \operatorname{span}\{\omega_{2k-1} + \sqrt{-1}\omega_{2k}, \quad k = 1,\ldots,n\}$$

$$\oplus \operatorname{span}\{\omega_{2k-1} - \sqrt{-1}\omega_{2k}, k = 1,\ldots,n\} .$$

This decomposition is easily checked to be invariant under changes of the $X_1, JX_1,\ldots,X_n, JX_n$ basis (to another of the same form). This invariance can be in fact checked without computation as follows: The span of $X_1 + \sqrt{-1}\, JX_1,\ldots,X_n + \sqrt{-1}\, X_n$ which corresponds to $\partial/\partial\bar{z}$'s is the $-\sqrt{-1}$ eigenspace of J extended by complex linearity to $V^* \otimes \mathbf{C}$: e.g.,

$$J(X_1 + \sqrt{-1}\, JX_1) = \sqrt{-1}\, J^2 X_1 + JX_1 = -\sqrt{-1}\, X_1 + JX_1$$

$$= -\sqrt{-1}\,(X_1 + \sqrt{-1}\, JX_1) .$$

Similarly

$$J(X_1 - \sqrt{-1}\, JX_1) = -\sqrt{-1}\, J^2 X_1 + JX_1 = \sqrt{-1}\, X_1 + JX_1$$

$$= \sqrt{-1}\,(X_1 - \sqrt{-1}\, JX) .$$

So

$$V \otimes \mathbf{C} = (-\sqrt{-1}) \text{ eigenspace of } J \mathbin{\dot{+}} \sqrt{-1} \text{ eigenspace of } J .$$

The decomposition of $V^* \otimes \mathbf{C}$ is then determined by

$$V^{1,0} \overset{\text{det}}{=} \text{span}\{\omega_{2k-1} + \sqrt{-1}\,\omega_{2k}, \quad k = 1,\ldots,n\}$$

$$= \{\omega \in V^* \otimes \mathbf{C} \mid \omega(X) = 0, \text{ all } X \in$$
$$-\sqrt{-1} \text{ eigenspace of } J \text{ on } V \otimes \mathbf{C}\}$$

[e.g. $dz_1((\partial/\partial x_1) + \sqrt{-1}\,(\partial/\partial y_1)) = 0$].

$$V^{0,1} \overset{\text{def}}{=} \text{span}\{\omega_{2k-1} + \sqrt{-1}\,\omega_{2k}, k = 1,\ldots,n\}$$

$$= \{\omega \in V^* \otimes \mathbf{C} \mid \omega(x) = 0, \text{ all } X \in$$

$$+\sqrt{-1} \text{ eigenspace of } J \text{ on } V \otimes \mathbf{C}\}.$$

[e.g. $d\bar{z}_1((\partial/\partial x_1) - \sqrt{-1}\,(\partial/\partial y_1)) = 0$].

Thus $V^* \otimes = V^{1,0} \oplus V^{0,1}$ and this is basis-choice independent.

This decomposition of $V^* \otimes \mathbf{C}$ gives rise to a decomposition of the whole complex exterior algebra (over $\mathbf{C}$) $\Lambda_{\mathbf{C}}(V^* \otimes \mathbf{C})$ and hence to a type (p,q) decomposition as before.

Now suppose M is a C^∞ manifold on which there is a C^∞ family of endomorphisms $J_p : M_p \to M_p$, $p \in M$, $\ni$. $J_p^2 = -1$ for all $p \in M$. Such a family is called an *almost complex structure* on M. *(Note: Not every manifold admits an almost complex structure.) We specifically now do not* assume that the almost complex structure J comes from a complex structure on M, i.e., a covering of M by coordinate charts with holomorphic overlaps. The question then naturally arises, when does such a general almost complex structure in fact come from a complex structure? Necessary conditions are not hard to find:

Let $\mathcal{F}^{p,q}$ = the set of C^∞ $\mathbf{C}$-valued differential forms ω on $M \ni \omega|_p$ is type (p,q) at p for all $p \in M$. Then $\mathcal{F}^{0,0}$ = $\mathbf{C}$-valued functions on M, etc. Note that

$\mathcal{F}^{0,0}$, $\mathcal{F}^{1,0}$ and $\mathcal{F}^{0,1}$ generate (under $\wedge$) locally $\mathcal{F} = \cup_{p,q} \mathcal{F}^{p,q}$. Also

$$d\mathcal{F}^{0,0} \subset \mathcal{F}^{1,0} \oplus \mathcal{F}^{0,1} = \text{all} \quad 1-\text{forms}$$

$$d\mathcal{F}^{1,0} \subset \mathcal{F}^{1,1} \oplus \mathcal{F}^{2,0} \oplus \mathcal{F}^{0,2} = \text{all} \quad 2-\text{forms}$$

$$d\mathcal{F}^{0,1} \subset \mathcal{F}^{1,1} \oplus \mathcal{F}^{2,0} \oplus \mathcal{F}^{0,2}$$

It follows (from the local generation of $\mathcal{F}$) that

$$d\mathcal{F}^{p,q} \subset \mathcal{F}^{p+1,q} \oplus \mathcal{F}^{p,q+1} \oplus \mathcal{F}^{p+2,q-1} \oplus \mathcal{F}^{p-1,q+2}\,.$$

For a complex structure

$$d\mathcal{F}^{p,q} \subset \mathcal{F}^{p+1,q} \oplus \mathcal{F}^{p,q+1}\,.$$

So a *necessary condition* for an almost complex structure J to come from a complex structure is that

$$(*) \qquad d\mathcal{F}^{p,q} \subset \mathcal{F}^{p+1,q} \oplus \mathcal{F}^{p,q+1}$$

when $\mathcal{F}$ is decomposed according to J types.

(Note: If $\dim_R M = 2$, then $\mathcal{F}^{p,q} = 0$ if $p \geq 2$ or $q \geq 2$. Hence the necessary condition (*) is automatically satisfied.)

It is a fact — not easily proved, however — that the necessary condition (*) is in fact sufficient. An almost complex structure satisfying (*) is said to be *integrable*. Then we have Theorem (Newlander-Nirenberg): If J is a C^∞ integrable almost complex structure on a C^∞ manifold M, then there is a complex structure on M such that the J-mapping associated to this complex structure $=$ the given almost complex structure J.

It is of interest to look at what this theorem says in the $\dim_{\mathbb{R}} M = 2$ case: J determines an orientation on M. Moreover, we can construct a J-invariant metric (Riemannian) on M by setting

$$g(X,Y) = G(X,Y) + G(JX,JY)$$

where G is an arbitrary Riemannian metric on M. Now if $z = x + iy$ is a J-holomorphic coordinate system on M i.e. if $J(\partial/\partial x) = \partial/\partial y$ then the metric g is given by

$$\lambda^2(dx^2 + dy^2)$$

where λ is a C^∞ function. The converse also holds: If (x,y) is an oriented real coordinate system and if $g(\partial/\partial x,\partial/\partial x) = g(\partial/\partial y,\partial/\partial y)$ and $g(\partial/\partial x,\partial/\partial y) = 0$, then $x + iy$ is a holomorphic coordinate on M, i.e., $J(\partial/\partial x) = \partial/\partial y$. Thus the problem of finding a complex structure on M compatible with the given almost complex structure J is equivalent to finding real local coordinates (x,y) such that $g(\partial/\partial x,\partial/\partial x) = g(\partial/\partial y,\partial/\partial y)$ and $g(\partial/\partial x,\partial/\partial y) = 0$, i.e., so called "isothermal coordinates". This problem can always be solved, for any C^∞ Riemannian metric g (partial differential equation methods). In our setting, this solvability corresponds to the fact that an almost complex structure on a manifold M with $\dim_{\mathbb{R}} M = 2$ is always integrable.

There are other conditions that are equivalent to integrability of an almost complex structure. One of the more useful of these is as follows:

Define a tensor field N by

$$N(X,Y) = 2\{[JX,JY] - [X,Y] - J[X,JY] - J[JX,Y]\} ,$$

X, Y real vector fields. A calculation shows that N is function-linear in X and Y and hence that N is a (pointwise) tensor. If there is a complex J-holomorphic coordinate system defined in a neighborhood of $p \in M$ (i.e., a coordinate system $x_1 + \sqrt{-1}\, y_1,\ldots,x_n + \sqrt{-1}\, y_n \ni J(\partial/\partial x_j) = \partial/\partial y_j$), then

$$N_p = 0$$

because by inspection N has 0 components in the given coordinates. Thus the J of a complex structure has $N = 0$.

Another condition that we could impose on J has to do with "vector fields of type (1,0) or type (0,1)". We define a $\mathbf{C}$-vector field. (i.e., a vector field X with $X_p \in M_p \otimes \mathbf{C}$ for each $p \in M$) to be of type

$$(1,0) \quad \text{if} \quad JX = \sqrt{-1}\,X$$

$$(0,1) \quad \text{if} \quad JX = -\sqrt{-1}\,X\,.$$

The condition for the integrability of J becomes that type (1,0) vector fields are a Lie subalgebra, i.e. $[Z,W]$ is type (1,0) if z and W are of type (1,0). Since $[Z,W] = [\bar{Z},\bar{W}]$, this condition is equivalent to type (0,1) vector fields being a Lie subalgebra. It can be shown that either of these (trivially) equivalent conditions, as well as $N \equiv 0$, is equivalent to integrability in our previous sense and hence equivalent to the existence of an associated complex structure.

References

The journal literature of complex differential geometry is far too large to admit a quick summary or even a brief representative sampling. The following books will enable the reader to continue further with the topics introduced here, and will provide an introduction to the remainder of the literature.

Items [4] and [6] cover in detail the basic results about holomorphic functions of several complex variables, and a great deal beyond the basic results.

Chapter IX of item [5] is a treatment of basic complex differential geometry much along the lines of the discussion here; an appendix to [5] contains the proof of the integribility theorem for complex structures in the real analytic case; [5] also has a considerable bibliography up to its publication date. Item [2] gives a treatment of complex manifolds with a somewhat different emphasis involving a more cohomological viewpoint including characteristic classes, etc.

Item [9], though not specifically directed toward complex differential geometry, will provide some indications of recent developments in the field.

Item [1] provides a detailed treatment of the most important recent development in the field, S. T. Yau's proof of the Calabi conjecture.

The algebraic geometric viewpoint on complex manifolds is discussed extensively in [3]; this includes the proof of the fundamental Kodaira theorems mentioned here in §11. The Kodaira theorems are also treated in [7] and [8].

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