Lecture VIII a: Proof of Jordan Canonical Form

Linear trans. $T: V \to V$, V finite dimensional over $\mathbb{C}$

We have already shown that it is enough to obtain Jordan form for $T \,|\,$ a generalized eigenspace, that is $T \,\Big|\, \bigcup_{\ell=1}^{+\infty} \ker(T-\lambda I)^{\ell}$.

So we assume $V =$ a generalized eigenspace for a fixed λ. We do induction dimension. When $\dim V = 1$, everything is clear.

Suppose we know Jordan form works for $n-1$ on down, and $\dim V = n$, $n \geq 2$. Since λ is an eigenvalue of T on V, range$(T-\lambda I)$ has dimension $k < n$. We consider two cases:

(1) $\ker(T-\lambda I) \cap \text{range}(T-\lambda I) = \{\vec{0}\}$.

By the rank-nullity theorem, $\dim \ker(T-\lambda I) + \dim \text{range}(T-\lambda I) = \dim V = n$.

So $V \cong \ker(T-\lambda I) \oplus \text{range}(T-\lambda I)$, and each summand is T invariant

$T((T-\lambda I)v) = (T-\lambda I)(Tv)$ so range is T-invariant. Similarly for ker.

By induction, range$(T-\lambda I)$ has Jordan form. And of course $\ker(T-\lambda I)$ does. So the direct sum does.

[Note: This case does occur! If $T = \lambda I$, then $\dim \text{range}(T-\lambda I) = 0$ and $\ker(T-\lambda I) = V$!]

(2) $\ker(T-\lambda I) \cap \text{range}(T-\lambda I) \neq \{\vec{0}\}$

Suppose $\dim(\ker \cap \text{range}) = s$, dimension $\ker(T-\lambda I) = k$ (so $s \leq k$). Then $\dim \text{range}(T-\lambda I) = n-k$.

Now consider the Jordan form of $T|\text{range}(T-\lambda I)$. There are exactly s Jordan blocks, since the kernel $(T-\lambda I) \cap \text{range}(T-\lambda I) = \ker(T-\lambda I)\big|_{\text{range}(T-\lambda I)}$

and this has as basis the first basis vector in each Jordan block. Now we build a basis for V as follows:

First, for each of the s Jordan blocks with "right hand" most basis w [so the Jordan basis is (in reverse order) $w, (T-\lambda I)w, \ldots (T-\lambda I)^l w$, some l where $(T-\lambda I)^{l+1} w = 0$], we choose $\hat{w}$ such that $w = (T-\lambda I)\hat{w}$. There are s such $\hat{w}$'s, say $\hat{w}_1, \ldots, \hat{w}_s$. Note that these are linearly independent since their $T-\lambda I$ images are!

To these $\hat{w}_1 \ldots \hat{w}_s$, we adjoin all the Jordan basis vectors of the s blocks in the the Jordan form of $T|\text{range}(T-\lambda I)$. Finally, note that if $s<k$, then the s-dimensional subspace $\ker \cap \text{range}$ does not fill out all of $\ker(T-\lambda I)$. So we add $v_1, \ldots, v_{k-s}$ vectors, a basis for a complement of $\ker(T-\lambda I) \cap \text{range}(T-\lambda I)$ in

ker $(T-\lambda I)$. For completeness of notation, let $u_1 \ldots u_{n-k}$ be the complete Jordan basis for $T|$ range $(T-\lambda I)$. So we have

$u_1 \ldots u_{n-k}$ spanning range $(T-\lambda I)$
$\hat{w}_1 \ldots \hat{w}_s$ with $(T-\lambda I)\hat{w}_j = w_j \quad j=1,\ldots s$.
linearly independent
$v_1, \ldots, v_{k-s}$ spanning a complement in ker $(T-\lambda I)$ of ker $(T-\lambda I) \cap$ range $(T-\lambda I)$.

There are $(n-k)+s+(k-s) = n$ vectors so, to see they form a basis for V, we need only check linear independence. ~~So~~ suppose

$$\Sigma \alpha_j u_j + \Sigma \beta_k v_k + \Sigma \gamma_\ell w_\ell = 0$$

Apply $T-\lambda I$. This annihilates $\Sigma \beta_k v_k$. It maps $\Sigma \gamma_\ell w_\ell$ into range $(T-\lambda I)$ as a linear combination of lead Jordan basis vectors. On the u's, $T-\lambda I$ pushes the "left-hand" most u's to 0, but moves other u's down one in Jordan level (to the left). It follows that all the $\gamma_\ell = 0$ (since the Jordan basis vectors are linearly independent). So now we have

$$\Sigma \alpha_j u_j + \Sigma \beta_k v_k = \vec{0}$$

But

$\Sigma \alpha_j u_j \in$ range $(T-\lambda I)$ while $\Sigma \beta_k v_k$ are in a vector space complement of ker $(T-\lambda I) \cap$ range $(T-\lambda I)$ in ker $(T-\lambda I)$

Since $\Sigma\beta_k v_k$ is thus in $\ker(T-\lambda I)$ and since $-\Sigma\beta_k v_k = \Sigma\alpha_j u_j$, it follows that

$$\Sigma\alpha_j u_j \in \ker(T-\lambda I) \cap \operatorname{range}(T-\lambda I).$$

The $\oplus$ decomposition of $\ker(T-\lambda I)$ thus implies

$$\Sigma\beta_k v_k = 0$$

and

$$\Sigma\alpha_j u_j = 0.$$

Thus (since the v's are independent and the u's are independent)

all β's $= 0$ and all α's $= 0$.

We already had all γ's $= 0$. Linear independence of the u's, v's and $\hat{w}$'s is thus established.

That T has Jordan form in this basis is clear (Jordan form on the whole space) □

This seems almost too good to be true as the proof of a famous theorem. But it works! This proof was found by A.F. Filippov (1971).

It is instructive to look at a matrix already in Jordan form (with only one eigenvalue) and see how the proof works back from the Jordan form thus obtained on $\operatorname{range}(T-\lambda I)$ to get the original Jordan form for the whole original generalized eigenspace.