# Lecture III: Dual Spaces, Adjoints & Inner Products

If $V$ is a vector space over a field $F$, then a <u>linear functional</u> $L$ on $V$ is a linear transformation $L: V \to F$ (where $F$ is regarded as a vector space over $F$ in the obvious way).

There is a natural notion of adding linear functionals : $(L_1 + L_2)(v) = L_1(v) + L_2(v)$, $v \in V$. and of multiplying them elements of $F$ $(\alpha L)(v) = L(\alpha v) [= \alpha L(v)]$. These operations make the set of all linear functionals a vector space over $F$, called the dual space of $V$, denoted $V^*$.

If $T: V \to W$ is a linear transformation of vector spaces over $F$, then the mapping $T^*$ defined by $\theta \to \theta(T(\cdot))$ is a linear transformation of $W^*$ to $V^*$. (In detail:

$$(T^* \theta)(v) = \theta(T(v)) \qquad \theta \in W^* ).$$

This linear transformation $T^*$ is called the adjoint of $T$.
(Note reversal of direction: $T$ takes $V$ to $W$, $T^*$ takes $W^*$ to $V^*$).

If $V$ is finite dimensional with basis $v_1, \ldots, v_n$ then there is an associated "dual basis" of $V^*$ defined as follows: Let $v_j^*$ be the linear functional defined by $v_j^*\left(\sum_{i=1}^{n} \alpha_i v_i\right) = \alpha_j$.

Then $v_1^*, \ldots v_n^*$ are linearly independent and generate $V^*$. Proof:

(1) independence: If $\sum_1^n \beta_j v_j^* = 0$, then for each $i = 1, \ldots n$

$\left(\sum_1^n \beta_j v_j^*\right) v_i = \sum_1^n \beta_j v_j^*(v_i) = \beta_i$ since

$v_j^*(v_i) = 0$ if $i \neq j$, $v_i^*(v_i) = 1$, all $i = 1, \ldots n$.

So $\sum \beta_j v_j^* = 0 \implies \beta_i = 0$, all $i = 1, \ldots n$.

(2) generation: Given a linear function $L_0 : V \to F$, let $L_1 = \sum_{j=1}^{n} L_0(v_j)\, v_j^*$. Then

$L_1(v_i) = L_0(v_i)$ for each $i = 1, \ldots n$.

Linearity and the fact that $v_1, \ldots, v_n$ generate $V \implies L_1 = L_0$ on all of $V$, namely

$$L_0\left(\sum_1^n \alpha_j v_j\right) = \sum_1^n \alpha_j L_0(v_j)$$

$$= \sum_1^n \alpha_j L_1(v_j) = L_1\left(\sum_1^n \alpha_j v_j\right). \quad \square$$

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If $S \subset V$, the annihilator $S^a$ of $S$ is
(subset, maybe not subspace)

(by definition) $\{\theta \in V^* : \theta(s) = 0, \forall s \in S\}$.

Note: $S^a = U^a$, if $U$ is the subspace generated by $S$ ( $= \cap$ of all subspaces

containing $S$ = set of all finite linear combinations of elements of $S$). Proof is easy using linearity.
Note: $S^a$ is a subspace of $V^*$ (clear).
Theorem: If $U$ is a subspace of dimension $k$ in a vector space $V$ of dimension $n$, then dimension $U^a = n-k$.

Proof: Choose $v_1, \dots, v_k$ a basis for $U$ and $v_{k+1}, \dots, v_n$ vectors in $V$ such that $v_1 \dots v_k\ v_{k+1} \dots v_n$ are a basis for $V$.

Then $v_{k+1}^* \dots v_n^*$ are a basis for $U^a$.
(1) independence: clear since $v_1^*, \dots, v_n^*$ are independent
(2) "generation": Need $U^a \supset \text{span}(v_{k+1}^*, \dots, v_n^*)$
[where span( ) = subspace generated]
This is clear since $v_{k+1}^*(v) = 0, \dots, v_n^*(v) = 0$

for every $v$ which is a linear combination of $v_1, \dots, v_k$. So $v_l^* \in U^a$ if $l \geq k+1$, and hence $\text{span}(v_{k+1}^*, \dots, v_n^*) \subset U^a$.

Also need $U^a \subset \text{span}(v_{k+1}^*, \dots, v_n^*)$:
For this, suppose $L = \sum_{j=1}^{n} \beta_j v_j^* \in U^a$.

Then $0 = L(v_i) = \beta_i$ if $i \leq k$.
(since $v_i \in U$). Hence
$$L = \sum_{k+1}^{n} \beta_j v_j^* . \quad \square$$

We now apply this to adjoints, using this preliminary observation:

Theorem: $\ker T^* = (\operatorname{im} T)^a$.

Proof: $L \in \ker T^* \iff (T^*L)(v) = 0, \ \forall v \in V$

$\iff L(T(v)) = 0, \ \forall v \in V \iff L \in (\operatorname{im} T)^a$. $\square$

From there, we can prove the following somewhat surprising result:

Theorem: If $W$ is a finite-dimensional vector space and $T: V \to W$ is a linear transformation, then

$$\dim(\operatorname{im} T^*) = \dim(\operatorname{im} T)$$

Proof: $\dim(\operatorname{im} T^*) = \dim W - \dim \ker T^*$.
But $\dim(\ker T^*) = \dim((\operatorname{Im} T)^a)$
$= \dim W - \dim \operatorname{Im} T$.
So $\dim(\operatorname{im} T^*) = \dim W - (\dim W - \dim \operatorname{Im} T)$
$= \dim(\operatorname{Im} T)$. $\square$

(This is mostly interesting if $V$ is also finite-dimensional, but the finite-dimensionality of $V$ is not needed for the statement and its proof).

As we shall see later, this result gives (when $V, W$ are both finite dimensional), interpreted in matrix terms, that "row rank" = "column rank" (row rank = dimension of span of the rows column rank = dimension of span of the columns). We shall return to this in detail later on.

If $V$ is a vector space, then the elements of $V$ give rise to linear functionals on $V^*$ as follows: With $v_0 \in V$ fixed, we define $L_{v_0} : V^* \to F$ by $L_{v_0}(\theta) = \theta(v_0)$.

This gives an injective linear transformation of $V \to V^{**}$: we think of $V$ as "contained in" $V^{**}$.

If $V$ is finite-dimensional so that $\dim V = \dim V^* = \dim V^{**}$, then the image of this injective map $V \to V^{**}$ is all of $V^{**}$ by dimension considerations: one says that $V$ "=" $V^{**}$ (in the same abuse of language that $V \subset V^{**}$ in general).

Optional digression:

If $V$ is infinite-dimensional, the image of $V$ in $V^{**}$ may fail to be all of $V^{**}$. For example, if $V$ = eventually 0 $\mathbb{R}$-valued sequences (over $\mathbb{R}$) then the linear functionals on $V$ have the form $(x_1, \ldots, x_n \ldots) \to \sum \alpha_j x_j$ for some sequence $(\alpha_1 \ldots \alpha_n \ldots)$. and every

sequence $(\alpha_1, \alpha_2, \ldots)$ gives rise to a linear functional, as is easy to check (both parts).
Now the image of $V$ in $V^{**}$ is the set of linear functionals of the form

$$\overset{\swarrow \text{arbitrary sequence}}{(\alpha_1, \alpha_2 \ldots)} = \sum_{j=1}^{+\infty} \alpha_j x_j$$

where $(x_1, x_2, \ldots x_n, \ldots)$ is an eventually zero sequence. Note that the $x_j$'s are determined, for a given linear functional $L$ of this sort, by $L$'s values on eventually 0 sequences, namely if $L(\alpha_1, \ldots \alpha_n \ldots)$

$= \sum \alpha_j x_j$ for all sequences $(\alpha_1, \alpha_2 \ldots)$
then $x_j = L((0, 0, 0, \ldots 0, 1, 0 \ldots$

1 in $j$th spot.

Now let $W =$ eventually 0 sequences in the space of all sequences. $W$ is a subspace of $V^*$. By Axiom of Choice etc. $\exists$ a subspace $U$ such that $U \cap W = \{\vec{0}\}$ by $U \neq \{0\}$. Indeed, we can make $V^* = W \oplus U$. Let $L_W$ be the restriction to $W$ of each linear functional $L : V^* \to \mathbb{R}$.
Note that if $L$ has the form $\sum \alpha_j x_j$, $(x_1, \ldots x_n \ldots)$ eventually 0, then $L_W$ determines $(x_1, \ldots x_n \ldots)$. But $L$ on $W \oplus U$ is <u>not</u> determined by $L_W$: $L$ can be arbitrary chosen on $U$! So $\exists L$ not in image of $V$ in $V^{**}$ $\square$

Inner products:

An <u>inner product</u> on a real vector space $V$ is a function $V \times V \to \mathbb{R}$ such that
(1) it is linear in each "slot"
(2) it is symmetric:

$$\langle v, w \rangle = \langle w, v \rangle$$

where we write the image of $v, w$ as $\langle v, w \rangle$ and
(3) $\langle v, v \rangle \geq 0$ and $= 0 \iff v = 0$.

Every $\mathbb{R}$-vector space admits an inner product: If $v_\lambda$, $\lambda \in \Lambda$ (some possibly uncountable index set) is a basis then

$$\left\langle \sum \alpha_{\lambda_i} v_{\lambda_i}, \sum \beta_{\lambda_j} v_{\lambda_j} \right\rangle$$

(finite sums) $$= \sum_{\lambda \in \Lambda} \alpha_\lambda \beta_\lambda$$

works (right hand side sum is finite). But inner products will interest us primarily for finite dimensional vector spaces (and "Hilbert spaces" as we shall discuss them later).

Basic example: $\mathbb{R}^n$ finite dimensions

$$\langle (x_1, \ldots, x_n), (y_1, \ldots, y_n) \rangle = \sum_{i=1}^{n} x_i y_i$$

Hilbert space $l_2 = \{(x_1, \ldots) : \text{sequences with } \sum x_i^2 < +\infty\}$

Inner product

$$\langle (x_1, \ldots), (y_1, \ldots) \rangle = \sum_{i=1}^{+\infty} x_i y_i$$

Exercise $\sum_{i=1}^{+\infty} x_i y_i$ is absolutely convergent.

Definition: An orthonormal basis $v_1, \ldots, v_n$ for a (finite dimensional) vector space $V$ with inner product is a basis (in the usual sense) such that

$$\langle v_i, v_j \rangle = 0 \quad \text{if } i \neq j$$
$$\langle v_i, v_i \rangle = 1 \quad i = 1, 2, \ldots n.$$

Example: $\mathbb{R}^n$ usual inner product "standard basis" $(1, 0, \ldots 0)$ $(0, 1, 0 \ldots)$ $(0, 0, 1, 0 \ldots)$ $\ldots$ $(0, 0, \ldots, 0, 1)$

Theorem: If $V$ is a finite-dimensional vector space with an inner product $\langle , \rangle$, then there is an orthonormal basis $v_1, \ldots, v_n$ for $V$ (relative to the inner product $\langle , \rangle$).

Preliminary observation: A set $v_1, \ldots v_k$ with $\langle v_i, v_j \rangle = 0$, $i \neq j$, all $v_i \neq 0$, is necessarily linearly independent

Reason $\sum \alpha_i v_i = 0 \Rightarrow \langle v_i, \sum \alpha_i v_i \rangle = 0 \Rightarrow \langle v_i, v_i \rangle \alpha_i = 0$ $\square$

Proof (Gram Schmidt process):
Choose a basis $w_1, \dots, w_n$
Set

$$v_1 = \frac{w_1}{\langle w_1, w_1 \rangle^{\frac{1}{2}}} \quad \left( = \frac{1}{\langle w_1, w_1 \rangle^{\frac{1}{2}}} w_1 \text{ in more usual looking notation!} \right)$$

Then $\langle v_1, v_1 \rangle = 1$. (Note that $\langle w_1, w_1 \rangle \neq 0$ since $w_1 \neq \vec{0}$).

Let

$$\hat{v}_2 = w_2 - \beta_1 v_1 \quad \text{where } \beta_1 \text{ is chosen}$$

so that

$$\langle \hat{v}_2, v_1 \rangle = 0 \quad \text{namely}$$

$$\langle w_2, v_1 \rangle - \beta_1 \langle v_1, v_1 \rangle = \langle w_2, v_1 \rangle - \beta_1 = 0.$$

Set

$$v_2 = \frac{1}{\langle \hat{v}_2, \hat{v}_2 \rangle^{\frac{1}{2}}} \hat{v}_2 .$$

Note that $\hat{v}_2 \neq \vec{0}$ since $w_2$ and $v_1$ are linearly independent.

Set $\hat{v}_3 = w_3 - \gamma_1 v_1 - \gamma_2 v_2$ where $\gamma_1, \gamma_2$ are chosen to make

$$\langle \hat{v}_3, v_1 \rangle = 0 \quad \langle \hat{v}_3, v_2 \rangle = 0$$

namely

$$\gamma_1 = \langle w_3, v_1 \rangle \quad \gamma_2 = \langle w_3, v_2 \rangle$$

$\hat{v}_3 \neq 0$ since $w_3 \notin \operatorname{span}(v_1, v_2) \subset \operatorname{span}(w_1, w_2)$

Set $v_3 = \dfrac{1}{\langle \hat{v}_3, \hat{v}_3 \rangle^{\frac{1}{2}}} \hat{v}_3$

Note: $v$'s are independent. So process terminates! $\square$

If $V$ is a vectorspace with inner product $\langle \, , \rangle$ then there is a linear mapping $V \to V^*$ defined by

$$v \to \langle \bullet , v \rangle.$$

(linear function $w \to \langle w, v \rangle$).

Since $\langle v, v \rangle \neq 0$ (If $v \neq 0$) this is injective.

In case $V$ is finit-dimensional, the image of this map is thus all of $V^*$.

[Note: Not every linear isomorphism $V \to V^*$ arises this way!]

In this situation, if we "identify" $V$ with $V^*$ in this way, then $U^a$ corresponds to an item called $U^\perp$ the "orthogonal complement" of $U$, namely

$$U^\perp = \{ v \in V : \langle v, u \rangle = 0, \ \forall u \in U \}$$

An orthonormal basis $v_1, \ldots, v_n$ is then its own dual: $\langle v_i, v_j \rangle = 0$ $i \neq j$, $\langle v_i, v_i \rangle = 1$

An if $U = \text{span}(v_1, \ldots v_\ell)$ with $v_1, \ldots, v_\ell, v_{\ell+1} \ldots v_n$ an orthonormal basis, $U^\perp = \text{span}(u_{\ell+1}, \ldots, u_n)$. So dimension $U^\perp = \dim V - \dim U$

(note that an orthonormal basis of $U$ can be thus expanded to be an orthonormal basis of $V$ (use Gram Schmidt!).

Also, the adjoint of $T: V \to V$ now corresponds to a linear transformation $T^*: V \to W$ defined by

$$\langle Tv, w \rangle_W = \langle v, T^* w \rangle_V, \quad \begin{matrix} v \in V \\ w \in W \end{matrix}$$

both $V, W$ with inner products

This "identification" of $V$ with $V^*$ (when $V$ is finite-dimensional with an inner product) is useful. But one should not lose sight of the fact that annihilators in $V^*$ and adjoints $T^*: W^* \to V^*$ are more fundamental in a real sense.

Still, the complementary dimension result in orthogonal complement form is important to remember (as proved, essentially, above):

Theorem: If $V$ is finite-dimensional with inner product $\langle\ ,\ \rangle$ and if $U$ is a subspace, then

$$\dim U^\perp = \dim V - \dim U.$$

Proof (again): Choose an orthonormal basis $v_1, \ldots, v_\ell$ of $U$, $\ell = \dim U$, extend to a basis $v_1, \ldots, v_\ell, \hat{v}_{\ell+1}, \ldots, \hat{v}_n$ of $V$, apply Gram-Schmidt to get o.n. basis $v_1, \ldots, v_\ell, v_{\ell+1}, \ldots v_n$ on $V$. Then check $U^\perp = \text{span}(v_{\ell+1}, \ldots, v_n)$. $\square$

Recall:
~~Recall~~ An orthonormal set is necessarily linearly independent. (proof easy and already done)

More on $l_2$: The Hilbert space $l_2$ contains a large orthonormal set analogous to the "standard basis" of $\mathbb{R}^n$, namely, $(1,0\ldots\;)$, $(0,1,0\ldots)$, $(0,0,1,0\ldots)$ etc. This set is linearly independent of necessity (also easy to check this explicitly). But it is not a vector space basis: e.g. $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots) \in l_2$ but is not a finite linear combination of the one 1, rest 0 vectors. The vectors $(1,0\ldots)$, $(0,1,0\ldots)$ etc. are a basis for a subspace of $l_2$, namely, the space of eventually 0 sequences. But there are many other things in $l_2$!

In fact, no vector space basis of $l_2$ could be orthonormal. [Reason: An orthonormal set in $l_2$ has to be countable because each point in it is distance $\sqrt{2}$ to each other one in $d(v,w) = \sqrt{\langle v-w, v-w\rangle}$ distance, but $l_2$ is seperable metric in this distance function so every uncountable set has an accumulation point, But $l_2$, being a complete normed vector space in $\|v\| = \sqrt{\langle v,v\rangle}$, has to have finite or uncountable dimension. And is so not finite dimensional! Details of all this (much) later