Practice Problems II

[Optional Problem 1 is not as such needed for Basic Exam!]

1. Suppose $J : V \to V$ is an "operator" (linear transformation) on a real vector space J with "$J^2 = -1$", that is $J(J(v)) = -v$ for $v \in V$. Assuming V is finite dimensional

(1) Prove that there is an inner product $\langle\, , \rangle$ on V such that $\langle Jv, Jv\rangle = \langle v, v\rangle$ for all $v \in V$. [Suggestion: try "averaging"]

(2) Show that if $\langle\, , \rangle$ is such an inner product (from here on) and W is a subspace of V with $JW \subset W$ (W is "J invariant") then $J(W^\perp) \subset W^\perp$.

(3) Show that $\langle Jv, v\rangle = 0$.

(4) Show that V has a basis of the form (indeed an orthonormal basis) $v_1, Jv_1, v_2, Jv_2 \ldots v_k, Jv_k$ and in particular that V is even-dimensional.

(5) Let $V^{\mathbb{C}}$ = the complexification of V, i.e. the set of formal linear combinations

$\alpha v + i\beta w \quad \alpha, \beta \in \mathbb{R} \quad v, w \in V$

with the "obvious" rules of addition

$$(\alpha_1 v_1 + i\beta_1 w_1) + (\alpha_2 v_2 + i\beta_2 w_2) = (\alpha_1 v_1 + \alpha_2 v_2) + i(\beta_1 w_1 + \beta_2 w_2) \text{ etc.}$$

and multiplication

$$(a+bi)(\alpha v + i\beta w) = a\alpha v - \beta b w + i(\alpha b v + a\beta w).$$

Extend J to $V^{\mathbb{C}}$ by "complex linearity". Then show that

(a) $J : V^{\mathbb{C}} \to V^{\mathbb{C}}$ is diagonalizable

(b) The eigenvalues of $J : V^{\mathbb{C}} \to V^{\mathbb{C}}$ are $\pm i$ with eigenspaces

$$\{v - iJv : v \in V\} \qquad (+i \text{ eigenvalue})$$

and

$$\{v + iJv : v \in V\} \qquad (-i \text{ eigenvalue})$$

(b) Show that $V^{\mathbb{C}}$ is n-dimensional ($n = \dim_{\mathbb{R}} V$) over $\mathbb{C}$ (n is even here) and that the two eigenspaces each have dimension $n/2$ over $\mathbb{C}$.

(1) Show that the Hermitian inner products on $\{v - iJv : v \in V\}$ are in 1-1 correspondence with the J-invariant inner products on V via

V inner product

$$\langle v, w\rangle \overset{=}{\Longleftrightarrow} \langle\langle v - iJv, w - iJw\rangle\rangle$$

where $\langle\ \rangle$ = real J invariant inner product and $\langle\langle\ \rangle\rangle$ Hermitian inner product.

[Idea: If use $=$ as definition, then

$$\langle Jv, Jw\rangle = \langle\langle Jv - iJ^2 v, Jw - iJ^2 w\rangle\rangle$$

$$= \langle\langle i(v - iJv), i(w - iJw)\rangle\rangle$$

$$= i\bar{i}\, \langle\langle v - iJv, \cancel{i}(w - iJw)\rangle\rangle$$

$$= \langle v, w\rangle$$ etc.]

(8) How is $\langle v, w\rangle$ related to $\langle\langle\ ,\ \rangle\rangle$?
(9) How is $\langle\langle\ ,\ \rangle\rangle$ related to $\langle\ ,\ \rangle$

(7,8,9 are really one problem).

2. Suppose $T: V \to V$ is a linear transformation of a $\mathbb{C}$-vector space (finite dimensional) with, for some $k \geq 2$, $T^k = \text{Identity}$. Show that T is diagonalizable
[Idea: Average to get T-invariant Hermitian inner product $\langle\langle\ ,\ \rangle\rangle$. Then get eigenvalue & associated eigenspace, show T-invariant $\Rightarrow$ $(\ )^{\perp}$ is T-invariant, continue].

3. Suppose A is an $n \times n$ orthogonal matrix with $\det A > 0$. Prove that $+1$ is an eigenvalue of A if n is odd (with a real eigenvector). Show that if n is even, this may fail (via example).
[Hint: Think about diagonalization of A as a normal operator].

4. Show that the $n \times n$ orthogonal matrices ($A^{-1} = A^{t}$) form a group. Show that those with $\det = 1$ are a normal subgroup [You may assume det product = product of determinants]

5. ~~Prove~~: A ($\mathbb{C}$-valued) matrix A ~~is~~ $n \times n$ with n distinct eigenvalues is diagonalizable over $\mathbb{C}$. [Hint: Recall that set of eigenvectors for all different eigenvalues are independent — proved in notes].

*6. Think about why the set of matrices A, $n \times n$, $\mathbb{C}$-valued, with n distinct eigenvalues is an open dense subset of $\mathbb{C}^{n^2}$ (set of all $n \times n$ $\mathbb{C}$-matrices). This requires some knowledge of algebra, namely the theorem that if $P(z)$ is a polynomial in one variable of degree n, then there is a multivariable polynomial in the coefficients of P, the "discriminant" $\Delta(P)$ such that P has distinct roots $\Leftrightarrow$ $\Delta(P) \neq 0$. [Example: $az^2 + bz + c$, $\Delta = b^2 - 4ac$] (You do not need to know what Δ is in order to do the problem — only that it exists (plus general things about polynomials that vanish on open sets, etc.)).

7. Prove the "essential uniqueness of determinants": If F: $n \times n$ matrices $\to \mathbb{R}$ (also works over $\mathbb{C}$!) is linear in each column & antisymmetric in column interchange then $F = \lambda_0 \det$ for some λ_0.

[Idea of proof: Condition $\Rightarrow$ F(rank < n matrix) = 0 since one column = linear comb of others in that case. Use "column operations" to reduce to $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ keeping tracking of what each one does to F and to det — the same thing — to show $F = F(I_n) \det$.]
Related to earlier problem on $\det = 0 \Leftrightarrow$ columns are dependent, part when you showed independent columns $\Rightarrow \det \neq 0$

[Same thing works for rows!]

8. Prove: If P is an $n \times n$ matrix then

$\det(\cdot P$ ("column matrix")) is

("column matrix": $n \times n$ matrix thought of as matrix of n column vectors)

antisymmetric in columns & linear in columns. (Reason: P acts as linear transformation on column vectors)

9. Combine 7 & 8 to decide (column matrix) $\longrightarrow \det(P \times (\text{column matrix}))$ has properties of det hence $= \lambda_0 \det$. To determine λ_0, evaluate on I_n to get $\lambda_0 = \det P$.

Deduce that $\det(PA) = \det P \det A$ any two $n \times n$ matrices P, A.