Problem Set I : Review of determinants and related matters

1. Show that the powers of the permutation $1 \to 2$, $2 \to 3$, $3 \to 4$, ... $n-1 \to n$, $n \to 1$ form a subgroup of S_n (the "symmetric group" of all permutations on $1, \ldots, n$) and that this subgroup $\subset A_n$ if and only if n is odd. [A_n = subgroup of S^n consisting of permutations with sign = +1)

2. Verify carefully that every interchange is obtainable as a product of an odd number of adjacent interchanges so that $\operatorname{sgn} \pi = -1$ if π is an interchange.

3. Work out in detail the proof (using $\operatorname{sgn} \pi = \operatorname{sgn} \pi^{-1}$) that
$$\det A = \det(A^t).$$

4. Check carefully the linearity of $\det A$ and $\det A^t$ with respect to rows & columns (i.e. ~~replacing~~ if A has a column ~~by~~ like $\alpha \, col_* + \beta \, col_{**}$, say the kth column, then
$$\det A = \alpha \det A_* + \beta \det A_{**}$$
where $A_* = A$ except with kth column $= col_*$

$A_{**} = A$ except with kth column $= col_{**}$.

and same for rows.

5. Show that if A ~~is~~ ($n \times n$ array, square matrix) has linearly dependent rows then $\det A = 0$ & same for columns.
(Suggestion: If lth row = linear combination of other rows, apply prob. 4 to get $\det A \overset{?}{=}$ linear combination of determinants each of which has a repeated row).

6. Prove the converse of problem 5:
[If $\det A = 0$, then the rows (and columns) are linearly dependent] by using the following outline:
This is the same as independent rows $\Rightarrow \det \neq 0$.
Suppose rows independent. Then apply successively in the right way the following operations, which do not change the zeroness or nonzeroness of the determinant:
(a) interchange/permutation of rows
(b) multiplying row by nonzero constant
(c) adding a multiple of some row to another row.
Noting that rows linearly independent $\Rightarrow$ 1st column has a nonzero entry, change to form

$$\begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ 0 & & & & \\ 0 & & \boxed{} & & \\ \vdots & & & & \\ 0 & & & & \end{pmatrix} \longleftarrow \text{ } n-1 \times n-1 \text{ block with rows independent}$$

Continue to get $\begin{pmatrix} 1 & & & & \\ 0 & 1 & & & \\ 0 & 0 & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$, determinant $\neq 0$.
1's on diagonal, 0's below diagonal

7. Prove that

$$\det\begin{pmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ \vdots & & & & \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{pmatrix} = \prod_{1 \le i < j \le n} (a_j - a_i)$$

(Suggestion: Try subtracting 1st row from all others, then factoring out $a_2 - a_1$ from second row, $a_3 - a_1$ third row, etc. Then repeat process. Try this for $n = 2, 3, 4$ to see how it is going to work! It gets slightly messy but it works out.

Alternatively, consider that the determinant must be divisible by $a_j - a_i$, $j \ne i$ since it vanishes if $a_i = a_j$ and go from there).

8. Formulate & prove "expansion by minors": (first column)

$$\det\begin{pmatrix} a_{11} & & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & & a_{nn} \end{pmatrix} = \sum_{i=1}^{n} a_{i1} (-1)^{i+1} M_i$$

where $M_i = \det((n-1) \times (n-1)$ square matrix obtained by dropping first column of A and ith row of $A)$.

Same for first row expansion.

jth row expansion? jth column expansion?

(Save for later, after matrix multiplication is reviewed, if you wish)

9. Use problem 8 to show that

$$A \cdot \hat{A} = (\det A)\, \mathrm{Id}$$

where $\mathrm{Id} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ usual "identity matrix"
(1's down diagonal, 0's elsewhere)

and

$\hat{A}$ = transpose(d matrix) of the matrix the ijth entry of which =

$$(-1)^{i+j} \det \begin{pmatrix} \text{matrix of size } n-1 \times n-1 \\ \text{obtained by erasing} \\ i\text{th row \& } j\text{th column} \\ \text{of } A \end{pmatrix}$$

"signed minor").

Example

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ signed minor matrix $\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$

$\hat{A} = \begin{pmatrix} a & -b \\ -c & a \end{pmatrix}$ ← (transpose of signed minor matrix)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad-bc & -ba+ba \\ cd-dc & -bc+ad \end{pmatrix}$$

$$= (ad-bc) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$