Lecture XI: Inverse Function Theorem Examples, Implicit Function Theorem, Contraction Mapping Ideas, and Equality of Mixed Partials

Inverse function theorem examples:

(1) polar coordinates $(r,\theta) \xrightarrow{F} (r\cos\theta, r\sin\theta)$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{(x(r,\theta), y(r,\theta))}$

matrix for $F' = \begin{pmatrix} \frac{\partial x(r,\theta)}{\partial r} & \frac{\partial x(r,\theta)}{\partial \theta} \\ \frac{\partial y(r,\theta)}{\partial r} & \frac{\partial y(r,\theta)}{\partial \theta} \end{pmatrix}$ "Jacobian matrix"

Jacobian determinant = det (matrix for F')

$$= \det\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = r(\cos^2\theta + \sin^2\theta) = r$$

So F' is nonsingular on $\{(r,\theta) : r \neq 0\}$
So polar coordinate map is "locally smoothly invertible" in a neighborhood of every (x_0, y_0)
$= F(r_0, \theta_0)$, $r_0 \neq 0$.

(2) $z \xrightarrow{F} z^2$, $z \in \mathbb{C}$. As a real map from $\mathbb{R}^2$ to $\mathbb{R}^2$
this is $(x,y) \xrightarrow{F} (x^2 - y^2, 2xy)$
Its Jacobian matrix is $\begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}$

so its Jacobian determinant is $4(x^2+y^2)$.
This is nonzero except at $(x,y) = (0,0)$.
This corresponds to (two) smooth "branches" of $\sqrt{z}$, $z \in \mathbb{C}$ defined locally except at $z=0$: namely F is locally smoothly invertible except at $(0,0)$ $(= F(0,0))$.

(3) $z \to P(z)$ P a polynomial with $\mathbb{C}$ coefficients, $z \in \mathbb{C}$.
As a real mapping, the Jacobian of P (Jacobian determinant that is)*

$$= \left|\frac{\partial P}{\partial z}\right|^2.$$ So if $\left.\frac{\partial P}{\partial z}\right|_{z_0} \neq 0$, then

for each w near $P(z_0)$, there is a unique z near z_0 with $P(z) = w$, and this w depends smoothly on w, w near z_0. This is familiar from algebra: $\frac{\partial P}{\partial z} \neq 0$ means the $z = z_0$ root of $P(z) - P(z_0) = 0$ is simple, etc.

(4) Illustration: a proof technique similar to the minimization proof for local surjectivity in the Inverse Function Theorem:
Fundamental Theorem of Algebra: If $P(z)$ is a polynomial of positive degree, then $\exists z_0 \in \mathbb{C}$ such that $P(z_0) = 0$
Proof: $\exists$ M such that $\min_{|z|=M} |P(z)| > |a_0|$
So $\min_{|z| \le M} |P(z)$ is "interior"
(since $|P(0)| = |a_0| < \min_{|z|=M} |P(z)|$.
Let z_0 be such that $|z_0| < M$ and

$|P(z_0)| = \min_{|z| \le M} |P(z)|$. Claim: $P(z_0) = 0$.
If not, write P as a polynomial

*This is from complex analysis. We take this for granted for purposes of this illustration (not used elsewhere, however)

in $z-z_0$ by $P(z-z_0+z_0)$ expansion so
$P(z) =$ $Q(z-z_0)$. If P has degree $n>0$, so does Q. Write

$$Q(z-z_0) = b_0 + 0 + 0 + b_k(z-z_0)^k + \dots b_n(z-z_0)^n$$

where (the contradiction hypothesis) $b_0 \neq 0$ and $k \geq 1$ is the first coefficient above b_0 which is nonzero. Choose $\varepsilon \neq 0$ such that ($|\varepsilon|$ is small and) $b_k \varepsilon^k$ is a negative real multiple of b_0. Then if $z = z_0 + \varepsilon$,

$$P(z) = Q(z-z_0) = b_0 + b_k \varepsilon^k \quad + o(\varepsilon^k)$$

By choice of ε, if $0 < |b_k \varepsilon^k| < b_0$,

$$|b_0 + b_k \varepsilon^k| = |b_0| - |b_k \varepsilon^k|$$

Hence for ε with $|\varepsilon|$ small enough (but >0) (and $b_k\varepsilon^k$ as described)
$|P(z)| < |b_0|$. This contradicts $|b_0| = |P(z_0)|$ having minimum $|P(z)|$ among z with

$|z| \leq M$, since $|z_0| < M$ and hence $|z| < M$ for $|\varepsilon|$ small enough (along with the other condition) $\square$.

Inverse Function Theorem Example
(Modified Newton's Method in Action)

$F(x,y) = (x + xy, y)$. Solve: $F(x,y) = (a,b$

Exact solution: $y = b$ $\quad x + bx = a$ or $x = a/(1+b)$
(for $|b| < 1$).

Newton's Method: $\vec{y}_1 = (a,b)$ $\quad$ = "x" in lecture notatio

$$\vec{y}_2 = \vec{y}_1 - F(\vec{y}_1) + \vec{x} = (a,b) - (a + ab, b) + (a,b)$$
$$= (a - ab, b)$$

$$\vec{y}_3 = \vec{y}_2 - F(\vec{y}_2) + \vec{x} = (a - ab, b) - (a - ab + b(a-ab), b) + (a,b)$$

$$= (a - ab + ab^2, b)$$

Inductive proof that $y_k = (a - ab + \cdots \pm ab^k, b)$: $\quad + (-1)^k ab^k$

k to k+1 step:

$$\vec{y}_{k+1} = (a - ab \ldots + (-1)^k ab^k, b) - (a - ab \ldots + (-1)^k ab^k, b) - b(a - ab \ldots + (-1)^k ab^k, 0) + (a, b)$$

$$= (a - ab + ab^2 \ldots + (-1)^{k+1} ab^{k+1}, b). \; \square$$

$$\lim \vec{y}_k = (\lim_{k \to +\infty} a - ab \ldots + (-1)^k ab^k, b)$$

$$\underset{(\text{if } |b|<1)}{=} (a/(1-(-b)), b) = (a/(1+b), b)$$

in agreement with exact solution

Contraction Mapping Idea (will be used later for, e.g. differential equations) re Inv Func Th. proof:

Recall our iteration scheme for solving $F(y) = w$ (x unknown, w given) by $y_{k+1} = y_k - F(y_k) + w$. We want to put this in a "fixed point" context. Namely, if we set $H(y) = y - F(y) + w$, then $y = H(y)$ is the same as $F(y) = w$. So solving $F(y) = w$ is the same as looking for a "fixed point" of H. Also $\|H(y') - H(y'')\| = \|y' - F(y') + w - (y'' - F(y'') + w)\|$
$= \|y' - F(y') - (y'' - F(y''))\| \leq \frac{1}{2}\|y' - y''\|$
when $\|F' - I\| \leq \frac{1}{2}$ over a ball containing all points involving (i.e. contain y', y'' in particular). This suggest using the following "Contraction Mapping Theorem" (with $X = \{x : \|x\| \leq \varepsilon/2\}$ in our previous set-up).

Contraction Mapping Theorem: If X is a complete metric space and $H : X \to X$ is a function such that, for some $\lambda < 1$ $d(H(x_1), H(x_2)) \leq \lambda\, d(x_1, x_2)$ for all $x_1, x_2 \in H$, then there is a unique point $x_0 \in X$ such that $H(x_0) = x_0$

Uniq. Proof: $H(x_1) = x_1$ and $H(x_2) = x_2$ together imply $d(H(x_1), H(x_2)) = d(x_1, x_2)$, which contradicts $d(H(x_1), H(x_2)) \leq \lambda\, d(x_1, x_2)$ unless $d(x_1, x_2) = 0$.

Proof of Existence: Pick x_1 in X arbitrarily. Set $x_{i+1} = H(x_i)$, $i \geq 1$. Then $d(x_{i+1}, x_i) \leq C\lambda^{i-1}$ for $C = d(x_1, f(x_1))$. So $\{x_i\}$ is a Cauchy sequence. If $x_0 = \lim x_i$, then clearly $f(x_0) = \lim f(x_i) = x_0$ $\square$

Equality of Mixed Partials Without Integration

Theorem: $f: U^{\text{open in } \mathbb{R}^2} \to \mathbb{R}$, continuous 1st & 2nd partials. Then

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right).$$

Proof: Consider, given $(x, y) \in U$, $h, k > 0$

$$\Delta(h,k) \overset{\text{def}}{=} f(x+h, y+k) - f(x+h, y) - f(x, y+k) + f(x,y)$$

where h, k are required to satisfy $\exists$ closed ball radius $\sqrt{h^2+k^2}$, center (x,y) $\subset U$.

Define $\phi(y) = f(x+h, y) - f(x,y)$

Then

$$\Delta(h,k) = \phi(y+k) - \phi(y)$$

So by MV Theorem for Derivatives

$$\Delta(h,k) = k\, \phi'(y+k_1) \qquad 0 \le k_1 \le k$$

But $\phi'(y+k_1) = \left.\frac{\partial f}{\partial y}\right|_{(x+h, y+k_1)} - \left.\frac{\partial f}{\partial y}\right|_{(x, y+k_1)}$

$$= h\, \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\Big|_{(x+h_1, y+k_1)} \qquad 0 \le h_1 \le h$$

and hence

$$\Delta(h,k) = kh\, \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\right)\Big|_{(x+h_1, y+k_1)}$$

Interchanging the roles of y and x we write

$\psi(x) = f(x, y+k) - f(x,y)$ so $\Delta(h,k) = \psi(x+h) - \psi(x)$

and as before, mutatis mutandis,

$$\Delta(h,k) = hk\, \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\Big|_{(x+h_2, y+k_2)} \qquad \begin{array}{l} 0 \le h_2 \le h \\ 0 \le k_2 \le k. \end{array}$$

So

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\right)\Big|_{(x+h_1, y+k_1)} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\right)\Big|_{(x+h_2, y+k_2)}$$

Letting $h, k \to 0^+$ so $h_1, k_1, h_2, k_2 \to 0^+$ and using continuity LHS $\to \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\Big|_{x,y}$, RHS to $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\Big|_{(x,y)}$. So mixed partials $\overset{=}{\text{at}}$ (x,y) $\square$