Lecture X: The Inverse Function Theorem

Definition: A function $f: U \to V$, U, V open sets in $\mathbb{R}^n$ is a C^1 <u>diffeomorphism</u> if

(1) f is differentiable at each point $x \in U$ and $f'|_x$ is a continuous function on U.

(2) f is one-to-one and onto.

(3) $f^{-1}: V \to U$ is differentiable at each point $y \in U$ and $(f^{-1})'|_y$ is a continuous function on V.

The Inverse Function Theorem: If $f: U \to \mathbb{R}^n$ is a function on an open set U in $\mathbb{R}^n$ such that f is differentiable at each point $x \in U$ and such that $f'|_x$ is a continuous function on U, then for each $x_0 \in U$ such that $f'|_{x_0}$ is invertible, there is an open set U_0 with $x_0 \in U_0 \subset U$ such that

(a) $f|U_0$ is one-to-one

(b) $f(U_0)$ is open

(c) $f|U_0 : U_0 \to f(U_0)$ is a C^1 diffeomorphism.

The proof of the Inverse Function Theorem involves two stages. First, one uses the Mean Value Estimate to show that there is an open neighborhood of x_0 on which f is one-to-one. This is relatively straightforward. Then one needs to show that f is "locally onto", e.g., there is an open

set $W \subset f(U)$ with $f(x_0) \in W$.
Interestingly, this second step is actually an automatic consequence of the (local) injectivity, even for functions which are just continuous, not necessarily (continuously) differentiable:

The "Invariance of Domain" Theorem (L.E.J. Brower): If $f: U \to \mathbb{R}^n$, $U^{\text{open}} \subset \mathbb{R}^n$ is one-to-one and continuous, then $f(U)$ is open in $\mathbb{R}^n$ [and $f^{-1}: f(U) \to U$ is continuous].

However, the proof of this theorem involves techniques of algebraic topology (it was one of the first of many major triumphs of algebraic topology) and it is thus beyond the scope of these notes. So we shall prove the "local ontoness" of f in the Inverse Function Theorem by using differentiability to construct "f^{-1}" (locally) directly.

[Note that the [ ] bracketed part of the "Invariance of Domain" Theorem is an automatic consequence of the first part. The crucial point is that the injective image of an open set is open, and continuity of the inverse follows easily (exercise)].

From now on, we take $\vec{x}_0 = \vec{0}$ and $F(\vec{x}_0) = \vec{0}$ without loss of generality. We turn first to the local injectivity of f in a neighborhood of x_0. Namely, assume again without loss of generality that $f'|_{x_0}$ = the identity map I_n from $\mathbb{R}^n$ to $\mathbb{R}^n$ (if not, replace f by the composition of f and a nonsingular linear map $\mathbb{R}^n \to \mathbb{R}^n$).

Now choose $\varepsilon > 0$ such that $B(\vec{x}_0, \varepsilon) = B(\vec{0}, \varepsilon) \subset U$ and such that $\| I_n - f'|_x \| < 1$ for all $x \in B(\vec{x}_0, \varepsilon)$. (Here $\| \ \|$ means operator norm as usual). This choice is possible by the continuity of f' and the wolog hypothesis that $f'(x_0) = f'(\vec{0}) = I_n$.

Lemma: $f|_{B(\vec{x}_0, \varepsilon)}$ is one-to-one.

Proof: Set $G = I - f$, i.e.
$G(\vec{x}) = \vec{x} - f(\vec{x})$, $\vec{x} \in B(\vec{x}_0, \varepsilon) = B(\vec{0}, \varepsilon)$.
Then for $\vec{x}_1, \vec{x}_2 \in B(\vec{0}, \varepsilon)$,

$$\| G(\vec{x}_1) - G(\vec{x}_2) \| \leq M \| x_1 - x_2 \|$$

where $M = \sup\limits_{x \ni \|x\| \leq \|x_1\|, \|x\| \leq \|x_2\|} \| I_n - f'|_x \|$ since $G' = I - f'$

Note that $M < 1$ (since $cl(B(\vec{0}, \min(\|x_1\|, \|x_2\|))$ is a compact subset of $B(\vec{0}, \varepsilon)$).
But $G(\vec{x}_1) - G(x_2) = x_1 - x_2 - (f(x_1) - f(x_2))$
so if $f(x_1) = f(x_2)$, $\| G(x_1) - G(x_2) \| = \| x_1 - x_2 \|$.
But $\| G(x_1) - G(x_2) \| \leq M \| x_1 - x_2 \|$, $M < 1$
and $\| G(x_1) - G(x_2) \| = \| x_1 - x_2 \|$
are consistent only if $x_1 = x_2$. Thus
$f(x_1) = f(x_2)$, $x_1, x_2 \in B(\vec{0}, \varepsilon) \Rightarrow x_1 = x_2$ $\square$.

This disposes of "local injectivity".
Now we turn to "local surjectivity" (local onto-ness). We shall eventually exhibit a construction by an iterative process of a "local inverse". But first we give a "trick" proof that $f(B(\vec{0}, \varepsilon)) \supset B(0, \delta)$ for some $\delta > 0$ (still assuming $f(\vec{0}) = \vec{0}$, $f'|_{\vec{0}} = I_n$).

For this, consider $f \,|\, \{\vec{x} : \|\vec{x}\| = \varepsilon/2\}$. Since f is injective on $B(\vec{0}, \varepsilon)$, $f(x)$ cannot equal $\vec{0}$ for x with $\|x\| = \varepsilon/2 > 0$. So $\min_{\|x\| = \varepsilon/2} \|f(x)\| > 0$, say $\min_{\|x\| = \varepsilon/2} \|f(x)\| = \delta_1$. We claim $f(B(\vec{0}, \varepsilon)) \supset B(\vec{0}, \delta_1/2)$.
To check this claim, suppose $w \in B(\vec{0}, \delta_1/2)$ and consider the function $H(x) = \|f(x) - w\|^2$ on the closed ball $\{x : \|x\| \leq \varepsilon/2\}$.
Since $f(x) \geq \delta_1$ if $\|x\| = \varepsilon/2$, $H(x) \geq (\delta_1 - \delta_1/2)^2 = \delta_1^2/4$. But $f(0) = \delta_1^2/4$. So $H(x) \,|\, \{x : \|x\| \leq \varepsilon/2\}$ attains its minimum at some y_0 with $\|y_0\| < \varepsilon/2$ (the minimum is "interior"), say $f(y_0) = (\alpha_1, \ldots, \alpha_n)$.
Since H is a local minimum at y_0, the $x_1, \ldots x_n$ partial derivatives of H must be 0 at y_0. Direct calculation shows that

$$\left.\frac{\partial H}{\partial x_j}\right|_{(\alpha_1, \ldots, \alpha_n)} = 2 \sum_{i=1}^{n} \left.\frac{\partial f_i}{\partial x_j}\right|_{\vec{y}_0} (\alpha_i - w_i)$$

$(\vec{w} = (w_1, \ldots, w_n))$.

But the matrix $\left.\frac{\partial f_i}{\partial x_j}\right|_{y_0}$ is nonsingular.

Hence it must be that $\alpha_i = w_i$ for $i = 1 \ldots n$.
And hence $f(y_0) = \vec{w}$. $\square$

Exercise: Interpret this proof geometrically (idea: f' onto means that at any point in the image of f one can move – to first order – in any direction. So if one were as close to w as possible, one would have to be at w — otherwise one could move (more) towards w!

Now we look at the iterative scheme for finding a solution of $f(y) = w$ when $\|w\|$ is small (retaining our $f(\vec{0}) = \vec{0}$ and $f'|_{\vec{0}} = I_n$ situation).

We start with $y_1 = w$ as our first approximation. This is a good start since f near $\vec{0}$ is close to the identity, namely $\| f(y) - y \| \leq \frac{1}{2} \|y\|$ if $y \in B(\vec{0}, \varepsilon)$, ε as before.

For an improvement, we put

$$y_2 = y_1 + w - f(y_1)$$

Why is this a good idea? Since f' is close to I ($\| I - f' \| < \frac{1}{2}$ on $B(\vec{0}, \varepsilon)$), $f(y_2)$ should be close to

$$f(y_1) + I(w - f(y_1)) = w.$$

To make this precise, let us choose $\varepsilon > 0$ so that $\| f' - I \| < \frac{1}{2}$ on $B(\vec{0}, \varepsilon)$ (smaller than before). Suppose $\| w \| < \frac{1}{4}\varepsilon$.
With $y_1 = y$ and $y_2 = y_1 - F(y_1) + w$
and inductively $y_{k+1} = y_k - F(y_k) + w$,
we have estimates as follows:

First note that if $x, x+h \in B(0, \varepsilon)$ then

$$\| F(x+h) - F(x) - h \| = \| F(x+h) - (x+h) - (F(x) - x) \|$$
$$\leq \tfrac{1}{2} \| h \| \qquad (\text{since } \| F' - I \| \leq \tfrac{1}{2}).$$

Thus $F(y_{k+1}) = F(y_k) + w - F(y_k) + \underset{\leq \frac{1}{2}\| w - F(y_k)\|}{\text{term of norm}}$

So $\| F(y_{k+1}) - w \| \leq \frac{1}{2} \| w - F(y_k) \|$
and
$$\| y_{k+1} - y_k \| = \| w - F(y_k) \| \leq \tfrac{1}{2} \| w - F(y_{k-1}) \|.$$

So, since $\| y_2 - y_1 \| \leq \| F(y_1) - w \| = \| F(w) - w \|$
$\leq \frac{1}{2} \| w \| < \varepsilon/4$
we get that $\| y_{k+1} - y_k \| < \varepsilon / 2^{k+1}$

So $\| y_{k+1} - w \| < \varepsilon/2$ and $y_{k+1} \in B(\vec{0}, \varepsilon)$
all k. Also
$\{ y_k \}$ is Cauchy with limit in $B(\vec{0}, \frac{3\varepsilon}{4})$
(since $\| w \| < \varepsilon/4$). Similarly $F(y_k) \in B(\vec{0}, \varepsilon)$
for all k. So all estimates are valid
for all k. Clearly $F(\lim y_k) = w$. $\square$