Arzela - Ascoli Theorem:
Details of the Final Steps of the Proof:

(1) Uniform Cauchy given convergence at countable dense subset:

Set up: $f_{n_j} : X \to Y$, X compact, $\{f_{n_j}\}$ converges at each point $s_1, s_2, s_3 \ldots$ of a countable dense subset of X, $\{f_{n_j}\}$ equicontinuous.

Hope to prove: $\{f_{n_j}\}$ converges uniformly on X. This is the same as $\{f_{n_j}\}$ being uniformly Cauchy. By compactness of X, it is enough to prove that given $\varepsilon > 0$ and $x_0 \in X$, there is a $\delta_{\varepsilon, x_0} > 0$ and an N_{ε, x_0} such that: $l, k \geq N$ and $d_X(x_0, x) < \delta_{\varepsilon, x_0} \Rightarrow$

$d_Y(f_{n_l}(x), f_{n_k}(x)) < \varepsilon$ if $l, k \geq N$.

Why this suffices: $B(x_0, \delta_{\varepsilon, x_0})$, $x_0 \in X$ is an open cover of X. So there is a finite subcover. And $N =$ max of N's for each element of the finite subcover will work for uniform Cauchy for the given ε. (that is, $l, k \geq$ that $N \Rightarrow d_Y(f_{n_l}(x), f_{n_k}(x)) < \varepsilon$).

We find, given $\varepsilon > 0$ and x_0, the required $\delta_{\varepsilon, x_0}$ and N_{ε, x_0} as follows: Choose $\delta_{\varepsilon, x_0}$ such that $d_X(x_0, x) < \delta_{\varepsilon, x_0} \Rightarrow$

$d_Y(f_{n_j}(x), f_{n_j}(x_0)) < \varepsilon/6$ for all n_j (i.e., all j).

This is possible by equicontinuity.

Then choose N_{ε, x_0} as follows:

Choose some s in the countable dense subset (on which $\{f_{n_j}\}$ converges) such that $d_X(x_0, s) < \delta_{\varepsilon, x_0}$ ← already chosen.

Then choose N_{ε, x_0} such that $l, k \geq N_{\varepsilon, x_0} \Rightarrow$

$d_Y(f_{n_l}(s), f_{n_k}(s)) < \varepsilon/6$.

Then for $l, k \geq N_{\varepsilon, x_0}$ and $x \in B(x_0, \delta_{\varepsilon, x_0})$, $d_Y(f_{n_l}(x), f_{n_k}(x))$ can be estimated by:

$f_{n_l}(x)$ is $\varepsilon/6$-close to $f_{n_l}(x_0)$ which is $\varepsilon/6$ close to $f_{n_l}(s)$ which is $\varepsilon/3$-close to $f_{n_k}(s)$ which is $\varepsilon/6$-close to $f_{n_k}(x_0)$ which is $\varepsilon/6$ close to $f_{n_k}(x)$

$\Rightarrow f_{n_l}(x)$ is $\varepsilon/6 + \varepsilon/6 + \varepsilon/3 + \varepsilon/6 + \varepsilon/6 = \varepsilon$-close to $f_{n_k}(x)$. [Here: "a ε-close to b" means $d(a, b) < \varepsilon$].

Thus these $\delta_{\varepsilon, x_0}$ and N_{ε, x_0} suffice □

(2) Every sequentially compact metric space X has a countable dense subset.

Def: An ε-net in X, $\varepsilon > 0$, is a maximal set S such that $s_1, s_2 \in S$, $s_1 \neq s_2 \Rightarrow d_X(s_1, s_2) \geq \varepsilon$.
(i.e. S has this property, that every two distinct points are at least ε-separated and no $S' \supset S$, $S' \neq S$, has this property).

Lemma: $\varepsilon > 0$, X seq compact $\Rightarrow \exists$ an ~~finite~~ ε-net in X. ~~The~~ An ε-net is necessarily a finite set.

Proof: Finiteness is clear: an infinite ε-net would yield an infinite sequence with no convergent subsequence. For existence: choose $x_0 \in X$. If $X \subset B(x_0, \varepsilon)$, done. If not, choose $x_1 \ni d_X(x_0, x_1) \geq \varepsilon$. If $B(x_0, \varepsilon) \cup B(x_1, \varepsilon) = X$, done. If not, choose $x_2 \in (X - B(x_0, \varepsilon)) \cap (X - B(x_1, \varepsilon))$ so that $d_X(x_1, x_2) \geq \varepsilon$ & $d_X(x_0, x_2) \geq \varepsilon$. Continue this process. It must terminate since otherwise one would obtain an infinite ~~set with~~ sequence with no convergent subsequence $\boxtimes$

To get a countable dense subset S, let S_n = a $1/n$-net, $n = 1, 2, 3 \ldots$ and set

$$S \doteq \bigcup_{n=1}^{+\infty} S_n$$

Since each S_n is finite, S is countable and clearly dense. $\square$