Uniform Convergence, Uniform Continuity, Uniform Equicontinuity, Arzela-Ascoli Theorem

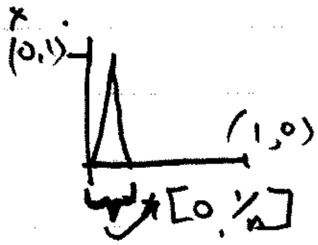
Definition: A sequence $\{f_n\}$ of functions from a metric space (X, d_X) to a metric space (Y, d_Y) converges uniformly to $f_0 : X \to Y$ if, for each $\varepsilon > 0$, $\exists N$ such that, if $n \geq N$,

$$d_Y(f_n(x), f_0(x)) < \varepsilon \quad , \forall x \in X.$$

The point is that ε is independent of $x \in X$, as opposed to "pointwise convergence", which means only that $\forall \varepsilon > 0$, $\exists N_{\varepsilon, x} \ni$ $d_Y(f_n(x), f_0(x)) < \varepsilon$ if $n \geq N_{\varepsilon, x}$ but $N_{\varepsilon, x}$ can depend on x.

Examples:

(1)



$X = [0,1] = Y$

$f_n(x) = 0 \qquad x \geq \frac{1}{n}$

$$f_n(x) = 2nx \quad , \quad 0 \leq x \leq \frac{1}{2n}$$

$$f_n(x) = 1 - 2n\left(-\frac{1}{2n} + x\right) \qquad \frac{1}{2n} < x < \frac{1}{n}$$

$$= 2 - 2nx \qquad \frac{1}{2n} < x < \frac{1}{n}$$

Converges pointwise to 0 function, but not uniformly

(2) $\quad f_n(x) = x^n \qquad X = [0,1] = Y$

converges to $f_0(x) = 0 \quad 0 < x < 1$, $\quad f_0(1) = 1$, pointwise not uniformly

Fundamental fact: "Uniform limit of continuous functions is continuous"

Exact statement:

Theorem: If $\{f_n : X \to Y\}$ is a sequence of continuous functions from a metric space (X, d_X) to a metric space (Y, d_Y) and if $\{f_n\}$ converges uniformly to $f_0 : X \to Y$, then $f_0 : X \to Y$ is continuous.

Proof: Given $\varepsilon > 0$, choose N such that $d_Y(f_N(x), f_0(x)) < \varepsilon/3$ for all $x \in X$ (possible by definition of uniform convergence). Then, for a given $x_0 \in X$, choose $\delta > 0$ such that $d_X(x, x_0) < \delta \Rightarrow d_Y(f_N(x), f_N(x_0)) < \varepsilon/3$: possible because f_N is continuous. Then if $d_X(x, x_0) < \delta$,

$$\begin{aligned} d_Y(f_0(x), f_0(x_0)) \le\ & d_Y(f_0(x), f_N(x)) \\ &+ d_Y(f_N(x), f_N(x_0)) \\ &+ d_Y(f_N(x_0), f_0(x_0)) \\ <\ & \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned}$$

So f_0 is continuous at x_0. $\square$

Study this proof carefully. The result is fundamental

Exercises: (1) Formulate a definition of a sequence of functions $f_n : X \to Y$ being "uniformly Cauchy"

(2) Prove that if Y is complete and $\{f_n : X \to Y\}$ is uniformly Cauchy, then $\exists\, f_0 : X \to Y$ such that $\{f_n\}$ converges uniformly to f_0.

Definition: A function $f: X \to Y$, (X, d_X) and (Y, d_Y) metric spaces is <u>uniformly continuous</u> on X if, for each $\varepsilon > 0$, there is a $\delta > 0$ such that $d_X(x_1, x_2) < \delta \Rightarrow d_Y(f(x_1), f(x_2)) < \varepsilon$.

Proposition: If (X, d_X) is a compact metric space (and (Y, d_Y) a metric space), then every continuous function $f: X \to Y$ is uniformly continuous.

Proof: Given $\varepsilon > 0$, we produce a $\delta > 0$ that works in the definition of uniform continuity as follows:
For each $x \in X$, $\exists$ $\alpha_x > 0$ such that $x' \in B(x, \alpha_x) \Rightarrow d_Y(f(x'), f(x)) < \varepsilon/2$. The sets $B(x, \alpha_x/2)$ are an open cover of X (since $x \in B(x, \alpha_x/2)$!).
Choose a finite subcover $B(x_1, \alpha_{x_1}/2), \ldots, B(x_n, \alpha_{x_n}/2)$.
Let $\delta = \min(\alpha_{x_1}/2, \ldots, \alpha_{x_n}/2)$.
Claim: This δ works.
Reason: If $d_X(x', x'') < \delta$, then $x' \in B(x_j, \alpha_{x_j}/2)$ for some j and then

$$d_X(x'', x_j) \le d(x_j, x') + d(x', x'') < \alpha_{x_j}/2 + \delta \le \alpha_{x_j}$$

since $\delta = \min(\alpha_{x_1}/2, \ldots, \alpha_{x_n}/2)$
So $d_Y(f(x''), f(x_j)) < \varepsilon/2$ and $d_Y(f(x'), f(x_j)) < \varepsilon/2$.
Hence $d_Y(f(x''), f(x')) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. $\blacksquare$

The proof just given is actually harder than the proof by contradiction and subsequences. This often happens!

Subsequence proof: Suppose f_0 is not uniformly continuous. Then for some $\varepsilon > 0$, $\delta = 1/n$ $n = 1, 2 \ldots$ all fail. So $\exists$ x_n', x_n'' with $d_X(x_n', x_n'') < \frac{1}{n}$ but $d_Y(f(x_n'), f(x_n'')) \geq \varepsilon$ (>0). Choose a convergent subsequence of $\{x_n'\}$, say $\{x_{n_j}'\}$ converges to $x_0 \in X$. Choose

$\lambda > 0$ such that $d_X(x_0, x) < \lambda \Rightarrow d_Y(f(x_0), f(x)) < \frac{\varepsilon}{2}$.

Then when j is large $d_X(x_{n_j}', x_0) < \lambda/2$

and $d_X(x_{n_j}', x_{n_j}'') < \frac{1}{n_j} < \lambda/2$ so

x_{n_j}' and x_{n_j}'' both have f-images $\mathrm{dis}_Y < \varepsilon/2$

from $f(x_0)$, contradicting $d_Y(f(x_{n_j}'), f(x_{n_j}'')) \geq \varepsilon$. $\square$

Exercise (important): Use the subsequence method to prove

"Existence of Lebesgue Numbers":

If X is a compact metric space and if $\{U_\lambda : \lambda \in \Lambda\}$ is a open cover of X, then $\exists$ $\alpha > 0$ such that, for each $x \in X$, the open ball $B(x, \alpha)$ is contained (entirely) in some U_λ. [Point is that α does not depend on x!

Exercise: If $f: A \to Y$, $A \subset X$, X, Y metric spaces, and if f is uniformly continuous and Y is complete, then f extends to the closure of A as a continuous function, i.e. $\exists\ F: cl(A) \to Y$ continuous such that $F|A = f$. ($cl(A) =$ closure of A). Suggestion: First show that, if f is uniformly continuous, then the f-image of a Cauchy sequence is a Cauchy sequence.

Definition: A family $\mathcal{F}$ of functions from a metric space X to a metric space Y is <u>equicontinuous</u> at $x_0 \in X$ if, for $\forall\, \varepsilon > 0$,

$\exists\, \delta > 0$ such that $d_X(x, x_0) < \delta \Rightarrow$

$d_Y(f(x), f(x_0)) < \varepsilon$ for all $f \in \mathcal{F}$.

A family $\mathcal{F}$ of functions is <u>equicontinuous</u> on X if $\mathcal{F}$ is equicontinuous at each point x_0 in X.

Exercise: Formulate the (more or less obvious) definition of $\mathcal{F}$ being uniformly equicontinuous on X. Then prove that if X is compact and $\mathcal{F}$ is a family of functions from X to Y, and if $\mathcal{F}$ is equicontinuous on X, then $\mathcal{F}$ is uniformly equicontinuous on X. (Suggestion: Use Lebesgue numbers).

Arzela-Ascoli Theorem: If $\mathcal{F}$ is a family of continuous functions from a compact metric space X into a compact metric space Y and if $\mathcal{F}$ is (uniformly) equicontinuous on X, then every sequence $\{f_n : f_n \in \mathcal{F}\}$ has a subsequence which converges uniformly on X.

In most applications, X and Y are subsets of euclidean spaces. But this is one of those theorems which are actually easier to think about in greater generality!

Idea of proof: Choose a countable dense subset S of X and a subsequence $\{f_{n_j}\}$ which converges at each point of S by our previously introduced "super-trick" (end of lecture $\overline{\underline{IV}}$). Then (uniform) equicontinuity $\Rightarrow$ convergence uniformly on X. $\square$