Summary of Lecture 4: Proof of Heine Borel Theorem

Proposition: $[0,1]$ is sequentially compact (Bolzano Weierstrass)
Proof: Suppose $\{x_j\}$ is a sequence in $[0,1]$.
Then either $[0,\frac{1}{2}]$ or $[\frac{1}{2},1]$ has the property that x_j is in it for infinitely many j values.
Choose $I_1 = [0,\frac{1}{2}]$ if it has this property, otherwise set $I_1 = [\frac{1}{2},1]$. And choose $x_{j_1} \in I_1$.
Choose I_2 = left hand closed half of I_1 if x_j is in that interval for ∞ly many j.
Otherwise choose I_2 = right hand half
(Then x_j is in I_2 for ∞ly many j).
Choose x_{j_2}, $j_2 > j_1$ in I_2.
Continue to get $I_1 \supset I_2 \supset I_3 \ldots$
I_n length $1/2^n$, $x_{j_n} \in I_n$ $j_{n+1} > j_n$.

Let α = l.u.b. left hand endpts of the I_n, $n=1,2,3\ldots$
Then $\alpha \in I_n$ for all n. Also
$|\alpha - x_{j_n}| \le 1/2^n$, $\forall n$. So $\{x_{j_n} : n=1,2\ldots\}$ subsequence of $\{x_j\}$ converges to α. $\square$

Cor: $M>0 \Rightarrow [-M,M]$ is seq compact in $\mathbb{R}$.

Proposition: For each $M>0$, $[-M,M]\times\ldots\times[-M,M]$ is seq. compact in $\mathbb{R}^n$.
Proof: $\{\vec{x}_j\}$ sequence in $[-M,M]\times\ldots\times[-M,M]$ has lth component, $l=1,\ldots,n$, $\in [-M,M]$.
Passing to subsequence n times can arrange that sequence of lth components converges (to pt in $[-M,M]$)
so whole subsequence converges to pt in $[-M,M]\times\ldots\times[-M,M]$

Proposition: A closed bounded set in $\mathbb{R}^n$ is sequentially compact.

Proof: A closed bound set $\Rightarrow \exists M > 0 \ni$
$A \subset [-M, M] \times \ldots \times [-M, M]$.
Sequence $\{a_j\}$ in A thus has subsequence that converges, and A being closed $\Rightarrow$ limit is in A. $\square$

Proposition: A sequentially compact set A in $\mathbb{R}^n$ is closed and bounded.

Proof: A not bounded $\Rightarrow \exists \vec{x}_n \in A \ni$ $d(0, \vec{x}_n) \geq n$. Sequence $\{x_n\}$ has no convergent subsequence.

A not closed $\Rightarrow \exists$ $a_n \in A$ such that $\{a_n\}$ converges to a_0, $a_0 \notin A$. Then $\{a_n\}$ has no subsequence converging to a point of A. $\square$

Proposition: A compact set A in $\mathbb{R}^n$ is closed and bounded:

Proof: If $x \notin A$, then $A \subset \bigcup_{j=1}^{+\infty} \{y \in \mathbb{R}^n : d(y, x) > \frac{1}{j}\}$

The set $\{y \in \mathbb{R}^n : d(y, x) > \frac{1}{j}\}$ is open. So A compact
$\Rightarrow A \subset \bigcup_{j=1}^{N} \{y \in \mathbb{R}^n : d(y, x) > \frac{1}{j}\}$

So $a \in A \Rightarrow d(a, x) > \frac{1}{N}$. Hence $B(x, \frac{1}{N}) \subset$ complement of A. So complement of A is open.

Boundedness: $A \subset \bigcup_{j=1}^{+\infty} \{y \in \mathbb{R}^n : d(y, \vec{0}) < j\}$. So

$A \subset B(\vec{0}, N)$, some N. So A bounded. $\square$

not done in class. Proof done in class is on next page. This one generalizes to prove seq. compact $\Downarrow$ compact for metric spaces.

Proposition: $A \subset \mathbb{R}^n$ seq. compact $\Rightarrow$ A compact.

Proof: If $A \subset \bigcup_{\lambda \in \Lambda} U_\lambda$ then

$A \subset \bigcup_{j=1}^{+\infty} U_{\lambda_j}$, some λ_j choices (proved earlier)

Suppose $\exists$ $x_j \in A$ but $x_j \notin \bigcup_{j=1}^{+\infty} U_{\lambda_j}$,

each $j = 1, 2, 3 \ldots$. Then $\exists$ subsequence $\{x_{j_k}\}$ converging to $x_0 \in A$.

Now $x_0 \in U_{\lambda_{j_0}}$ some j_0 (since

$x_0 \in A$ and $x_0 \in \bigcup_{j=1}^{+\infty} U_{\lambda_j}$). But

$\{x_{j_k}\}$ converges to x_0 $\Rightarrow$ then $x_{j_k} \in \bigcup_{j=1}^{j_0} U_{\lambda_j}$

for k large, contradicting choice of x_j. $\square$

Putting all this together gives H-B Theorem.
equivalences:

closed bounded $\Leftrightarrow$ seq compact $\Rightarrow$ compact

Proof 2 (one done in class) that seq compact $\Rightarrow$ compact. ?

We have proved already seq compact $\Rightarrow$ closed & bounded $\Rightarrow$ compact.

For this First note that if $B \subseteq A \subseteq \mathbb{R}^n$ and A is compact and B is closed then B is compact. Reason: If $\bigcup U_\lambda$ is an open cover of B then $\bigcup U_\lambda \cup (\mathbb{R}^n - B)$ is an open cover of A. Finite subcover looks like $U_{\lambda_1}, \ldots, U_{\lambda_n}$ and $\mathbb{R}^n - B$ where $U_{\lambda_1}, \ldots, U_{\lambda_n}$ has to cover B. $\square$.

for closed bounded $\Rightarrow$ compact

So it suffices to show $[-M, M] \times \ldots \times [-M, M]$ is compact in $\mathbb{R}^n$ and for this it is enough to show $[0,1] \times \ldots \times [0,1] \subset \mathbb{R}^n$ is compact. Suppose $\{U_\lambda : \lambda \in \mathcal{L}\}$ is an open cover with no finite subcover. Then $\exists\ A_1 \times \cdots \times A_n \subset [0,1]^n$ which is not covered by finitely many U_λ, where each $A_i = [0, \frac{1}{2}]$ or $[\frac{1}{2}, 1]$. Choose one and call it S_1. Subdivide S_1 in similar way into 2^n "subcubes" of side length $1/4$. One, call it S_2, requires ∞ly many U_λ to cover it (not covered by finitely many). Continue $S_1 \supset S_2 \supset S_3 \ldots$ each requires ∞ly many U_λ to cover it. $\exists\ p_0 \in \bigcap S_j$ (nested intervals as in proof of Bolzano-Weierstrass, applied to each of the n coordinates). Since $p_0 \in [0,1]^n$, $p_0 \in U_{\lambda_0}$ for some λ_0. Since U_{λ_0} is open and "side length" of S_k is 2^{-k}, it follows (from $p_0 \in S_k$ all k) that for k sufficiently large, $S_k \subset U_{\lambda_0}$. This contradicts that S_k requires infinitely many U_λ to cover it $\square$.

Super-trick for $[0,1]\times[0,1]\times\cdots$ (infinite product):

If $\{\vec{x}_j\}$ is a sequence of elements of $[0,1]\times[0,1]\times\cdots$ (i.e. a sequence of sequences, each sequence having value in $[0,1]$) then $\exists$ a subsequence $\{\vec{x}_{j_k}\}$

such that for each $i = 1, 2 \ldots$, the sequence ith component of $\vec{x}_{j_k}$, $k = 1, 2, 3 \ldots$ converges (to a point of $[0,1]$).

Proof: There is a subsequence of $\{\vec{x}_j\}$ such that the first component converges. There is a (second) subsequence of that first subsequence $\ni$ the second component also converges. There is a (third) subsequence of that second subsequence such that the third component also converges. Etc.

Take the 1st element of the first subsequence, the second element of the second subsequence, the third element of the third subsequence, etc. This is a subsequence of the original sequence and it converges in every "slot", i.e. the ith component converges, for each fixed i; Reason is that from the ith term onward, it is a subsequence of the ith subsequence!

Application: f_j functions on $[0,1]$, value in $[0,1]$. $\exists$ subsequence $f_{j_k} \ni$ for each rational r in $[0,1]$, the sequence $\{f_{j_k}(r)\}$ converges.

Surprising! Think about, e.g. $f_j(x) = |\sin jx|$. Not easy to think of explicit subsequence that works!