Summary of Lecture III

$U \subset$ metric space (X,d) is <u>open</u> if, for $\forall x \in U$, $\exists\ \varepsilon_x > 0$ such $\{y \in X : d(x,y) < \varepsilon_x\} \subset U$.

Notation: $\{y \in X : d(x,y) < \varepsilon\} = B(x,\varepsilon)$ "the open ball of radius ε around x" (or "with center x").

Lemma: Open balls are open sets. (see G&G for (easy) proof)

ϕ, X are open, union of open sets is open, finite intersections $U_1 \cap \ldots \cap U_n$ of open sets are open (can be false for infinite intersection e.g. $\bigcap_{n=1}^{+\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = [0]$, not open in $\mathbb{R}$).

Closed: $\overset{\text{def}}{=}$ complement open. ϕ, X, finite unions arbitrary intersections of closed are closed!

<u>Rational balls and covers</u>

$X = \mathbb{R}^n$ "rational ball" $B((r_1, \ldots, r_n), \alpha)$
$r_1, \ldots, r_n \in Q$ $\alpha \in Q$ $\alpha > 0$.

There are countably many rational balls (Exercise)

Lemma: U open in $\mathbb{R}^n$ = union of the rational balls $\subseteq U$.

Proof: Suffices to show $\vec{x} \in U \Rightarrow \exists$ rat. ball B with $x \in B$ and $B \subset U$. Reason for this: $\exists\ \varepsilon > 0 \ni B(\vec{x}, \varepsilon) \subset U$. Choose $(r_1, \ldots, r_n) = \vec{r}$ (rationals) so that $d(\vec{x}, \vec{r}) < \varepsilon/400$. Choose rational $\alpha \ni \varepsilon/3 < \alpha < \varepsilon/2$. Then $\vec{x} \in B(\vec{r}, \alpha)$ and $B(\vec{r}, \alpha) \subset U$ (by triangle inequality). Application:

An open cover $\{U_\lambda, \lambda \in \Lambda\}$ of a set A is a collection of open sets $\ni \bigcup_{\lambda \in \Lambda} U_\lambda \supset A$.

Theorem (Lindelöf): Every open cover of a set $A \subset \mathbb{R}^n$ has a countable subcover, i.e. $\exists\ \lambda_1, \lambda_2, \lambda_3 \ldots \ni \bigcup_{j=1}^{\infty} U_{\lambda_j} \supset A$.

Proof: Let $\mathcal{B}$ = the set of rational balls $B(\vec{r}, \alpha)$ {such that α rational} $B(\vec{r}, \alpha) \subset U_\lambda$ for some λ. Then $\bigcup_{B(r,\alpha) \in \mathcal{B}} B(r, \alpha) = \bigcup U_\lambda \supset A$.

There are countably many balls in $\mathcal{B}$ (since set of all rational balls is countable!) So let $B_1, B_2 \ldots$ be a counting of the balls in $\mathcal{B}$. Choose $U_{\lambda_j} \ni$ to $B_j \subset U_{\lambda_j}$. Then $\bigcup_{j=1}^{\infty} U_{\lambda_j} \supset \bigcup_{j=1}^{\infty} B_j$

$= \bigcup_{\lambda \in \Lambda} U_\lambda \supset A$. $\square$.

Special importance is attached to those sets $A \subseteq \mathbb{R}^n$ such that every open cover of A has a finite subcover. Such a set is called <u>compact</u>.

Heine-Borel Theorem: The following are equivalent for a set $A \subseteq \mathbb{R}^n$:

(a) A is compact

(b) A is closed (in $\mathbb{R}^n$) and bounded (means: $\exists\ C > 0$ such that $A \subset B(\vec{o}, C)$).

(c) Every sequence in A has a subsequence that converges to a point in A.

Property (3) is called "sequential compactness": i.e. a set (or metric space) A is <u>sequentially compact</u> if every (infinite) sequence $\{a_j\}$ in A has a subsequence $\{a_{j_k}\}$ which converges to some $a_0 \in A$.

(Note: Compactness also makes sense for metric spaces: A metric space (X, d) is <u>compact</u> if $\bigcup_{\lambda \in \Lambda} U_\lambda = X$, $U_\lambda \subset X$, U_λ open, all $\lambda \in \Lambda$ $\Rightarrow \exists\, \lambda_1, \dots, \lambda_n \ni \bigcup_{j=1}^{n} U_{\lambda_j} = X$).

Exercise: A set $A \subseteq \mathbb{R}^n$ is compact if and only if (A, d_A), $d =$ $\mathbb{R}^n$-metric restricted to $A \times A$, is compact as a metric space (unto itself). [For this, you need: $B \subset A$ is open in the metric space $B \Leftrightarrow B$ has the form $U \cap A$ for some open <u>set</u> $U \subset \mathbb{R}^n$].

The utility of the Heine-Borel Theorem is derived (mostly) from the following:

Theorem: If $f: X \to Y$ is continuous (X, Y metric spaces) and X is compact, then $f(X)$ is compact.

Here $f(X)$ being compact can be either as a subset of Y <u>or</u> as a metric space itself with distance function d_Y restricted to $f(X) \times f(X)$. These ideas coincide (cf. the exercise), indeed not just for $f(X)$ but for any $A \subset Y$: A is compact in Y is the same thing as A is compact as a metric space with the "induced" metric d, restricted to $A \times A$.

The proof of the Theorem follows easily by noting:
A function $f: X \to Y$, X, Y metric spaces is continuous (i.e., continuous at each $x \in X$) if and only if, for all $V \subset Y$, V open in Y, the set $f^{-1}(V) \stackrel{def}{=} \{x \in X : f(x) \in V\}$ is open in X.

This equivalence is just a matter of tracing through definitions (cf. G&G).

Proof of Theorem (that "continuous image of compact set is compact"): If $\bigcup_{\lambda \in \Lambda} V_\lambda \supset f(X)$, then

$\bigcup_{\lambda \in \Lambda} f^{-1}(V_\lambda) = X$. So by compactness of X,

$\exists\, \lambda_1, \ldots, \lambda_n \ni \bigcup_{j=1}^{n} f^{-1}(V_{\lambda_j}) = X$. Then

$\bigcup_{j=1}^{n} V_{\lambda_j} \supseteq f(X)$ $\square$

Cor: $f: X \to Y$ continuous, A compact in X
$\Rightarrow$ $f(A)$ compact in Y.

Application: $f: [a,b] \to \mathbb{R}$ continuous $\Rightarrow$
$\exists\, x_0 \in [a,b]$ such that $f(x_0) \geq f(x)$ for every $x \in [a,b]$.

Proof: $f([a,b])$ is compact in $\mathbb{R}$, hence closed and bounded by Heine-Borel Theorem. So $f([a,b])$ has a least upper bound α and $\alpha \in f([a,b])$. Choose $x_0 \ni f(x_0) = \alpha$. $\square$

Exercises:

1. Check carefully the business about $A \subset X$ is compact in X if and only if A as a metric space (with metric d_X restricted to $A \times A$) is compact. Slogan: Compactness is an "absolute" property. ("Relative" property of A depends on what A is thought of as being part of. E.g. $A = (0,1)$ is closed as a metric space unto itself (with $d(x,y) = |x-y|$ metric) but is not closed as included in $\mathbb{R}$).
2. Prove carefully that if B is a closed bounded set in $\mathbb{R}$, it contains its own least upper bound.
3. (Do before problem 2?) Look carefully at the relationship between being closed in a metric space and containing all limits of sequences (e.g., Theorem 1.8, Chap. 1, G&G)

*4 Suppose $A \subset \mathbb{R}^n$ is an uncountable set. Show there is a point $a_0 \in A$ such that for each $\varepsilon > 0$, $B(a_0, \varepsilon) \cap A$ is uncountable. (Such a point is called a "condensation point" of A. More precisely, a point x, whether in A or not!, is called a condensation point of A if $\forall \varepsilon > 0$, $B(x, \varepsilon) \cap A$ is uncountable). So you are proving that every uncountable set contains a condensation point of itself).