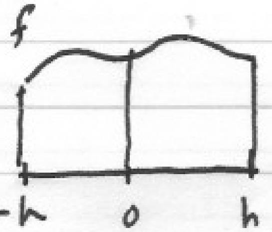
Simpson's Rule

Introductory idea:

Given a function $f: [-h, h] \to \mathbb{R}$, Simpson proposed the sum $\frac{h}{3}(f(-h) + 4 f(0) + f(h))$ as a good approximation to $\int_{-h}^{+h} f(t)\,dt$, supposing that f is continuous (so the integral definitely exists) and even better if f is differentiable a number of times. This idea is based on observing that the formula is exact for quadratic (and in fact cubic) polynomials, as one checks by calculation. Indeed the coefficients are determined with this in mind. (cf. [Gamelin-Greene,]).

Thinking of this set-up as a "double panel" (because it looks like a double panel door — temporary terminology for here!), we can imagine interpreting an integral



$\int_a^b f(t)\,dt$, f continuous, as (a limit of) a sum of "double panels", namely a_0, a_0+h, a_0+2h, then a_0+2h, a_0+3h, a_0+4h, then .. finally $b-2h, b-h, b$ when $h = (b-a)/N$, N even, so there are N single panels, $N/2$ double panels. Simpson's Approximation "Simpson's Rule" for approximating $\int_a^b f(t)\,dt$, adding double panels, is

$$h/3 \big(f(a) + 4f(a+h) + 2f(a+2h) + 4f(a+3h) + \dots + 2f(b-2h) + 4f(b-h) + f(b) \big)$$

Simpson's Rule and Associated Estimates

We estimate how well Simpson's Rule works via a sequence of steps.

Step 1: If $f : [0,h] \to \mathbb{R}$ is differentiable, if $f(0)=0$ and if for some continuous function $g : [0,h] \to \mathbb{R}$, $|f'(t)| \le g(t)$ for each $t \in [0,h]$ then $|f(h)| \le \int_0^h g(t)\,dt$.

Proof: Set $G(t) = \int_0^t g(s)\,ds$. Then G is differentiable and $(G-f)'(t) = g(t) - f'(t) \geq 0$ for all $t \in [0,h]$. By the Mean Value Theorem, and the fact that $G(0) - f(0) = 0$,

$$(G-f)(h) \geq 0 \quad \text{or} \quad f(h) \le G(h).$$

Similar reasoning applied to $G+f$ shows that

$$G(h) + f(h) \geq 0 \quad \text{or} \quad f(h) \geq -G(h). \quad \square$$

Note: Continuity of f' is not required.

Step 2: If $E : [0,h] \to \mathbb{R}$ is four times differentiable with $0 = E(0) = E'(0) = E''(0) = E^{(3)}(0)$ and if $|E^{(iv)}(t)| \le Mt$ for all $t \in [0,h]$, then

$$E(h) \le Mh^5/5!$$

Proof: Apply step 1 to $E^{(3)}(t)$ $\quad t \geq 0, t \le h$ to get $\quad E^{(3)}(t) \le Mt^2/2$.

Then apply step 1 again to get $E''(t) \le Mt^3/3!$

Continue. $\square$

Step 3: If f is a polynomial of degree ≤ 3 then for all $h > 0$

$$\int_{-h}^{+h} f(t)\,dt = \frac{h}{3}\left(f(h) + 4f(0) + f(-h)\right).$$

Proof: Compute directly. $\square$

[The following estimate is optimal: use $f(x)=x^4$ as an example] ③

Simpson's Rule Estimate:

If f is four times differentiable on $[-h,h]$ and if $|f^{(iv)}(t)| \le M$ for all $t \in [-h,h]$, then

$$\left| \int_{-h}^{h} f(t)\,dt - \frac{h}{3}\left(f(h)+4f(0)+f(-h)\right)\right|$$

$$\le \frac{2M}{180} h^5 = \frac{Mh^5}{90}$$

Proof: Since the lefthand side $=0$ for any polynomial of degree ≤ 3, it suffices to check the case where

$0 = f(0) = f'(0) = f''(0) = f^{(3)}(0)$. For this,

set $E(t) = \int_0^t f(s)\,ds - \frac{t}{3} f(t)$.

Calculation gives $E(0) = E'(0) = E''(0) = E^{(3)}(0) = 0$

and $E^{(iv)}(t) = -\frac{1}{3} f'''(t) - \frac{t}{3} f^{(iv)}(t)$

So $|E^{(iv)}(t)| \le 2Mt/3$

(because $|f'''(t)| \le tM$ by the Mean Value Theorem)

By Step 2, $|E(h)| \le \frac{2M}{3} \cdot \frac{h^5}{5!} = \frac{M}{180}$.

Similarly $\left| \int_{-h}^{0} f(s)\,ds - \frac{h}{3} f(-h) \right| \le \frac{M}{180}$

Since $f(0)=0$ in our case, the estimate follows. ∎

Cor: If an interval $[a,b]$ is divided into N (N even) panels, $N/2$ "double panels", then error in Simpson's Rule

$\le \frac{1}{180} (b-a)^5 M N^{-4}$ if $|f^{iv}| \le M$ on $[a,b]$.