Exercise set II, 31BH: Proving basic calculus results.
and some things about trigonometric series (prob 9)

1. Prove that if f is differentiable on (c, d) and $[a, b] \subset (c, d)$, then $\exists\, \lambda \in [a, b] \ni$
$$f'(\lambda) = [f(b) - f(a)]/(b - a).$$
(Suggestion: Use Rolle's Theorem, proved in class.

2. Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function. Given a partition $P : a = a_0 < a_1 < \ldots < a_N = b$ of $[a, b]$ the Riemann sums for P are

$$\sum_{i=0}^{N-1} (a_{i+1} - a_i) \sup_{x \in [a_i, a_{i+1}]} f(x) \qquad \text{upper}$$

$$\sum_{i=0}^{N-1} (a_{i+1} - a_i) \inf_{x \in [a_i, a_{i+1}]} f(x). \qquad \text{lower}$$

A function is Riemann integrable if, given $\varepsilon > 0$, $\exists\, P \ni$ upper sum$_P$ − lower sum$_P$ $< \varepsilon$.

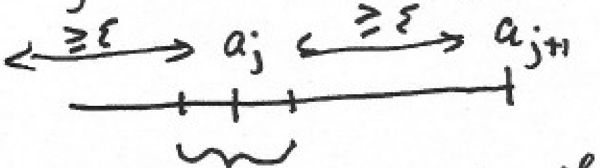
Prove: If f is Riemann integrable, then $\exists$ a unique α such that $\alpha \geq$ all lower sums (all P) and $\alpha \leq$ all upper sums (all P).
(By definition, α = the Riemann integral of f over $[a, b]$).

(Suggestion: Note that if P_1 and P_2 are partitions then $P_1 \cup P_2$ (all points from each) is a partition and
$$\text{lower sum}_{P_1} \leq \text{lower sum}_{P_1 \cup P_2} \leq \text{upper sum}_{P_1 \cup P_2} \leq \text{upper sum}_{P_1}).$$

3. Set $\text{mesh}(P) = \max\limits_{i=0 \text{ to } N} (a_{i+1} - a_i)$ where P is $a_0 \ldots a_N$.

Show that if f is bounded and Riemann integrable then given $\varepsilon > 0$, $\exists\ \delta > 0$ such that $\text{mesh}(P) < \delta \Rightarrow \text{upper sum}_P - \text{lower sum}_P < \varepsilon$.

(Suggestion: Choose P $a_0 \ldots a_N \ni \text{upper}_P - \text{lower}_P < \varepsilon/2$. Then note that if P_1 has mesh $< \frac{\text{mesh}(P)}{4}$ at most one subinterval of P_1 contains an a_j without being $\subset$ some $[a_i, a_{i+1}]$ and this subinterval contains no other a_j. Estimate $\text{upper}_{P_1} - \text{lower}_{P_1}$ by noting



that subintervals of this sort contribute no more than $N \cdot \text{mesh}(P_1) \cdot 2M$ to the difference and that the sum of other subinterval differences is $\leq$ difference for P itself).

4. Show that if $f : [a, b] \to \mathbb{R}$ is continuous, f is Riemann integrable.

5(a) Show that the uniform limit of (bounded) Riemann integrable functions is Riemann integrable and that the R. integral of the limit is the limit of the Riemann integrals.

(b) Show by example that part (a) fails to work if uniformity is omitted, even if the limit is bounded (i.e., that the limit may fail to be R. integrable)

(Suggestion for 5(b): Enumerate the rationals in $[0,1]$ as $r_1, r_2, r_3 \ldots$ and let $f_n(x)$ be defined by $f_n(x) = 0$ unless $x \in \{r_1, \ldots, r_n\}$ while $f_n(x) = 1$ if $x \in \{r_1, \ldots, r_n\}$. Then f_n is R. integrable with $\int = 0$.)

6. Prove that if f_n is a sequence of functions on $(0,1)$ such that
(a) f_n is continuous (b) f_n is differentiable with f_n' continuous on $(0,1)$ (c) f_n and f_n' converge uniformly on $(0,1)$.
Prove that $\lim f_n$ is differentiable on $(0,1)$ and
$(\lim f_n)' = \lim f_n'$.
(Suggestion: Note that $f_n(x) = \int_{1/2}^{x} f_n'(x) + f_n(\frac{1}{2})$ and use pr. 5 plus Fundamental Theorem of Calculus)

7(a) If $\sum_0^\infty a_n x^n$ converges for some $x = x_0$, prove that $\sum_0^\infty |a_n x^n|$ converges uniformly on $[-r, +r]$ for each $r < |x_0|$ and hence $\sum_0^{+\infty} a_n x^n$ converges uniformly on $[-r, +r]$.
(b) Prove under those circumstances that $\sum_{n=0}^{+\infty} n a_n x^{n-1}$ also converges uniformly on $[-r, +r]$
(c) Use prob. 6 to formulate and prove a Theorem on differentiation of power series "term by term".

8. Set $E(x) = 1 + x + x^2/2! + x^3/3! + \ldots$

(a) Show that $E(x)$ converges everywhere and $E'(x) = E(x)$. Deduce that $E(x) = e^x$ (Suggestion: Compute the derivative of $e^{-x}E(x)$).
[I am assuming here that e^x is defined as the inverse function of $\int_1^x \frac{1}{t}\,dx$ etc. not by power series!].

(b) Investigate showing directly (by multiplying out) that $E(x_1)E(x_2) = E(x_1 + x_2)$

9(a) Given a series $a_1 + a_2 + a_3 \ldots$, set $S_0 = 0$, $S_1 = a_1$, $S_2 = a_1 + a_2$, etc. Prove (analogously to integration by parts) the following "partial summation" formula

$$\sum_{n=1}^{N} a_n b_n = S_N b_N - \sum_{n=0}^{N-1} S_n (b_{n+1} - b_n)$$

(Suggestion: Induction on N)

(b) Prove that, for each fixed $x \in [-\pi, +\pi]$, there is an M_x such that

$$\left| \sum_{n=1}^{N} \sin nx \right| \le M_x \quad \text{for all } N = 1, 2, 3 \ldots$$

(Suggestion: This is obvious when $x = 0$. When $x \neq 0$, use $\sum^N \sin nx = \mathrm{Im}(e^{ix} + \ldots + e^{iNx})$ and such the geometric series explicitly).

(c) Use parts (a) and (b) to show that $\sum_1^{+\infty} \frac{\sin nx}{\ln n}$ converges (pointwise) for all $x \in [-\pi, \pi]$.

[(d) Show that the series in part (c) is not the Fourier series of any continuous function on $[-\pi, \pi]$.]
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