Weierstrass Approximation Theorem

Theorem:

If $f:[a,b]\to\mathbb{R}$ is a continuous function, then there is a sequence of polynomials $\{P_n(x)\}$ such that $\{P_n|_{[a,b]}\}$ converges uniformly to f on $[a,b]$.

We begin by a generali observation on C^∞ functions:

Lemma 1: If $h:\mathbb{R}\to\mathbb{R}$ is a continuous function with "compact support" ($\stackrel{\text{def}}{=}$ closure of $\{x: f(x)\neq 0\}$ is compact), and if $\rho:\mathbb{R}\to\mathbb{R}$ is a C^∞ function, then

$$\hat{h}(x) \stackrel{\text{def.}}{=} \int_{-\infty}^{+\infty} h(x+y)\,\rho(y)\,dy$$

is C^∞ on $\mathbb{R}$. (note: integral exists by compact support)

Proof: $$\hat{h}(x) = \int_{-\infty}^{+\infty} h(y)\,\rho(-x+y)\,dy$$

by the change of variables formula*. Since $h(y)=0$ for $|y|$ large, for each x in a neighborhood of a chosen $x_0 \in \mathbb{R}$, we can differentiate under the integral sign to get $\hat{h}'(x)$ exists and

$$\hat{h}'(x) = \int_{-\infty}^{+\infty} h(y)\,\rho'(-x+y)\,dy.$$

And similarly $\hat{h}'$ is differentiable, etc. $\square$

* $z = \underset{\text{fixed}}{x} + y$ So $dz = dy$ $\quad h(x+y)\rho(y) = h(z)\rho(y-x)$

Then replace the "dummy variable" z by y.

Lemma 2: If $p(y)$ is a polynomial and h has compact support, $h : \mathbb{R} \to \mathbb{R}$ continuous, then $\hat{h}(x)$ is a polynomial.

Proof: Let $p(y) = \sum_{n=0}^{N} a_n y^n$. Then

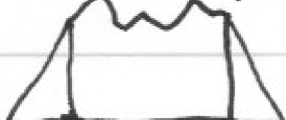
$$\hat{h}(x) = \int_{-\infty}^{+\infty} h(y) \sum_{n=0}^{N} a_n (y-x)^n \, dy$$

But $\sum_{n=0}^{N} a_n (y-x)^n = \sum_{n=0}^{N} b_n(x) y^n$ for some

polynomials $b_n(x)$. So $\hat{h}(x) = \sum_{n=0}^{N} \left(\int h(y) y^n \right) b_n(x)$ $\square$.

Proof of the Theorem: It is enough to treat the case $f : [-1,1] \to \mathbb{R}$ and $0 = f(-1) = f(+1)$ but $f \not\equiv 0$. (This is because any continuous function on a closed interval can be extended to a slightly larger closed interval as a continuous function which vanishes at the endpoints: see picture).

Extension by linear functions:



We consider f as extended to be a continuous function $f : \mathbb{R} \to \mathbb{R}$ by setting $f(t) = 0$ if $|t| > 1$.

Now define a sequence of polynomials p_n by

$$p_n(t) = c_n \left(1 - \tfrac{1}{25} x^2\right)^n \quad \text{where}$$

c_n is chosen so that

$$\int_{-5}^{+5} p_n(t) \, dt = 1.$$

(Note that $c_n \to +\infty$ as $n \to +\infty$ because $|1 - \frac{1}{25}x^2| \leq 1$ and $(1 - \frac{1}{25}x^2)^n \to 0$ uniformly, for each fixed $\delta \in (0,1)$, on $\{t : \delta \leq |t| \leq 5\}$). Also, for each fixed such δ

$\lim_{n\to+\infty} \int_{-\delta}^{+\delta} \rho_n(t)\,dt = 1$ while $\lim_{n\to+\infty} \int_{-5}^{-\delta} \rho_n(t)\,dt = 0$ and $\lim_{n\to+\infty} \int_{\delta}^{+5} \rho_n(t)\,dt = 0$. We shall verify this on the next page. Assuming this for the moment now.

Define h_n by $h_n(x) = \int_{-\infty}^{+\infty} \rho_n(t)\, f(x+t)\,dt$.
(f extended to all of $\mathbb{R}$ as already described).

By Lemma 2, $h_n(x)$ is a polynomial in x.

Note also that [since $f(x+t) = 0$ if $|x+t| \geq 1$] for $x \in [-1,1]$, it must be that

$$h_n(x) = \int_{-5}^{+5} \rho_n(t)\, f(x+t)\,dt.$$

We now show that given $\varepsilon > 0$, $\exists\, N \ni n \geq N \Rightarrow$ $|h_n(x) - f(x)| < \varepsilon$ for all $x \in [-1,1]$.

First, $\exists\, \delta > 0$ such that $|f(x_1) - f(x_2)| < \varepsilon/400$ for all $x_1, x_2 \ni |x_1 - x_2| \leq \delta$. Choose $N \ni n \geq N \Rightarrow$ $\left|1 - \int_{-\delta}^{+\delta} \rho_n(t)\,dt\right| \leq \frac{(\sup|f|)^{-1}}{100}\varepsilon$, so $\int_{-5}^{-\delta} \rho_n + \int_{+\delta}^{5} \rho_n < \frac{(\sup|f|)^{-1}}{100}\varepsilon$

Now $|h_n(x) - f(x)| = \left| \int_{-5}^{+5} \rho_n(t)\big(f(x+t) - f(x)\big)\,dt \right|$
since $\int_{-5}^{+5} \rho_n(t)\,dt = 1$.

So for all $x \in [-1,1]$ and $n \geq N$:

$$|h_n(x) - f(x)| \leq \left| \int_{-\delta}^{+\delta} \rho_n(t) \big|\big(f(x+t) - f(x)\big)\big|\,dt \right|$$

$$+ \left| \int_{-5}^{-\delta} \rho_n(t)\, \overbrace{|f(x+t) - f(x)|}^{\leq 2\sup|f|}\,dt \right|$$

$$+ \int_{+\delta}^{5} \rho_n(t)\, \overbrace{|f(x+t) - f(x)|}^{\leq 2\sup|f|}\,dt$$

$$\leq \frac{\varepsilon}{400} \int_{-\delta}^{+\delta} \rho_n(t)\,dt + 2\sup|f| \left(\int_{-5}^{-\delta} \rho_n + \int_{\delta}^{5} \rho_n \right) < \varepsilon. \quad \square$$

Each $< \frac{(\sup|f|)^{-1}}{100}\varepsilon$!

It remains to check that, for each $\delta \in (0,1)$

$$\lim_{n\to\infty}\left(\int_{-5}^{-\delta} \rho_n(x)\,dx + \int_{\delta}^{+5} \rho_n(x)\,dx\right) = 0$$

$\left(\Rightarrow \lim_{n\to+\infty} \int_{-\delta}^{+\delta} \rho_n(x)\,dx = 1\right)$. For this, it is enough to check (by symmetry of ρ_n) that

$$\lim_{n\to+\infty}\left(\int_{\delta}^{+5} \rho_n(x)\,dx \Big/ \int_{0}^{+5} \rho_n(x)\,dx\right) = 0.$$

Now $\int_0^{+5} \rho_n(x)\,dx \geq \int_0^{+\delta/2} \rho_n(x)$

$$\geq c_n \int_0^{+\delta/2}\left(1-\frac{\delta^2}{4\cdot 25}\right)^n dx \geq c_n \frac{\delta}{2}\left(1-\frac{\delta^2}{4\cdot 25}\right)^n$$

So $\ln \int_0^{+5} \rho_n(x)\,dx \geq \ln c_n + \ln\frac{\delta}{2} + n\cdot\ln\left(1-\frac{\delta^2}{4\cdot 25}\right)$

while $\ln \int_{\delta}^{+5} \rho_n(x)\,dx \leq \ln\left(c_n(5-\delta)\left(1-\frac{\delta^2}{25}\right)^n\right)$

$$= \ln c_n + \ln(5-\delta) + n\ln\left(1-\frac{\delta^2}{25}\right)$$

Now $(\ln(1-x))'|_{x=a} = -1/(1-x)$ if $0<x<1$. So $\ln(1-x)$ is decreasing strictly on $0<x<1$. In particular, $\ln\left(1-\frac{\delta^2}{25}\right) < \ln\left(1-\delta^2/4\cdot 25\right)$.

It follows that

$$\lim_{n\to+\infty}\left(\ln\int_{\delta}^{+5} \rho_n(x)\,dx - \ln\int_0^{\delta/2} \rho_n(x)\,dx\right) = -\infty$$

(as $n\to+\infty$) so $\lim_{n\to+\infty}\int_{\delta}^{+5} \rho_n(x)\,dx \Big/ \int_0^{\delta/2} \rho_n(x)\,dx = 0$

Hence $\lim_{n\to+\infty}\left(\int_{\delta}^{+5} \rho_n(x)\,dx \Big/ \int_0^{+5} \rho_n(x)\,dx\right) = 0$. $\square$