

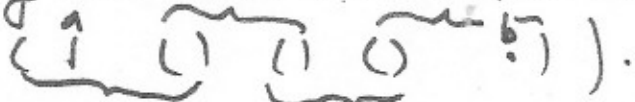
# A Baire Category Theorem Example: A Second Category Set of Measure 0.

Given  $\varepsilon > 0$ ,  $\exists$  an open cover of the <sup>(set of)</sup> rational numbers  $\mathbb{Q} \subset \mathbb{R}$  by (open) intervals  $\{I_j\}$  with  $\sum_{j=1}^{+\infty} l(I_j) < \varepsilon$ . [ $l(I) =$  length of the interval so that  $l((a,b)) = b-a$ .]

Proof: Let  $r_1, r_2, r_3 \dots$  be an enumeration of  $\mathbb{Q}$  and choose  $I_j$  with  $r_j \in I_j$  and  $l(I_j) < \varepsilon/2^{j+1}$ .

If we set  $U = \bigcup I_j$ , then the components of  $U$  are again a set of open intervals, necessarily countable (since they are disjoint) say  $J_1, J_2, \dots$  and  $\sum_{j=1}^{+\infty} l(J_j) < \varepsilon$  again.

(To prove this, note that if an open interval  $J$  is a union of open intervals  $K_1, K_2, K_3, \dots$  then  $l(J) \leq \sum_{j=1}^{+\infty} l(K_j)$ . Exercise: Prove this by choosing  $[a,b] \subset J$  arbitrarily, and showing  $l([a,b]) = b-a \leq \sum l(K_j)$  by choosing a finite collection of  $K$ 's that cover  $[a,b]$  and noting that some subcollection of that looks like this:



Let  $C_\varepsilon = \mathbb{R} \setminus \bigcup_{j=1}^{+\infty} J_j^c$ , where, for given  $\varepsilon > 0$ , the  $J_j^c$  are as above. The set  $C_\varepsilon$  is closed since the  $J_j^c$  are open sets (open intervals).  $C_\varepsilon$  is nowhere dense, too, since it is closed but contains no elements of  $\mathbb{Q}$ , hence has empty interior.

(over)

Now consider  $C = \bigcup_{n=1}^{+\infty} C_{1/n}$ .

2

The "measure" of  $\mathbb{R}-C$  is 0 in the sense that, for any  $\varepsilon > 0$ , there is a cover of  $\mathbb{R}-C$  by open intervals with  $\sum \text{lengths} < \varepsilon$ . This is clear since  $\mathbb{R}-C \subset \mathbb{R}-C_{1/n} = \bigcup_{j=1}^{+\infty} J_{\frac{1}{n}}^j$  with

$$\sum_j \text{length}(J_{\frac{1}{n}}^j) < \frac{1}{n} \quad (\text{by definition}).$$

The set  $\mathbb{R}-C = \bigcap_{n=1}^{+\infty} (\mathbb{R}-C_{1/n})$  contains the

set  $Q$  since each  $\mathbb{R}-C_{1/n} = \bigcup_{j=1}^{+\infty} J_{\frac{1}{n}}^j$  contains

$Q$  by definition of the  $J_{\frac{1}{n}}^j$ 's.

Intuitively, one might suppose that  $\mathbb{R}-C$  would equal  $Q$  since the total length of the  $J_{\frac{1}{n}}^j$  intervals goes to 0 as  $n \rightarrow +\infty$ . But actually  $\mathbb{R}-C$  is a large set in the "category" sense: it is "second category", that is it cannot be expressed as a countable union of nowhere dense sets. In particular, it is uncountable.

To see this, note (as before) that  $C_{1/n}$ ,  $n=1, 2, \dots$  are each closed and nowhere dense, since it contains no elements of  $Q$  (and is closed). So if  $\mathbb{R}-C$  were a countable union  $\bigcup_{j=1}^{+\infty} D_j$  of nowhere dense sets, then so would  $\mathbb{R}$  be:

$\mathbb{R}$  would equal  $(\bigcup_{j=1}^{+\infty} D_j) \cup (\bigcup_{n=1}^{+\infty} C_{1/n})$ , contradicting the Baire Category Theorem.