A Baire Category Theorem Example: A Second Category Set of Measure 0.

Given $\varepsilon > 0$, $\exists$ an open cover of the (set of) rational numbers $\mathbb{Q} \subset \mathbb{R}$ by (open) intervals $\{I_j\}$ with $\sum_{n=1}^{+\infty} \ell(I_j) < \varepsilon$. [$\ell(I)$ = length of the interval so that $\ell((a,b)) = b-a$.]

Proof: Let $r_1, r_2, r_3 \ldots$ be an enumeration of $\mathbb{Q}$ and choose I_j with $r_j \in I_j$ and $\ell(I_j) < \varepsilon/2^{n+1}$.

If we set $U = \bigcup I_j$, then the components of U are again a set of open intervals, necessarily countable (since they are disjoint) say $J_1, J_2, \ldots$ and $\sum_{j=1}^{+\infty} \ell(J_j) < \varepsilon$ again.

(To prove this, note that is an open interval J is a union of open intervals $K_1, K_2, K_3, \ldots$ then $\ell(J) \leq \sum_{j=1}^{+\infty} \ell(K_j)$. Exercise: Prove this by choosing $[a,b] \subset J$ arbitrarily, and showing $\ell([a,b]) = b-a \leq \sum \ell(K_j)$ by choosing a finite collection of K's that cover $[a,b]$ and noting that some subcollection of that looks like this:

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Let $C_\varepsilon = \mathbb{R} \setminus \bigcup_{j=1}^{+\infty} J^\varepsilon$, where, for given $\varepsilon > 0$, the J^ε are as above. The set C_ε is closed since the J are open sets (open intervals). C_ε is nowhere dense, too, since it is closed but contains no elements of $\mathbb{Q}$, hence has empty interior.

(over)

Now consider $C = \bigcup_{n=1}^{+\infty} C_{1/n}$.

The "measure" of $\mathbb{R}-C$ is 0 in the sense that, for any $\varepsilon>0$, there is a cover of $\mathbb{R}-C$ by open intervals with Σ lengths $< \varepsilon$. This is clear since $\mathbb{R}-C \subset \mathbb{R}-C_{1/n} = \bigcup_{j=+1}^{+\infty} J_j^{1/n}$ with

$\sum_j \text{length}(J_j^{1/n}) < \frac{1}{n}$ (by definition).

The set $\mathbb{R}-C = \bigcap_{n=1}^{+\infty} (\mathbb{R}-C_{1/n})$ contains the

set Q since each $\mathbb{R}-C_{1/n} = \bigcup_{l=1}^{+\infty} J_j^{1/n}$ contains

Q by definition of the $J_j^{1/n}$'s.

Intuitively, one might suppose that $\mathbb{R}-C$ would equal Q since the total lengths of the $J_l^{1/n}$ intervals goes to 0 as $n \to +\infty$. But actually $\mathbb{R}-C$ is a large set in the "category" sense: it is "second category", that is it cannot be expressed as a countable union of nowhere dense sets. In particular, it is uncountable.

To see this, note (as before) that $C_{1/n}$ $n=1,2,\ldots$ are each closed and nowhere dense, since it contains no elements of $\mathbb{Q}$ (and is closed). So if $\mathbb{R}-C$ were a countable union $\bigcup_{j=1}^{+\infty} D_j$ of nowhere dense sets, then so would $\mathbb{R}$ be:

$\mathbb{R}$ would equal $(\bigcup_{j=1}^{+\infty} D_j) \cup (\bigcup_{n=1}^{+\infty} C_{1/n})$, contradicting the Baire Category Theorem