## After the interlude:
## Details of Minimization of Total Curvature and Hermitian Yang Mills

In the interlude on Hermitian Yang Mills ideas, it was remarked that an Hermitian Yang Mills connection was obtained as the result of minimizing a curvature (squared) integral, an analogue of the constant curvature metrics on a compact Riemann surfaces being those that minimized the integral of squared Gauss curvature over a conformal class. In this addendum, we make this more explicit in the Hermitian Yang Mills case.

Following our previous notation, we denote the curvature of the type (1,0) Hermitian connection on a holomorphic vector bundle E with Hermitian metric h on the fibres, Kähler metric g on the base manifold M, by $F^{\beta}_{\alpha j\bar{k}}$ , Greek indices in the bundle, Roman indices in the base manifold. Recall that we considered then the trace of F with respect to the j, $\bar{k}$ indices, the result being a function trF on M with values in the endomorphisms of the fibres of the bundle. For notational compactness, denote this trace by $K^{\beta}{}_{\alpha}$
Now we consider an additional contraction, namely set

$$\|K\|^2 = \sum K^{\beta}_{\alpha} K^{\alpha}_{\beta}$$

Then consider the integral (called J, following the notation and treatment in Kobayashi's Differential Geometry of Complex Vector Bundles. Note, however, that the roles of Greek and Roman indices are interchanged here compared to Kobayashi's conventions!):

$$J(h) = \frac{1}{2}\int \|K\|^2 \omega^n$$

where ω is the Kähler form.

This J is a functional on the space of Hermitian metrics h on the vector bundle. The basic result here is this:

There is a topological constant (formula to be given later), determined by $c_1$( E) and by the cohomology class of ω , which is a lower bound for J(h). This lower bound has the property that it is attained by a given Hermitian metric h if and only if h is an Einstein metric in the sense that tr F is a constant multiple of the identity endomorphism of the vector bundle.

To see how this works, let σ be the scalar curvature, namely $\sum K^{\alpha}_{\alpha}$ .

Determine c by $rc\int \omega^n = \int \sigma\omega^n$ , where r is the rank of the vector bundle.

Then $0 \le \|K - cI\|^2 = \|K\|^2 + rc^2 - 2c\sigma$ .
Integrating this gives

$$\int \|K\|^2 \omega^n \geq rc^2 \int \omega^n = \frac{\left(\int \sigma\omega^n\right)^2}{\left(r\int \omega^n\right)}.$$

On the other hand,

$$\int \sigma\omega^n = 2n\pi \int c_1(E) \wedge \omega^{n-1}.$$

Thus $J(h) = \frac{1}{2}\int \|K\|^2 \omega^n \geq \frac{2\left(n\pi \int c_1(E)\wedge\omega^{n-1}\right)^2}{\int r\omega^n}$

where equality holds if and only if $\|K - cId\|^2 = 0$ everywhere.

More generally, one finds that the critical points of J are the metrics for which the scalar curvature K is parallel with respect to the Hermitian connection, analogously to the Yang Mills harmonic curvature idea (cf. Kobayashi, op. cit. for details).

We can see easily that for a complex vector bundle over a compact (real) 4-manifold, a connection with curvature 2-from that is self-dual or anti-self dual (the difference is only a matter of orientation reversal on M ) is a minimum of the curvature squared integral already indicated . This explains the on-going fascination with self-duality and anti-self duality.
One sees this as follows: Write $F = R^{\nabla}$ for convenience. Then the norm squared of F $\|F\|^2$ is $\int_M tr(F \wedge {*F})$ where tr is trace over manifold indices and *F is the dual two-form of F. Note that the second Chern class of F , interpreted as a number by evaluating on the fundamental cycle of M, or equivalently the integral of the second Chern form over M, is given by

$c_2(F) = (\text{constant}) \int_M tr(F \wedge F)$
(With the constant $1/8\ \pi^2$ , this is an integer).

Now write $F = F^+ + F^-$ , where $F^+$ is self dual and $F^-$ anti-self dual . This the eigenspace decomposition under * , which has square the identity. Or alternatively and equivalently, one can set $F^+ = (1/2)(F + F^*)$, $F^- = (1/2)(F - F^*)$ ). Since * acts self-adjointly on the 2-froms at a point with respect to the usual inner product, the two eigenspaces of * are orthogonal with respect to the pointwise inner product and hence with respect to the integrated inner product. (Note that the inner product is given at each point by wedging together and then dividing by the volume form).

Then the second Chern class formula just given yields, upon multiplying out, that $\|F^+\|^2 - \|F^-\|^2$ is a topological number(second Chern class of the bundle up to constants). The "cross terms" vanish by virtue of the orthogonality. This shows that from simple algebra that $\|F\|^2 = \|F^+\|^2 + \|F^-\|^2$ is minimal if and only if one of $F^+$ or $F^-$ is 0, so that F is self dual or anti self dual.

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