The Bunkbed Conjecture is False

Based on joint work with Aleksandr Zimin and Igor Pak

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LA probability forum, October 2024

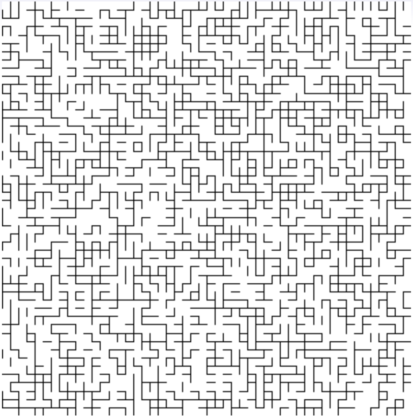

Figure: Percolation on $\mathbb{Z}^2$ with $p = 0.51$

Consider a graph $G = (V, E)$. (Bernoulli edge) percolation is a random graph obtained from the graph G, where each edge $e \in E$ is independently open (or survives) with probability $p_e \in (0, 1)$. This gives a spanning subgraph $H \subseteq G$ with probability

$$\prod_{e \in H} p_e \prod_{e \notin H} (1 - p_e).$$

A *cluster* is a set of vertices connected via open edges.

Theorem (Harris 1960 and Kesten 1980)

For $p \leq 0.5$, with probability 1 there is no infinite cluster in an edge percolation on $\mathbb{Z}^2$. For $p > 0.5$, with probability 1 there is such a cluster.

We call an event *closed upwards* if opening an extra edge never turns an event from true to false.

Theorem (Harris–Kleitman inequality)

Let $\mathbf{P}$ *be given by a Bernoulli percolation, and* $\mathcal{A}$ *and* $\mathcal{B}$ *are events closed upwards. Then*

$$\mathbf{P}(\mathcal{A} \cap \mathcal{B}) \geq \mathbf{P}(\mathcal{A})\mathbf{P}(\mathcal{B}).$$

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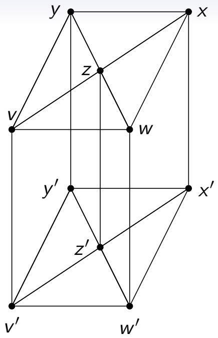

Conjecture (Bunkbed conjecture)

Probabilities of two copies of the same edge are equal. Probabilities of posts are arbitrary. Then

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Remark

The conjecture follows from its partial case where all posts have probability 0 *or* 1*.*

Proof.

Indeed, $\mathbf{P}_{G_b}(xy)$ and $\mathbf{P}_{G_b}(xy')$ are polynomials in p_e. If e is a post, $\mathbf{P}_{G_b}(xy) - \mathbf{P}_{G_b}(xy')$ is linear in p_e, so we can move it to 0 or 1, depending on the sign of the coefficient in it. □

We call vertices with posts *transversal*.

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Proposition

If there is only one transversal vertex v, the bunkbed conjecture is true.

Proof.

We can rewrite probabilities on G_b in terms of probabilities on G. So,

$$\mathbf{P}_{G_b}(xy) = \mathbf{P}_G(xy)$$

and

$$\mathbf{P}_{G_b}(xy') = \mathbf{P}_G(xv)\mathbf{P}_G(yv) \leq \mathbf{P}_G(xyv) \leq \mathbf{P}_G(xy).$$

□

Theorem (Linusson, 2008)

If x or y is transversal, then the bunkbed conjecture turns into equality. If any path from x to y in G passes through a transversal vertex, the bunkbed conjecture turns into equality.

Proof.

Look for the open component of y in $G \setminus T$ and switch the edges between the levels. □

In the *alternative bunkbed percolation*, each edge e in G is either deleted while the corresponding hyperedge e' in G' is retained with probability $\frac{1}{2}$, or vice versa: edge e is retained and e' is deleted.

Theorem (Linusson, 2008)

If BBC fails on some graph G for some probabilities p_e, then alternative BBC fails on some minor H of G.

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Despite intuitiveness, proving this conjecture is not straightforward and is an active area of research in percolation theory.[6] It was proved for specific types of graphs, such as wheels,[7] complete graphs,[8] complete bipartite graphs and graphs with a local symmetry.[9] It was also proved in the limit $p \to 1$ for any graph[10][11].

Figure: Known cases

We will use the notation like $\mathbf{P}(ad|b|c)$ to denote the probability, in this case, that vertices a and d belong to the same cluster, which is different from the clusters of b and c.
There are 5 elementary configurations on 3 vertices: $\mathbf{P}(abc)$, $\mathbf{P}(ab|c)$, $\mathbf{P}(a|b|c)$, $\mathbf{P}(a|bc)$, $\mathbf{P}(ac|b)$.

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Theorem (van den Berg–Haggström–Kahn)

$$\mathbf{P}(ab|cd)\mathbf{P}(a|d) \leq \mathbf{P}(ab|d)\mathbf{P}(a|cd)$$

Proof sketch.

We run a Markov chain process with a stable distribution being the uniform measure on $a|d$. Then we apply the Harris–Kleitman inequality to the events ab and cd which turn out to be closed upwards and downwards in the new coordinates. □

Proposition (Andrew Lohr)

If there are only two transversal vertices v, w, the bunkbed conjecture is true.

Proof (G., Zimin).

Add together some Harris–Kleitman and van den Berg–Haggström–Kahn inequalities.

$$\begin{aligned}\mathbf{P}_{G_b}(xy) - \mathbf{P}_{G_b}(xy') = \\ & \mathbf{P}(xy|v|w) + \mathbf{P}(xy|vw) \\ & + \mathbf{P}((xv \cup xw) \cap (yv \cup yw)) - \mathbf{P}(xv \cup xw)\mathbf{P}(yv \cup yw) \\ & + \mathbf{P}(xv|w)\mathbf{P}(yw|v) - \mathbf{P}(xv|yw)\mathbf{P}(v|w) \\ & + \mathbf{P}(xw|v)\mathbf{P}(w|yv) - \mathbf{P}(xw|yv)\mathbf{P}(v|w)\end{aligned}$$

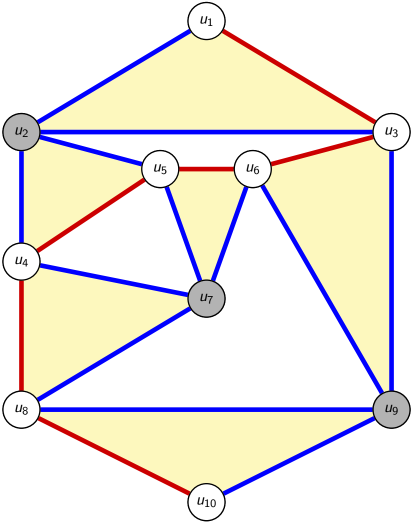
□

Question

What about 3 transversal vertices?

Theorem (Hollom, 2024)

For the following 3-regular hypergraph with 3 transversal vertices the alternative hypergraph bunkbed conjecture is false.

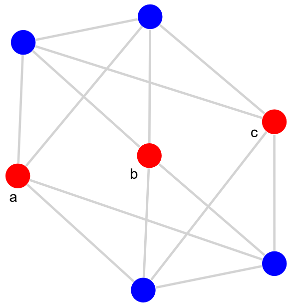


Question

Can it be proved that if $\mathbf{P}(ac|b) \approx \mathbf{P}(ab|c) \approx \mathbf{P}(a|bc) \approx 0$*, than* $\mathbf{P}(abc)$ *or* $\mathbf{P}(a|b|c)$ *is also* ≈ 0*?*

In particular, is $\min\big(\mathbf{P}(abc), \mathbf{P}(a|b|c)\big) < \frac{1}{2} - \varepsilon$?

The biggest minimum we can achieve is 0.29 on the graph in the Figure below. Each red-blue edge has probability 0.32537 and both blue-blue edges have probability 0.19231. This way we get $\mathbf{P}(abc) \approx \mathbf{P}(a|b|c) \approx 0.29065$.



Example (Decision tree techniques example)

Suppose I take cards from a shuffled deck one by one, until I get a spade. Then I take one more card. What are the chances that it is also a spade?

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Suppose I take cards from a shuffled deck one by one, until I get a spade. Then I take one more card. What are the chances that it is also a spade?

Solution: It is $\frac{1}{4}$, since we can invert the deck after the first spade without affecting the probability distribution. Under this transformation, the needed probability turns into a probability that the last card in the deck is a spade.

Definition

For two configurations $C_1, C_2 \in \Omega = 2^{[E]}$ and a set $S \subseteq E$ we denote by $C_1 \rightarrow_S C_2$ the configuration which coincides with C_1 on S and C_2 on its complement $\overline{S}$.

Lemma

Consider two independent Bernoulli percolations C_1 and C_2 having the same distribution μ on the same graph G. Let a decision tree T select each edge and reveal it in both C_1 and C_2. Furthermore, allow on each step, before revealing, decide if this edge will go to the set S (thus dependent on C_1 and C_2) or to its complement $\overline{S}$. Then $C_1 \rightarrow_S C_2$ is independent of $C_2 \rightarrow_S C_1 = C_1 \rightarrow_{\bar{S}} C_2$ and both of them are distributed as μ.

The key observation will be that when $C_1 \in a|b|c$ and $C_1 \rightarrow_{S_3} C_2 \in ab \cup ac$, one has $C_1 \rightarrow_{S_1} C_2 \in ab$ or $C_1 \rightarrow_{S_2} C_2 \in ac$.

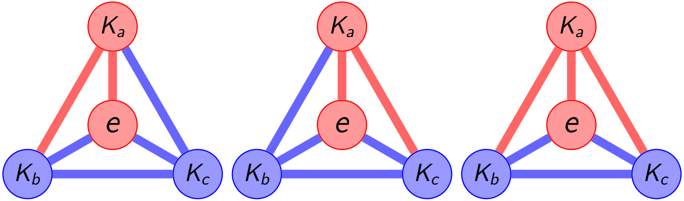


Figure: S_1, S_2 and S_3 for the case $C_1 \in a|b|c$. Regions surrounding a, b, c depict K_a, K_b and K_c. Respective sets are in blue and their complements are in red.

Theorem (G., Zimin)

$$\mathbf{P}(a|b \cap a|c)\mathbf{P}(ab \cup ac) \leq \mathbf{P}(ab|c) + \mathbf{P}(ac|b) + \mathbf{P}(a|bc).$$

Corollary

$\mathbf{P}(abc)$ *and* $\mathbf{P}(a|b|c)$ *can not be simultaneously greater than* 0.37586.

Theorem (G.)

$$\mathbf{P}(abc)^2 \le 8\mathbf{P}(ab)\mathbf{P}(ac)\mathbf{P}(bc).$$

Remark

On $\mathbb{Z}^2$ in a critical mode it is conjectured by Delfino and Viti that

$$\mathbf{P}(abc)^2 \to 1.044... \cdot \mathbf{P}(ab)\mathbf{P}(ac)\mathbf{P}(bc)$$

as a, b and c tend away from each other. Recently the proof was announced by Morris Ang, Gefei Cai, Xin Sun and Baojun Wu.

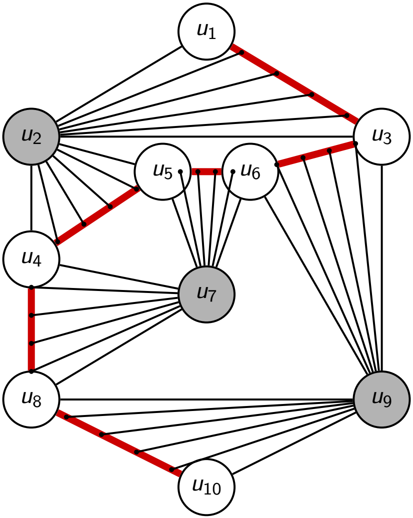
Theorem (G., Pak, Zimin)

There is a connected planar graph $G = (V, E)$ with $|V| = 7222$ vertices and $|E| = 14442$ edges, a subset $T \subset V$ with three transversal vertices, and vertices $u, v \in V$, s.t.

$$\mathbf{P}^{\mathrm{bb}}_{\frac{1}{2}}[u \leftrightarrow v] < \mathbf{P}^{\mathrm{bb}}_{\frac{1}{2}}[u \leftrightarrow v'].$$

In particular, the bunkbed conjecture is false.

The counterexample



Equality case of the Harris–Kleitman inequality:

$$\mathbf{P}(ab)\mathbf{P}(ac) = \mathbf{P}(abc)$$

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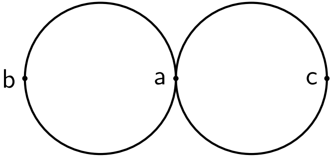


$$\Leftrightarrow$$

$$\mathbf{P}(a|bc) = 0$$

Theorem (G.)

$$\mathbf{P}(abc)\mathbf{P}(a|b|c) \geq \mathbf{P}(ab|c)\mathbf{P}(ac|b) + \mathbf{P}(ab|c)\mathbf{P}(a|bc) + \mathbf{P}(ac|b)\mathbf{P}(a|bc)$$

Corollary

The top-bottom direction is stable. If $\mathbf{P}(ab)\mathbf{P}(ac) \approx \mathbf{P}(abc)$, *then* $\mathbf{P}(a|bc) \approx 0$.

Conjecture

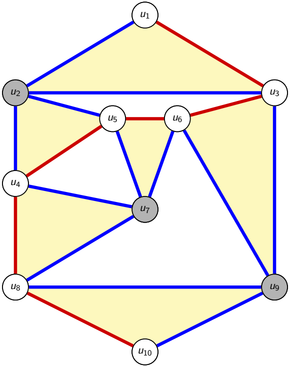
If $\mathbf{P}(a|bc) < \varepsilon$, *then* $\mathbf{P}(abc) - \mathbf{P}(ab)\mathbf{P}(ac) = O(\varepsilon \log\left(\frac{1}{\varepsilon}\right))$.

Lemma (G., Pak, Zimin)

Let H be Hollom's hypergraph with $T = \{u_2, u_7, u_9\}$. Consider the WZ hypergraph percolation where each hyperedge is replaced by a graph G with vertices a, b and c. Assume the connection probabilities satisfy

$$400\mathbf{P}(a|bc) \le \mathbf{P}(abc)\mathbf{P}(a|b|c) - \mathbf{P}(ab|c)\mathbf{P}(ac|b).$$

Then we have $\mathbf{P}_{G_b}(u_1 u_{10}) < \mathbf{P}_{G_b}(u_1 u'_{10})$.



Lemma (G., Pak, Zimin)

Let $n \geq 3$ and $0 < p < 1$. Consider a weighted graph G_n on $(n+1)$ vertices given in Figure 5. Denote $b := v_1$ and $c := v_n$. Then $\mathbf{P}(ab|c) = \mathbf{P}(ac|b)$ *and*

$$\mathbf{P}(abc)\,\mathbf{P}(a|b|c) - \mathbf{P}(ab|c)\,\mathbf{P}(ac|b) > \left(n\,\tfrac{1-p}{1+p} - 1\right)\mathbf{P}(a|bc)\,,$$

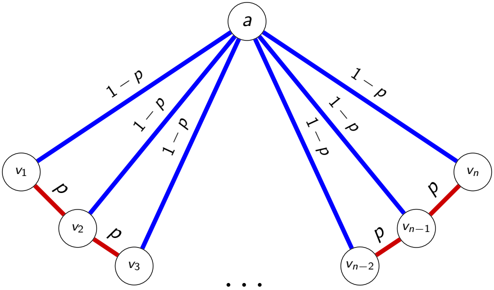
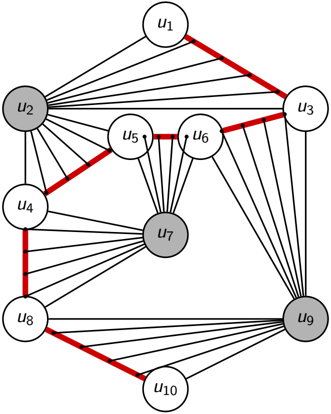


Figure: Graph G_n with $n+1$ vertices.

In the notation of Lemma, let $p = \frac{1}{2}$ and let $n := 3 \cdot 401 + 1 = 1204$. The resulting graph G_n is planar, has 1205 vertices and 2407 edges. Take Hollom's hypergraph H and substitute for each 3-hyperedge with a graph G_n from Lemma, placing it so a is a transversal vertex while $b = v_1$ and $c = v_n$ are the other two vertices. The resulting graph is still planar, has $10 + 6 \cdot 1202 = 7222$ vertices and $6 \cdot 2407 = 14442$ edges.



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A computer-assisted computation shows that one can use G_n with $p = \frac{1}{2}$ and $n = 14$, giving a relatively small graph on 82 vertices. However, even in this case, the difference of the probabilities in the BBC is on the order 10^{-47}.

For $p = 0.03$ one can take $n = 5$. In this case the alternative BBC is also violated.

In the notation of the Bunkbed Conjecture, one can ask if a version of the BBC holds for uniform $T \subseteq V$. This is equivalent to $\frac{1}{2}$-percolation on the product graph $G \times K_2$. To distinguish from BBC, we call this *Complete BBC*. Turns out the proof of Theorem extends to the proof of Complete BBC, since all nontransversal vertices helpfully lie on a single red path, but a counterexample is a little larger due to the added gadgets at transversal vertices, similar to [Hollom, 2024].

The difference of probabilities is even smaller in this case, and is on the order of 10^{-6500}.

Thank you for your attention!

geminger 1 hour ago | prev [–]

So it's been debunked.

reply

andrewflnr 51 minutes ago | parent [–]

Now it's just two beds.

reply

Conjecture (Kozma–Nitzan, 2024)

In a percolation on a graph having vertices a, b, c_1, ..., c_n one has

$$\mathbf{P}(ab) \geq \mathbf{P}(ac_1 \cup ac_2 \cup \cdots \cup ac_n) \min_i \mathbf{P}(c_i b)$$

Theorem (Kozma–Nitzan)

Conjecture above implies that there is no infinite cluster in percolation on $\mathbb{Z}^d$ at a critical probability. Interesting cases are $d = 3, \ldots, 9$.

Proposition

$$\mathbf{P}(ab) \geq \mathbf{P}(ac_1 \cup ac_2)\Big(\frac{\mathbf{P}(ac_1|c_2)}{\mathbf{P}(ac_1|c_2) + \mathbf{P}(ac_2|c_1)}\mathbf{P}(c_1 b) + \frac{\mathbf{P}(ac_2|c_1)}{\mathbf{P}(ac_1|c_2) + \mathbf{P}(ac_2|c_1)}\mathbf{P}(c_2 b)\Big)$$

Question

What about 3 c_i*'s?*