Inequalities in Graph Percolation

Based on joint work with Aleksandr Zimin and Igor Pak

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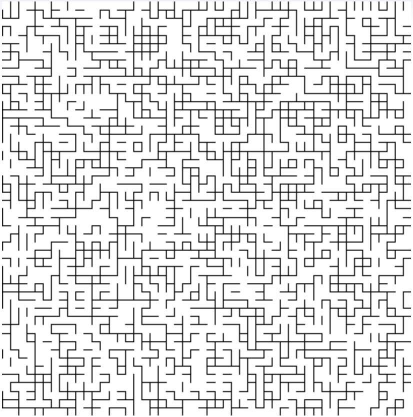

Figure: Percolation on $\mathbb{Z}^2$ with $p = 0.51$

Consider a graph $G = (V, E)$. (Bernoulli edge) percolation is a random graph obtained from the graph G, where each edge $e \in E$ is independently open (or survives) with probability $p_e \in (0, 1)$. This gives a spanning subgraph $H \subseteq G$ with probability

$$\prod_{e \in H} p_e \prod_{e \notin H} (1 - p_e).$$

A *cluster* is a set of vertices connected via open edges.

Theorem (Harris, 1960)

For $p \leq 0.5$, with probability 1 there is no infinite cluster in an edge percolation on $\mathbb{Z}^2$.

Theorem (Kesten, 1980)

For $p > 0.5$, with probability 1 there is such a cluster.

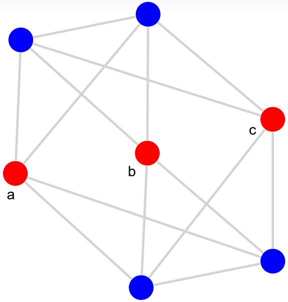

Figure: If we generate a subgraph randomly, what are the chances that a belongs to the same cluster as b, but not c?

We will use the notation like $\mathbf{P}(ad|b|c)$ meaning the probability, in this case, that vertices a and d belong to the same cluster, which is different from the clusters of b and c.
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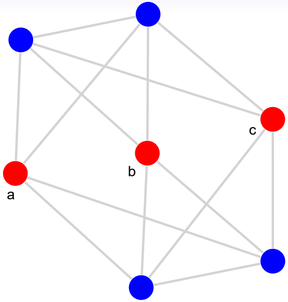

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$\mathbf{P}(abc)$, $\mathbf{P}(ab|c)$, $\mathbf{P}(a|b|c)$, $\mathbf{P}(a|bc)$, $\mathbf{P}(ac|b)$.

Question

What are the possible values of these probabilities for all possible graph percolations?

The restrictions come in forms of inequalities, the most prominent of which is the Harris–Kleitman inequality (sometimes called the FKG inequality).

Denote by H_n the n-dimensional discrete hypercube. We say that measure μ on H_n is a *product measure* if there exist probability measures $\mu_1, \mu_2, \ldots, \mu_n$ on $\{0,1\}$, such that μ coincides with the direct product $\mu_1 \times \mu_2 \times \cdots \times \mu_n$. So, a percolation gives us a measure on H_n, where $n = |E|$.

Theorem (Harris–Kleitman inequality)

Let μ be a probability product measure on H_n, and $\mathcal{A}$ and $\mathcal{B}$ are events closed upwards. Then

$$\mathbf{P}(\mathcal{A} \cap \mathcal{B}) \geq \mathbf{P}(\mathcal{A})\mathbf{P}(\mathcal{B}).$$

Corollary

If $\mathcal{A}$ is closed upwards and $\mathcal{B}$ is closed downwards,

$$\mathbf{P}(\mathcal{A} \cap \mathcal{B}) \leq \mathbf{P}(\mathcal{A})\mathbf{P}(\mathcal{B}).$$

Denote by H_n the n-dimensional discrete hypercube. We say that measure μ on H_n is a *product measure* if there exist probability measures μ_1, μ_2, ..., μ_n on $\{0, 1\}$, such that μ coincides with the direct product $\mu_1 \times \mu_2 \times \cdots \times \mu_n$. So, a percolation gives us a measure on H_n, where $n = |E|$.

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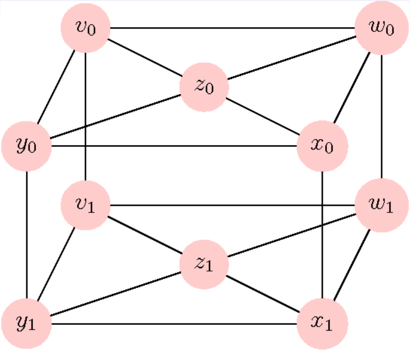
or

$$\mathbf{P}(abc)\mathbf{P}(a|b|c) \geq \mathbf{P}(ab|c)\mathbf{P}(a|bc) + \mathbf{P}(ab|c)\mathbf{P}(ac|b)$$

Theorem (G.)

$$\begin{aligned}\mathbf{P}(abc)\mathbf{P}(a|b|c) \geq \mathbf{P}(ab|c)\mathbf{P}(a|bc) + \mathbf{P}(ab|c)\mathbf{P}(ac|b)& \\ + \mathbf{P}(ac|b)\mathbf{P}(a|bc)&\end{aligned}$$



Conjecture (Bunkbed conjecture)

Probabilities of two copies of the same edge are equal. Probabilities of posts are arbitrary. Then

$$\mathbf{P}(x_0 y_0) \geq \mathbf{P}(x_0 y_1).$$

Remark

The conjecture follows from its partial case where all posts have probability 0 *or* 1*.*

Proof.

Indeed, $\mathbf{P}_{G_b}(x_0 y_0)$ and $\mathbf{P}_{G_b}(x_0 y_1)$ are polynomials in p_e. If e is a post, $\mathbf{P}_{G_b}(x_0 y_0) - \mathbf{P}_{G_b}(x_0 y_1)$ is linear in p_e, so we can move it to 0 or 1, depending on the sign of the coefficient in it. □

We call vertices with posts *transversal*.

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Proposition

If there is only one transversal vertex v, the bunkbed conjecture is true.

Proof.

We can rewrite probabilities on G_b in terms of probabilities on G. So,

$$\mathbf{P}_{G_b}(x_0 y_0) = \mathbf{P}_G(xy)$$

and

$$\mathbf{P}_{G_b}(x_0 y_1) = \mathbf{P}_G(xv)\mathbf{P}_G(yv) \leq \mathbf{P}_G(xyv) \leq \mathbf{P}_G(x_0 y_0).$$

$\square$

Theorem (van den Berg–Haggström–Kahn)

$$\mathbf{P}(ab|cd)\mathbf{P}(a|d) \leq \mathbf{P}(ab|d)\mathbf{P}(a|cd)$$

Proof sketch.

We run a Markov chain process with a stable distribution being the uniform measure on $a|d$. Then we apply the Harris–Kleitman inequality to the events ab and cd which turn out to be closed upwards and downwards in the new coordinates. □

Proposition (Andrew Lohr)

If there are only two transversal vertices v, w, the bunkbed conjecture is true.

Proof (G., Zimin).

Add together some Harris–Kleitman and van den Berg–Haggström–Kahn inequalities.

$$\begin{aligned}\mathbf{P}_{G_b}(xy) - \mathbf{P}_{G_b}(xy') = \\ & \mathbf{P}(xy|v|w) + \mathbf{P}(xy|vw) \\ & + \mathbf{P}((xv \cup xw) \cap (yv \cup yw)) - \mathbf{P}(xv \cup xw)\mathbf{P}(yv \cup yw) \\ & + \mathbf{P}(xv|w)\mathbf{P}(yw|v) - \mathbf{P}(xv|yw)\mathbf{P}(v|w) \\ & + \mathbf{P}(xw|v)\mathbf{P}(w|yv) - \mathbf{P}(xw|yv)\mathbf{P}(v|w)\end{aligned}$$

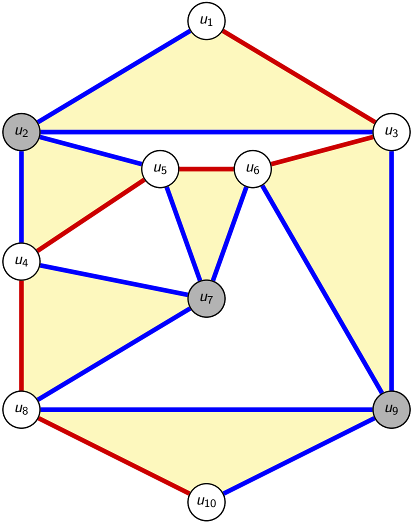
$\square$

Question

What about 3 transversal vertices?

Theorem (Hollom, 2024)

For the following 3-regular hypergraph with 3 transversal vertices the alternative hypergraph bunkbed conjecture is false.



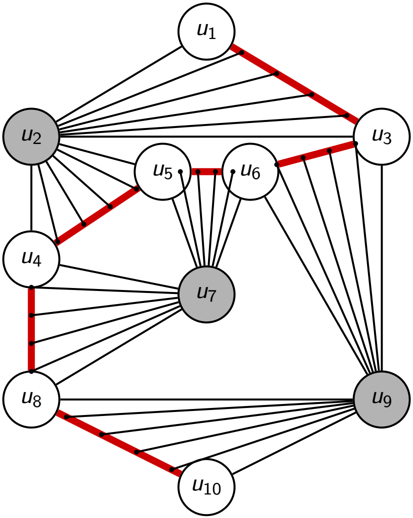
Theorem (G., Pak, Zimin)

There is a connected planar graph $G = (V, E)$ *with* $|V| = 7222$ *vertices and* $|E| = 14442$ *edges, a subset* $T \subset V$ *with three transversal vertices, and vertices* $u, v \in V$*, s.t.*

$$\mathbf{P}^{\mathrm{bb}}_{\frac{1}{2}}[u \leftrightarrow v] \,<\, \mathbf{P}^{\mathrm{bb}}_{\frac{1}{2}}[u \leftrightarrow v'].$$

In particular, the bunkbed conjecture is false.

The counterexample



Equality case of the Harris–Kleitman inequality:

$$\mathbf{P}(ab)\mathbf{P}(ac) = \mathbf{P}(abc)$$

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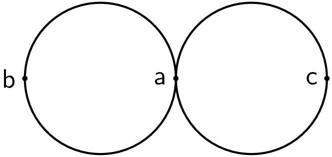


$$\Leftrightarrow$$

$$\mathbf{P}(a|bc) = 0$$

Theorem (G.)

$$\mathbf{P}(abc)\mathbf{P}(a|b|c) \geq \mathbf{P}(ab|c)\mathbf{P}(ac|b) + \mathbf{P}(ab|c)\mathbf{P}(a|bc) + \mathbf{P}(ac|b)\mathbf{P}(a|bc)$$

Corollary

The top-bottom direction is stable. If $\mathbf{P}(ab)\mathbf{P}(ac) \approx \mathbf{P}(abc)$, *then* $\mathbf{P}(a|bc) \approx 0$.

Conjecture

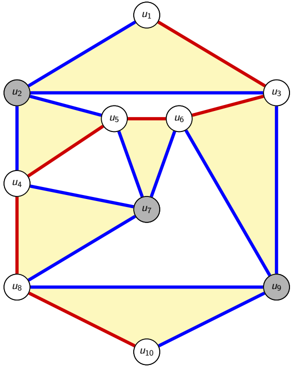
If $\mathbf{P}(a|bc) < \varepsilon$, *then* $\mathbf{P}(abc) - \mathbf{P}(ab)\mathbf{P}(ac) = O(\varepsilon \log\left(\frac{1}{\varepsilon}\right))$.

Lemma (G., Pak, Zimin)

Let H be Hollom's hypergraph with $T = \{u_2, u_7, u_9\}$. Consider the WZ hypergraph percolation where each hyperedge is replaced by a graph G with vertices a, b and c. Assume the connection probabilities satisfy

$$400\mathbf{P}(a|bc) \le \mathbf{P}(abc)\mathbf{P}(a|b|c) - \mathbf{P}(ab|c)\mathbf{P}(ac|b).$$

Then we have $\mathbf{P}_{G_b}(u_1 u_{10}) < \mathbf{P}_{G_b}(u_1 u_{10}')$.



Lemma (G., Pak, Zimin)

Let $n \geq 3$ and $0 < p < 1$. Consider a weighted graph G_n on $(n+1)$ vertices given in Figure 4. Denote $b := v_1$ and $c := v_n$. Then $\mathbf{P}(ab|c) = \mathbf{P}(ac|b)$ *and*

$$\mathbf{P}(abc)\,\mathbf{P}(a|b|c) - \mathbf{P}(ab|c)\,\mathbf{P}(ac|b) > \left(n\,\tfrac{1-p}{1+p} - 1\right)\mathbf{P}(a|bc)\,,$$

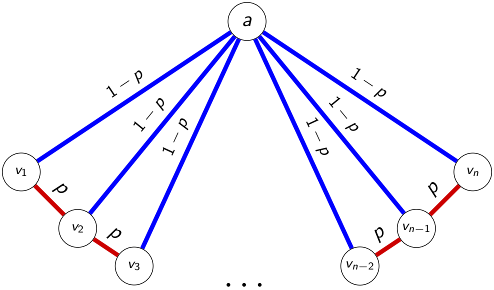
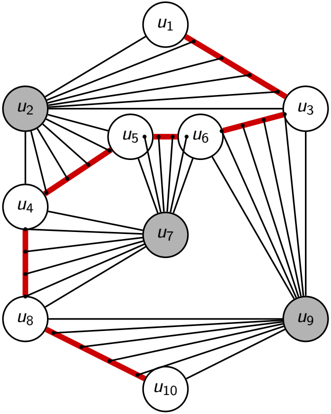


Figure: Graph G_n with $n+1$ vertices.

In the notation of Lemma, let $p = \frac{1}{2}$ and let $n := 3 \cdot 401 + 1 = 1204$. The resulting graph G_n is planar, has 1205 vertices and 2407 edges. Take Hollom's hypergraph H and substitute for each 3-hyperedge with a graph G_n from Lemma, placing it so a is a transversal vertex while $b = v_1$ and $c = v_n$ are the other two vertices. The resulting graph is still planar, has $10 + 6 \cdot 1202 = 7222$ vertices and $6 \cdot 2407 = 14442$ edges.



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Due to the multiple conditionings and the gadget structure, the difference of probabilities given by the counterexample is less than 10^{-4331}, out of reach computationally.
A computer-assisted computation shows that one can use G_n with $p = \frac{1}{2}$ and $n = 14$, giving a relatively small graph on 82 vertices. However, even in this case, the difference of the probabilities in the BBC is on the order 10^{-47}.
For $p = 0.03$ one can take $n = 5$. In this case the alternative BBC is also violated.

Conjecture (Kozma–Nitzan, 2024)

In a percolation on a graph having vertices a, b, $c_1, \ldots, c_n$ one has

$$\mathbf{P}(ab) \geq \mathbf{P}(ac_1 \cup ac_2 \cup \cdots \cup ac_n) \min_i \mathbf{P}(c_i b)$$

Theorem (Kozma–Nitzan)

Conjecture above implies that there is no infinite cluster in percolation on $\mathbb{Z}^d$ at a critical probability. Interesting cases are $d = 3, \ldots, 9$.

Proposition

$$\mathbf{P}(ab) \geq \mathbf{P}(ac_1 \cup ac_2)\Big(\frac{\mathbf{P}(ac_1|c_2)}{\mathbf{P}(ac_1|c_2) + \mathbf{P}(ac_2|c_1)}\mathbf{P}(c_1 b) + \frac{\mathbf{P}(ac_2|c_1)}{\mathbf{P}(ac_1|c_2) + \mathbf{P}(ac_2|c_1)}\mathbf{P}(c_2 b)\Big)$$

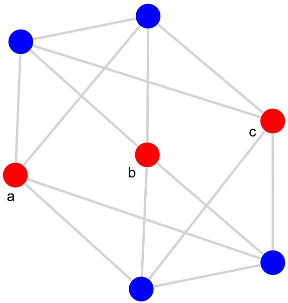
Question

What about 3 c_i's?

Question

Can it be proved that if $\mathbf{P}(ac|b) \approx \mathbf{P}(ab|c) \approx \mathbf{P}(a|bc) \approx 0$*, than* $\mathbf{P}(abc)$ *or* $\mathbf{P}(a|b|c)$ *is also* ≈ 0*?*

In particular, is $\min\left(\mathbf{P}(abc), \mathbf{P}(a|b|c)\right) < \frac{1}{2} - \varepsilon$?
The biggest minimum we can achieve is 0.29 on the graph in the Figure below. Each red-blue edge has probability 0.32537 and both blue-blue edges have probability 0.19231. This way we get
$\mathbf{P}(abc) \approx \mathbf{P}(a|b|c) \approx 0.29065$.



Definition

For two events $\mathcal{A}, \mathcal{B} \subseteq \Omega$, their *disjoint occurrence* $\mathcal{A} \,\square\, \mathcal{B}$ is defined as the event consisting of configurations x whose memberships in $\mathcal{A}$ and in $\mathcal{B}$ can be verified on disjoint subsets of indices.

Theorem (van den Berg–Kesten (vdBK))

$$\mathbf{P}(\mathcal{A} \,\square\, \mathcal{B}) \leq \mathbf{P}(\mathcal{A})\mathbf{P}(\mathcal{B})$$

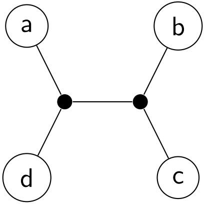
for every pair of closed upwards events $\mathcal{A}$ and $\mathcal{B}$.

Lemma (Hutchcroft)

$$\mathbf{P}(abcd) \leq \mathbf{P}(ab \cup ac \cup ad \cup bc \cup bd \cup cd)^2.$$

Proof.

Imagine that a, b, c, d are in one cluster. Then we can take a spanning tree of this cluster and find two nonintersecting path between a, b, c, d in it. Finally, we apply the vdBK inequality.



□

Corollary

$\min\big(\mathbf{P}(abcd), \mathbf{P}(a|b|c|d)\big)$ *is less than the root of* $x = (1-x)^2$ *equal to* $\frac{3-\sqrt{5}}{2} \approx 0.38$.

Sources of inequalities we have are:

1. Induction: van den Berg–Kesten inequality, union vdBK inequality (only for increasing events), Harris–Kleitman inequality and its colored percolation generalization (G., Pak);
2. Linear algebraic method: van den Berg–Kesten–Reimer inequality (for arbitrary events);
3. Ahlswede–Daykin inequality: used in the first proof of van den Berg–Haggström–Kahn inequality;
4. Markov chains passing to the uniform measure on $S|T$: van den Berg–Haggström–Kahn type inequalities;
5. Decision tree techniques: OSSS inequality, decision tree Harris–Kleitman and vdBK inequality.

Example (Decision tree techniques example)

Suppose I take cards from a shuffled deck one by one, until I get a spade. Then I take one more card. What are the chances that it is also a spade?

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Suppose I take cards from a shuffled deck one by one, until I get a spade. Then I take one more card. What are the chances that it is also a spade?

Solution: It is $\frac{1}{4}$, since we can invert the deck after the first spade without affecting the probability distribution. Under this transformation, the needed probability turns into a probability that the last card in the deck is a spade.

Definition

For two configurations $C_1, C_2 \in \Omega = 2^{[E]}$ and a set $S \subseteq E$ we denote by $C_1 \to_S C_2$ the configuration which coincides with C_1 on S and C_2 on its complement $\overline{S}$.

Lemma

Consider two independent bond Bernoulli percolations C_1 and C_2 having the same distribution μ on the same graph G. Let a decision tree T select each edge and reveal it in both C_1 and C_2. Furthermore, allow on each step, before revealing, decide if this edge will go to the set S (thus dependent on C_1 and C_2) or to its complement $\overline{S}$. Then $C_1 \to_S C_2$ is independent of $C_2 \to_S C_1 = C_1 \to_{\bar{S}} C_2$ and both of them are distributed as μ.

Theorem (G., Zimin)

$$\mathbf{P}(a|b \cap a|c)\mathbf{P}(ab \cup ac) \leq \mathbf{P}(ab|c) + \mathbf{P}(ac|b) + \mathbf{P}(a|bc).$$

Corollary

$\mathbf{P}(abc)$ *and* $\mathbf{P}(a|b|c)$ *can not be simultaneously greater than* 0.37586*.*

Theorem (G.)

$$\mathbf{P}(abc)^2 \leq 8\mathbf{P}(ab)\mathbf{P}(ac)\mathbf{P}(bc).$$

Remark

On $\mathbb{Z}^2$ in a critical mode it is conjectured by Delfino and Viti that

$$\mathbf{P}(abc)^2 \to 1.044... \cdot \mathbf{P}(ab)\mathbf{P}(ac)\mathbf{P}(bc)$$

as a, b and c tend away from each other. Recently the proof was completed by Morris Ang, Gefei Cai, Xin Sun and Baojun Wu.

Thank you for your attention!

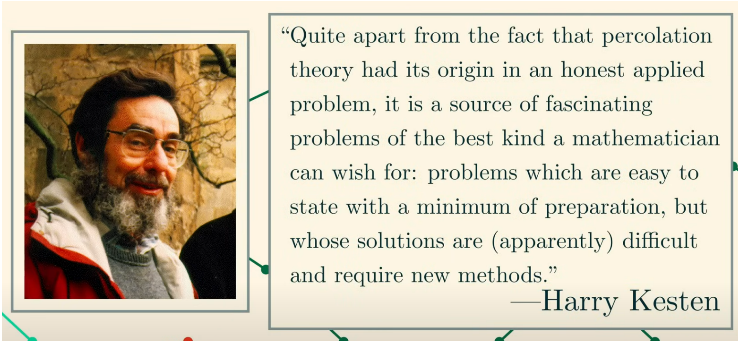

Figure: The critical probability for a site percolation on $\mathbb{Z}^2$ is around 0.592, so big QR-codes are unlikely to have a left to right path via black squares

In 1942, Rosalind Franklin, who then recently graduated in chemistry from the university of Cambridge, joined the BCURA. She started research on the density and porosity of coal. During the Second World War, coal was an important strategic resource. It was used as a source of energy, but also was the main constituent of gas masks.
Coal is a porous medium. To measure its 'real' density, one was to sink it in a liquid or a gas whose molecules are small enough to fill its microscopic pores. While trying to measure the density of coal using several gases (helium, methanol, hexane, benzene), and as she found different values depending on the gas used, Rosalind Franklin showed that the pores of coal are made of microstructures of various lengths that act as a microscopic sieve to discriminate the gases.