(n, k)—Monge–Kantorovich problem

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Eight or Nine Talks on Contemporary Optimal Transport Problems

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Primal and Dual (n, k)-problem

Suppose $X_1, X_2, \dots, X_n$ are measurable spaces with σ-algebras $\mathcal{B}_1$, $\mathcal{B}_2$, $\dots$, $\mathcal{B}_n$ respectively.
Let $Pr_{X_{i_1} \times \dots \times X_{i_k}}, Pr_I$ be the projection operator from $X = X_1 \times \dots \times X_n$ to the coordinate k-dimensional subspace $X_{i_1} \times \dots \times X_{i_k}$.

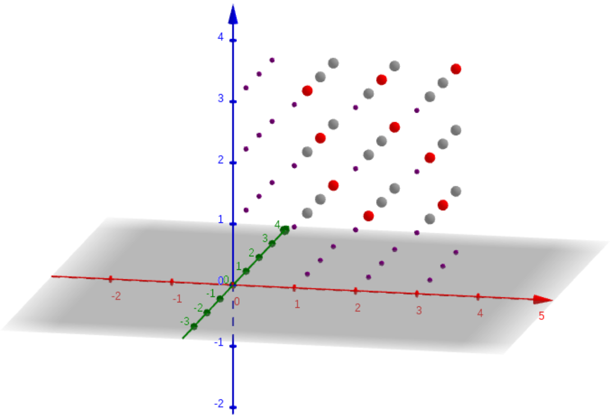
$$\mathcal{I}_k = \{(i_1, i_2, \dots, i_k) \mid 1 \le i_1 < i_2 \dots < i_k \le n\}.$$

For any multi-index $I = (i_1, \dots, i_k) \in \mathcal{I}_k$, there is a given finite measure μ_I on the space $X_{i_1} \times \dots \times X_{i_k}$. Denote $Pr_{I*}\mu$ by $Pr_I(\mu)$.

$$\mathcal{P}_\mu = \{\mu \mid \mu \text{ is a measure on } X \text{ such that } Pr_I(\mu) = \mu_I \text{ for any } I \in \mathcal{I}_k\}$$

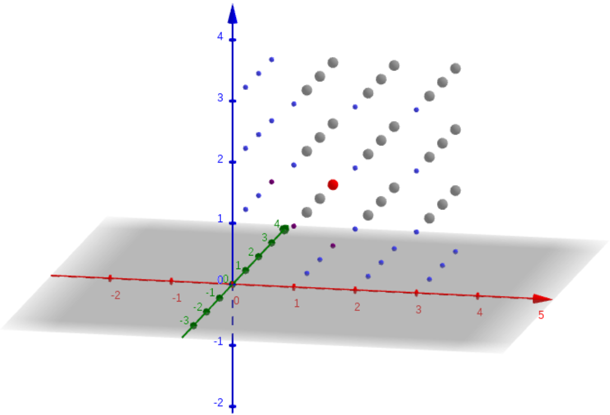
Also, assume $c : X \to \mathbb{R} \cup \{+\infty\}$ is a measurable cost function.

Latin squares as elements of $\mathcal{P}_\mu$ in $(3, 2)$



1	2	3
2	3	1
3	1	2

Latin squares as elements of $\mathcal{P}_\mu$ in $(3, 2)$



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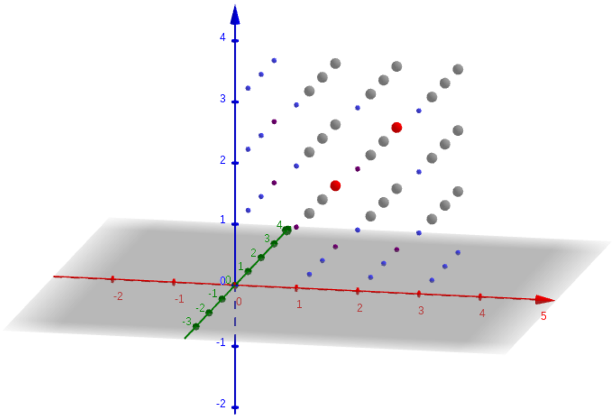
Latin squares as elements of $\mathcal{P}_\mu$ in $(3, 2)$



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Primal and Dual (n, k)-problem

Definition

The Primal (n, k)-problem [G, Zimin, Kolesnikov, 18] is to minimize the functional

$$P(\pi) = \int_X c(x_1, \ldots, x_n)\, d\pi$$

over $\pi \in \mathcal{P}_\mu$.

Definition

The Dual (n, k)-problem is to maximize the functional

$$D(\{f_I\}) = \sum_{I \in \mathcal{I}_k} \int f_I(x_{i_1}, \ldots, x_{i_k})\, d\mu_I$$

over (integrable) functions $\{f_I\}$ such that $\sum_I f_I(x_{i_1}, \ldots, x_{i_k}) \leq c(x_1, \ldots, x_n)$.

By standard duality reasoning under some additional assumptions (in particular $\mathcal{P}_\mu \neq \varnothing$), the minimum in the primal problem exists and is equal to the supremum in the dual problem. By Komlós theorem and Fatou's lemma, the supremum in the dual problem can be achieved.

Discrete motivations

$(2,1)$ – assignment problem

$(2,1)$ – maximal bipartite matching

$(3,2)$ – completing Latin squares

$(3,2)$ well-posedness – completing Latin squares

$(4,2)$ well-posedness – completing Graeco-Latin squares

$(4,2)$ well-posedness – sudoku

$(6,1)$ – Zebra Puzzle, a.k.a. Einstein's Riddle

(2, 1) with limited density as (3, 2)

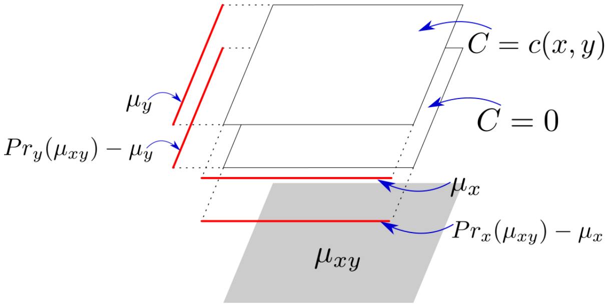
In [Corman, McCann, 2012], the following problem is considered:

Definition

The Monge-Kantorovich problem with limited density is to minimize the functional

$$P(\pi) = \int_{X \times Y} c(x, y) \, d\pi$$

over $\pi \in \mathcal{P}_{\{\mu_x, \mu_y\}}$ such that $\pi \leq \mu_{xy}$.



Definition of primal $(3,2)-$problem

Our main example will be the $(3,2)$-problem, although most properties can be extended to the general case. Let us denote the base spaces by X, Y, and Z. The spaces $X \times Y$, $X \times Z$, and $Y \times Z$ are equipped with finite measures $\mu_{xy}, \mu_{xz}, \mu_{yz}$. Elements of $\mathcal{P}_\mu$ are called **uniting measures**. To define this explicitly,

Definition

A measure μ on $X \times Y \times Z$ is called **uniting** if

$$Pr_{XY}(\mu) = \mu_{xy}, Pr_{XZ}(\mu) = \mu_{xz}, Pr_{YZ}(\mu) = \mu_{yz}.$$

Assume $c : X \times Y \times Z \to \mathbb{R} \cup \{+\infty\}$ is a cost function.

Definition

The Primal $(3,2)$-problem is to minimize the functional

$$P(\pi) = \int_{X \times Y \times Z} c(x,y,z) d\pi$$

over uniting measures π.

Primal $(3,2)-$problem with the cost function xyz.

Primal problem

Consider the $(3,2)$-problem on I^3 with Lebesgue measures on the coordinate planes. Our goal is to find a measure π minimizing $P(\pi) = \int xyz \; d\pi$.

Let $T_x : \mathbb{R}^3 \to \mathbb{R}^3$ be an involution such that:

$$T_x(x,y,z) = (1-x, y, z).$$

T_y and T_z are defined in the same way. Then for any measure μ with the Lebesgue marginals:

$$P(\mu \circ T_x) = \int xyz \; d(\mu \circ T_x) = \int (1-x)yz \; d\mu = \int yz \; d\mu_{yz} - \int xyz \; d\mu = \frac{1}{4} - P(\mu).$$

Thus, involutions $T_x \circ T_y$, $T_x \circ T_z$, and $T_y \circ T_z$ do not change P, and the primal solution π can be assumed to be invariant under these involutions.

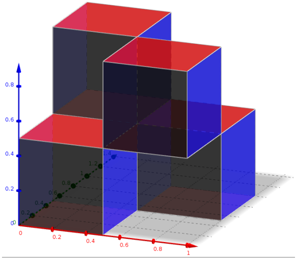
Primal $(3,2)-$problem with the cost function xyz.

$$S_1 = \left[0,\frac{1}{2}\right]^3 \bigcup \left[\frac{1}{2},1\right]^2 \times \left[0,\frac{1}{2}\right] \bigcup \left[0,\frac{1}{2}\right] \times \left[\frac{1}{2},1\right]^2 \bigcup \left[\frac{1}{2},1\right] \times \left[0,\frac{1}{2}\right] \times \left[\frac{1}{2},1\right].$$

From the symmetries of π under T's, one can obtain

$$\int_{I^3 - S_1} xyz \ d\pi \geq \int_{I^3 - S_1} (1-x)(1-y)(1-z) \ d\pi,$$

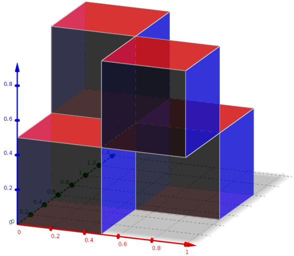
so the optimal π is concentrated on S_1.



Set S_1

By the same argument, $\mu(S_k) = 1$.

$$S_k = \bigcup_{\substack{a \oplus b \oplus c = 0 \\ 0 \le a, b, c < 2^k}} \left[\frac{a}{2^k}, \frac{a+1}{2^k}\right] \times \left[\frac{b}{2^k}, \frac{b+1}{2^k}\right] \times \left[\frac{c}{2^k}, \frac{c+1}{2^k}\right]$$



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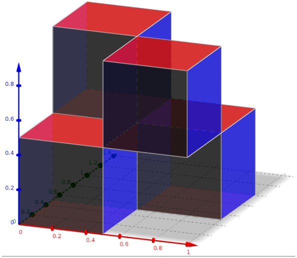
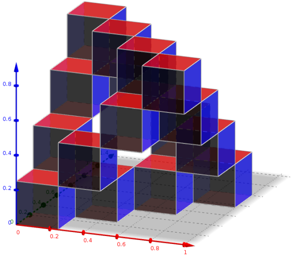


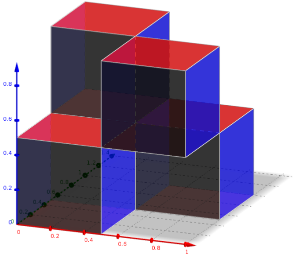
Set S_1



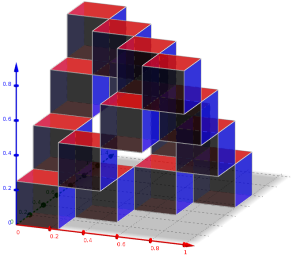
Set S_2

By the same argument, $\mu(S_k) = 1$.

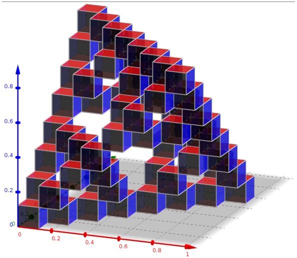
$$S_k = \bigcup_{\substack{a \oplus b \oplus c = 0 \\ 0 \le a,b,c < 2^k}} \left[\frac{a}{2^k}, \frac{a+1}{2^k}\right] \times \left[\frac{b}{2^k}, \frac{b+1}{2^k}\right] \times \left[\frac{c}{2^k}, \frac{c+1}{2^k}\right]$$



Set S_1



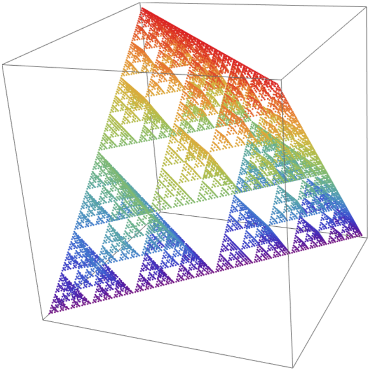
Set S_2



Set S_3

Support of the optimal measure

The optimal measure π is concentrated on $S = \bigcap S_k = \{(x, y, z) \in X \times Y \times Z \mid x \oplus y \oplus z = 0\}$ — the Sierpiński tetrahedron. Although S is highly non-smooth, S is a graph of the function $z = x \oplus y$.



Sierpiński tetrahedron

Dual $(3,2)-$problem with the cost function xyz.

Dual problem

Find Lebesgue integrable functions $f(x,y)$, $g(x,z)$, and $h(y,z)$ such that:

- $f(x,y)+g(x,z)+h(y,z) \leq xyz$ for all $(x,y,z) \in X \times Y \times Z$,
- $D(f,g,h) = \int_{X\times Y} f \ dxdy + \int_{X\times Z} g \ dxdz + \int_{Y\times Z} h \ dydz$ is maximal.

Remark: Due to the symmetry of $c(x,y,z)$, one can assume $f = g = h$.

Denote by $I(a,b)$ the integral $\int_0^a \int_0^b x \oplus y \ dx \ dy$.

$$f(x,y) = I(x,y) - \frac{1}{4}I(x,x) - \frac{1}{4}I(y,y).$$

Uniting measure

Remark

While solving the $(3,2)$*-problem, the following question arises: Is it true that for given measures* μ_{xy}*,* μ_{xz}*, and* μ_{yz} *there exists at least one uniting measure?*

Proposition (Weak sufficient condition)

The set of uniting measures is non-empty if $\mu_{xy} = \mu_x \times \mu_y$*,* $\mu_{xz} = \mu_x \times \mu_z$*, and* $\mu_{yz} = \mu_y \times \mu_z$ *for some measures* μ_x, μ_y, μ_z *on* X, Y, Z*.*

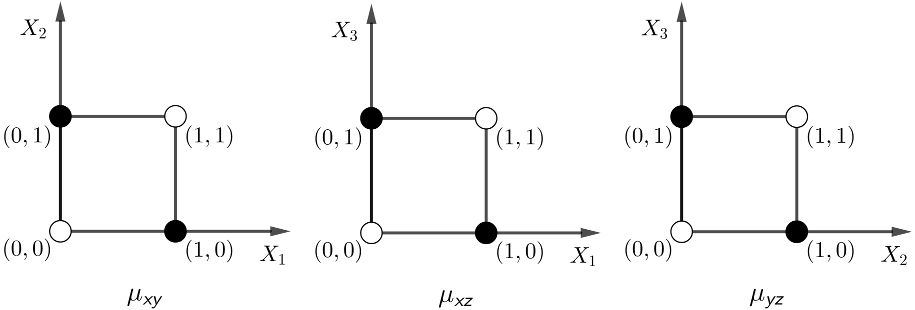
Proposition (Weak necessary condition)

For the existence of a measure μ *on* $X \times Y \times Z$ *with projections* μ_{xy}*,* μ_{xz}*,* μ_{yz}*, the following equalities must hold:*

$$Pr_X(\mu_{xy}) = Pr_X(\mu_{xz}) = \mu_x,$$
$$Pr_Y(\mu_{xy}) = Pr_Y(\mu_{yz}) = \mu_y,$$
$$Pr_Z(\mu_{xz}) = Pr_Z(\mu_{yz}) = \mu_z.$$

Existence of a uniting measure in $(3, 2)$

The weak necessary condition is not sufficient. Suppose $X = Y = Z = \{0, 1\}$. Define measures on $X \times Y$, $X \times Z$, and $Y \times Z$ by the picture (measure 0.5 in black points and 0 in white points):



There is no uniting measure for μ_{xy}, μ_{xz}, and μ_{yz}, but there exists a uniting signed measure.

Existence of a uniting measure in $(3, 2)$

We recall the existence of the uniting measure in the case $\mu_{xy} = \mu_x \times \mu_y$, $\mu_{xz} = \mu_x \times \mu_z$, $\mu_{yz} = \mu_y \times \mu_z$. For example, the measure $\mu_x \times \mu_y \times \mu_z$ fits. The following theorem generalizes this construction:

Theorem (Density condition)

Suppose X, Y, Z are spaces equipped with finite measures ν_x, ν_y, ν_z. Suppose that $\mu_{xy}, \mu_{xz}, \mu_{yz}$ are absolutely continuous with respect to $\nu_x \times \nu_y$, $\nu_x \times \nu_z$, $\nu_y \times \nu_z$ respectively. Assume p_{xy}, p_{xz}, p_{yz} are the respective densities. If for $\lambda \leq \frac{3}{2}$ the following holds:

$$1 \leq p_{xy}, p_{xz}, p_{yz} \leq \lambda,$$

then there exists a uniting measure for μ_{xy}, μ_{xz}, and μ_{yz}.

Existence of a uniting measure in $(3, 2)$

It's sufficient to prove the density condition theorem for $\lambda = \frac{3}{2}$. Without loss of generality, let ν_x, ν_y, ν_z be probability measures.

$$M = \mu_{xy}(X \times Y) = \mu_{xz}(X \times Z) = \mu_{yz}(Y \times Z).$$

Assume p_x, p_y, p_z are the densities of μ_x, μ_y, μ_z with respect to ν_x, ν_y, ν_z. There holds $1 \leq p_x, p_y, p_z, M \leq \lambda$.

For example, if $M = \lambda$, the following equalities hold: $\mu_{xy} = \lambda(\nu_x \times \nu_y)$, $\mu_{xz} = \lambda(\nu_x \times \nu_z)$, $\mu_{yz} = \lambda(\nu_y \times \nu_z)$. The measure $\mu = \lambda(\nu_x \times \nu_y \times \nu_z)$ has projections μ_{xy}, μ_{xz}, and μ_{yz}. The same argument works for $M = 1$.

Existence of a uniting measure in $(3, 2)$

The following signed measure is uniting:

$$\mu = \frac{4}{M^2}\mu_x \times \mu_y \times \mu_z - \frac{2}{M}\left(\nu_x \times \mu_y \times \mu_z + \mu_x \times \nu_y \times \mu_z + \mu_x \times \mu_y \times \nu_z\right) + 2\left(\mu_{xy} \times \nu_z + \mu_{xz} \times \nu_y + \mu_{yz} \times \nu_x\right) - \frac{1}{M}\left(\mu_{xy} \times \mu_z + \mu_{xz} \times \mu_y + \mu_{yz} \times \mu_x\right)$$

since

$$Pr_{XY}(\mu) = \frac{4}{M}\mu_x \times \mu_y - 2\nu_x \times \mu_y - 2\mu_x \times \nu_y - \frac{2}{M}\mu_x \times \mu_y + 2\mu_{xy} + 2\mu_x\nu_y + 2\nu_x\mu_y - \mu_{xy} - \frac{2}{M}\mu_x \times \mu_y = \mu_{xy}.$$

Existence of a uniting measure in $(3, 2)$

Check the non-negativity of this measure. To this end, check

$$\frac{4}{M^2} a_1 b_1 c_1 - \frac{2}{M}(a_1 b_1 + a_1 c_1 + b_1 c_1) + 2(a_2 + b_2 + c_2) - \frac{1}{M}(a_1 a_2 + b_1 b_2 + c_1 c_2) \geq 0$$

for $1 \leq a_1, b_1, c_1, a_2, b_2, c_2, M \leq \frac{3}{2}$. This expression is greater than $\varepsilon(M) > 0$ for all $a_1, b_1, c_1, a_2, b_2, c_2$, and $M \in \left(1, \frac{3}{2}\right)$.

Proposition

In the assumptions of the density condition, there exists a uniting measure μ that is absolutely continuous with respect to $\nu_x \times \nu_y \times \nu_z$, and the density of this measure is bounded and separated from zero.

Existence of a uniting measure in (n, k)

Theorem

For $\lambda \leq 2$, there exists a (not necessarily absolutely continuous) uniting measure μ.

For $\lambda > 2$, this theorem fails.
This result can be generalized to the (n, k)-problem.

Theorem

Suppose $\{\mu_I \mid I \in \mathcal{I}_k\}$ satisfy the weak necessary conditions. Then there exists a signed measure μ such that

$$Pr_I \mu = \mu_I, \quad I \in \mathcal{I}_k.$$

There exists an analogue of the density condition in the (n, k)-problem for some $\lambda_{n,k}$.

Dual $(3,2)$-problem

Definition

A function $F : X \times Y \times Z \to \mathbb{R}$ is called a $(3,2)$-function if there exist functions f_{xy}, f_{xz}, f_{yz} such that

$$F(x,y,z) = f_{xy}(x,y) + f_{xz}(x,z) + f_{yz}(y,z)$$

for all $(x,y,z) \in X \times Y \times Z$.

Definition

The Dual $(3,2)$-problem is the problem of maximizing the functional

$$D(F) = \int f_{xy}\ d\mu_{xy} + \int f_{xz}\ d\mu_{xz} + \int f_{yz}\ d\mu_{yz}$$

over $(3,2)$-functions $F \leq c$.

Boundedness of a dual solution

Remark

In the classical Monge-Kantorovich problem, if the cost function is (essentially) bounded, then there exists a (essentially) bounded dual solution.

Theorem

Assume $X = Y = Z = \mathbb{N}$, *and* μ_x, μ_y, μ_z *are probability measures on* X, Y, *and* Z. *Suppose* $\mu_{xy} = \mu_x \times \mu_y$, $\mu_{xz} = \mu_x \times \mu_z$, $\mu_{yz} = \mu_y \times \mu_z$, *and* c *is a cost function such that* $0 \leq c \leq 1$. *Denote by* F *a dual solution of the* $(3, 2)$*-problem with projections* μ_{xy}, μ_{xz}, μ_{yz} *and the cost function* c. *Then* $-12 \leq F$ *almost everywhere.*

Corollary

In the $(3, 2)$*-problem for compact metric spaces* X, Y, Z, *and bounded* c, *there exists a bounded dual solution.*