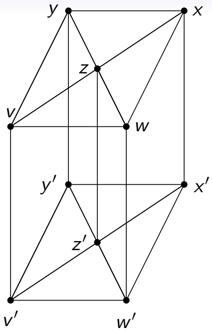
Percolation Inequalities and Decision Trees

Based on joint work with Aleksandr Zimin and Igor Pak

Nikita Gladkov, UCLA

Percolation Today, November 19, 2024



Conjecture (Bunkbed conjecture)

The probabilities of two copies of the same edge are equal. The probabilities of posts are arbitrary. Then:

$$\mathbf{P}(x \leftrightarrow y) \geq \mathbf{P}(x \leftrightarrow y').$$

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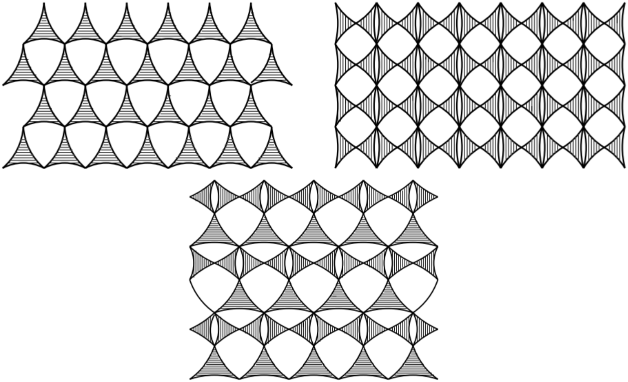
Gadgets

We can abstract away some part of the graph into a "gadget".
If the "interface" of a gadget consists of three vertices a, b, and c, then there are five probabilities governing its relationship with the rest of the graph:
$\mathbf{P}(abc)$, $\mathbf{P}(ab|c)$, $\mathbf{P}(a|b|c)$, $\mathbf{P}(a|bc)$, $\mathbf{P}(ac|b)$.



Sources of inequalities

1. Induction: Harris–Kleitman inequality and its colored percolation generalization (G., Pak), van den Berg–Kesten inequality, union vdBK inequality (only for increasing events).
2. Linear algebraic method: van den Berg–Kesten–Reimer inequality (for arbitrary events).
3. Ahlswede–Daykin inequality: used in the first proof of the van den Berg–Haggström–Kahn inequality.
4. Markov chains passing to the uniform measure on $S|T$: van den Berg–Haggström–Kahn type inequalities.
5. Decision tree techniques: OSSS inequality, decision tree Harris–Kleitman, and vdBK inequality.

Example (Decision tree techniques example)

Suppose I draw cards from a shuffled deck one by one until I get a spade. Then I take one more card. What is the probability that it is also a spade?

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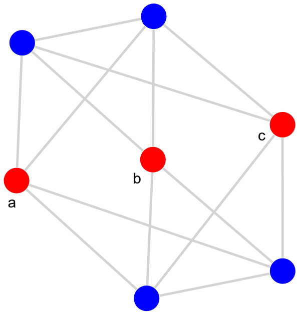
Solution: It is $\frac{1}{4}$, since we can invert the deck after the first spade without affecting the probability distribution. Under this transformation, the needed probability turns into a probability that the last card in the deck is a spade.

Example (Decision tree techniques example)

Consider a percolation configuration C on a graph $G = (V, E)$. Let S be the set of edges with both ends in the cluster of vertex a. Resample the edges in S. Will the probability that vertices b and c belong to the same cluster increase or decrease?



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Solution: It will decrease since if b and c belonged to different clusters, they will continue to be in different clusters. But if they both belonged to the cluster of a, with nonzero probability, they will be disconnected after the resampling.

Definition

For two percolation configurations $C_1, C_2 \in \Omega$ and a set $S \subseteq E$, we denote by $C_1 \to_S C_2$ the configuration that coincides with C_1 on S and C_2 on its complement $\overline{S}$.

$$C_1 \to_S C_2(e) = \begin{cases} C_1(e), & \text{if } e \in S, \\ C_2(e), & \text{otherwise.} \end{cases}$$

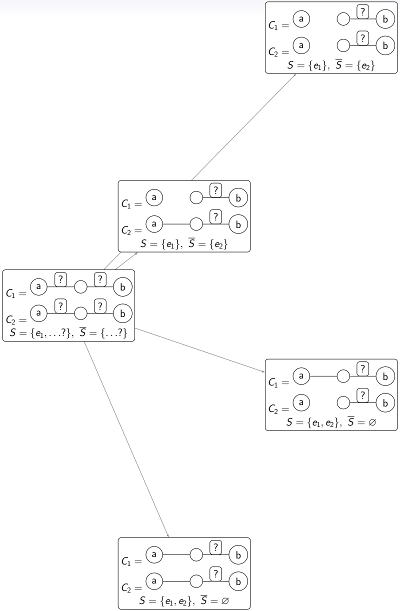
Lemma

Let G be finite. Let $S(C_1, C_2)$ be built by some decision tree. Then $C_1 \to_S C_2$ is independent of $C_2 \to_S C_1 = C_1 \to_{\bar{S}} C_2$, and both are distributed as μ.

Set built by decision tree

Algorithm 1 Building Set S by Decision Tree T

procedure BuildSet(T, C_1, C_2)
    $S \leftarrow \emptyset$
    $N \leftarrow N_0$ ▷ Start from the root node N_0 of the decision tree T
    while N is a decision node **do**
        $e \leftarrow$ edge queried by N
        if decision of N is "S" **then**
            $S \leftarrow S \cup \{e\}$
        end if
        if $C_1(e) = 1$ and $C_2(e) = 1$ **then**
            $N \leftarrow N_{11}$ ▷ Both configurations have edge e open
        else if $C_1(e) = 1$ and $C_2(e) = 0$ **then**
            $N \leftarrow N_{10}$ ▷ Configuration C_1 has edge e open and C_2 has it closed
        else if $C_1(e) = 0$ and $C_2(e) = 1$ **then**
            $N \leftarrow N_{01}$ ▷ Configuration C_1 has edge e closed and C_2 has it open
        else
            $N \leftarrow N_{00}$ ▷ Both configurations have edge e closed
        end if
    end while
    return S
end procedure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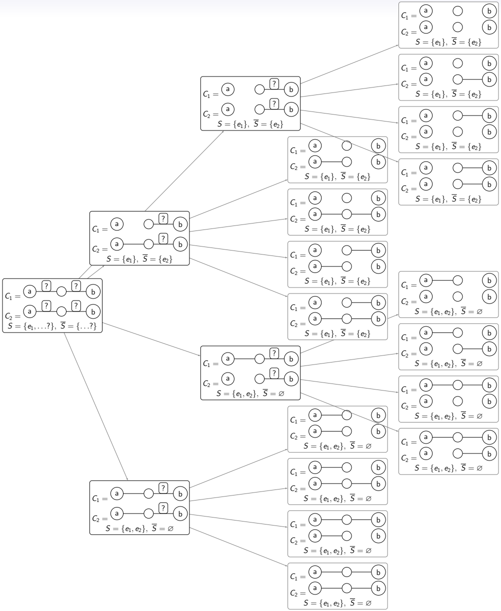
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Quiz

Which of the following sets are built by a decision tree?

a) Set of edges with both ends in the cluster of a;

b) Set of edges with one end in the cluster of a;

c) Set of edges with both ends *not* in the cluster of a;

d) Set of edges with one end in the cluster of a and one end in the cluster of b.

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Answer: b, c.

Site percolation does not simulate bond percolation

The key observation is that when $C_1 \in a|b|c$ and $C_1 \to_{S_3} C_2 \in ab \cup ac$, one has $C_1 \to_{S_1} C_2 \in ab$ or $C_1 \to_{S_2} C_2 \in ac$.

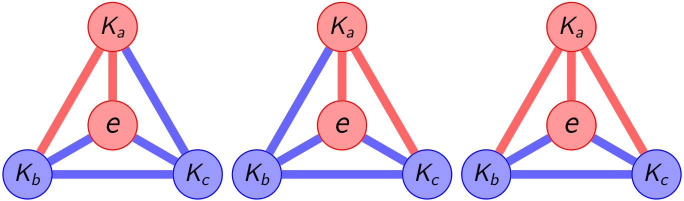

Figure: S_1, S_2 and S_3 for the case $C_1 \in a|b|c$. Regions surrounding a, b, c depict K_a, K_b and K_c. Respective sets are in blue and their complements are in red.

Theorem (G., Zimin)

$$\mathbf{P}(a|b \cap a|c)\mathbf{P}(ab \cup ac) \leq \mathbf{P}(ab|c) + \mathbf{P}(ac|b) + \mathbf{P}(a|bc).$$

Corollary

$\mathbf{P}(abc)$ *and* $\mathbf{P}(a|b|c)$ *cannot simultaneously be greater than* 0.37586*.*

Fisher, Essam, 1961

To show this, we note that the bond problem on any lattice L is isomorphic to the site problem on a suitably defined covering lattice L^C. The covering lattice is constructed by replacing each bond of L with a site (placed at its center) and linking these sites together by sufficient new bonds to ensure that if two bonds of L meet (at a vertex), then the corresponding two sites of L^C are joined by a direct bond and vice versa.

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Lemma (Partial case of a lemma by Hutchcroft)

$$\mathbf{P}(abcd) \leq \mathbf{P}(ab \cup ac \cup ad \cup bc \cup bd \cup cd)^2.$$

Similar arguments

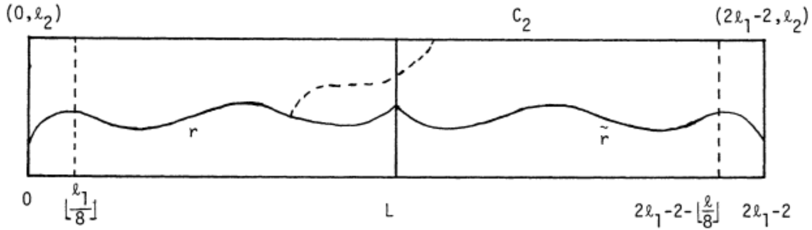


Figure 6.1

Remark on RSW (V. Tassion)

For Bernoulli percolation, the original proof of the Russo–Seymour–Welsh theorem relies on the spatial Markov property and independence: assuming that a left-right crossing exists in a square, one can first find the lowest crossing by exploring the region below it. Then the configuration can be sampled independently in the unexplored region (above the path).

Theorem (Decision tree HK inequality)

Let G be finite. Let $S(C_1, C_2)$ be built by some decision tree. Assume A and B are some events in Ω closed upward. Then:

$$\mathbf{P}(C_1 \in A, C_1 \to_S C_2 \in B) \geq \mathbf{P}(C_1 \in A)\mathbf{P}(C_1 \to_S C_2 \in B) = \mu(A)\mu(B).$$

Proof.

Induction by the decision tree. Take the lowest vertex in the decision tree assigning its edge to S and change the decision to $\bar{S}$. □

The natural generalization of the vdBK inequality to decision trees involves the set S.

Definition

For the decision tree setup, the disjoint occurrence $A \,\square_S\, B$ is defined as:

$$A \,\square_S\, B := \{C_1, C_2 \in \Omega, \text{ such that there exist } I, J \subset [n]$$
$$\text{such that } I \text{ is a witness of } A \text{ in } C_1,\ J \text{ is a witness of } B \text{ in } C_1 \to_S C_2 \text{ and } I \cap J \subseteq \overline{S}\}.$$

For $S = E$, this definition turns into the usual disjoint occurrence of A and B in C_1. For $S = \varnothing$, the event $A \,\square_S\, B$ coincides with $A \times B$.

Theorem (Decision tree vdBK inequality)

Let G be finite. Let the decision tree T build a set $S(C_1, C_2)$ and A and B be two closed upward events. Then:

$$\mathbf{P}(A \,\square_S\, B) \leq \mathbf{P}(A)\mathbf{P}(B).$$

Cauchy–Schwarz estimate

By definition, the decision tree can have some leaf nodes where not all edges are queried on the path leading to them. According to Algorithm 1, such edges are not assigned to S and therefore are assigned to $\overline{S}$. If we replace some of the leaf nodes with subtrees, we obtain a new tree. We say the new tree is a *continuation* of the old tree.

Theorem

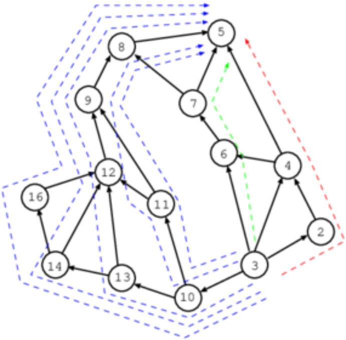
Let G be finite. Let T_1 and T_2 be decision trees for the events $C_1 \in A$ and $C_1 \in B$ respectively, such that T_2 continues T_1 and $B \subset A$ is an intersection of A with an increasing or decreasing event in Ω. Additionally, assume all nodes of T_1 assign edges to S. Then:

$$\mathbf{P}(C_1 \in B, C_1 \to_{S_2} C_2 \in B) \geq \frac{\mathbf{P}(B)^2}{\mathbf{P}(A)}.$$

Theorem

Let G be planar. Suppose a and b belong to the same face. Then:

$$\mathbf{P}(ab^{\square 3})^2 \leq \mathbf{P}(ab^{\square 2})^3.$$



Three-point function bound

Theorem

Let a, b, c be distinct vertices of graph G. Then:

$$\mathbf{P}(abc)^2 \leq 8\mathbf{P}(ab)\mathbf{P}(ac)\mathbf{P}(bc),$$

where $\mathbf{P}(abc)$ is the probability that a, b and c are in the same percolation cluster.

Remark

On $\mathbb{Z}^2$ in the critical mode, it is conjectured by Delfino and Viti that:

$$\mathbf{P}(abc)^2 \to 1.044... \cdot \mathbf{P}(ab)\mathbf{P}(ac)\mathbf{P}(bc)$$

as a, b and c tend away from each other. Recently the proof was completed by Morris Ang, Gefei Cai, Xin Sun and Baojun Wu.
The bound is conjecturally sharp up to a constant factor for $d < 6$.

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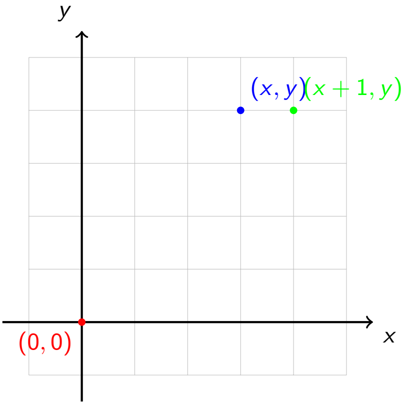
Questions?

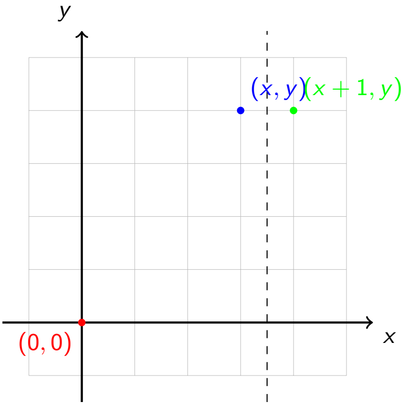
Bunkbed Conjecture

Monotonicity conjecture



$$\mathbf{P}\big(0 \leftrightarrow (x, y)\big) \geq \mathbf{P}\big(0 \leftrightarrow (x+1, y)\big) \text{ for } x \geq 0.$$

Monotonicity conjecture



$$\mathbf{P}\big(0 \leftrightarrow (x, y)\big) \geq \mathbf{P}\big(0 \leftrightarrow (x+1, y)\big) \text{ for } x \geq 0.$$

Remark

The bunkbed conjecture follows from its partial case where all posts have probability 0 or 1.

Proof.

Indeed, $\mathbf{P}_{G_b}(xy)$ and $\mathbf{P}_{G_b}(xy')$ are polynomials in p_e. If e is a post, $\mathbf{P}_{G_b}(xy) - \mathbf{P}_{G_b}(xy')$ is linear in p_e, so we can move it to 0 or 1, depending on the sign of the coefficient. □

We call vertices with posts *transversal*.

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Proposition

If there is only one transversal vertex v, the bunkbed conjecture is true.

Proof.

We can rewrite probabilities on G_b in terms of probabilities on G. Thus:

$$\mathbf{P}_{G_b}(xy) = \mathbf{P}_G(xy)$$

and

$$\mathbf{P}_{G_b}(xy') = \mathbf{P}_G(xv)\mathbf{P}_G(yv) \leq \mathbf{P}_G(xyv) \leq \mathbf{P}_G(xy)$$

by the Harris–Kleitman inequality. □

Theorem (Linusson, 2008)

If x or y is transversal, then the bunkbed conjecture turns into equality. If any path from x to y in G passes through a transversal vertex, the bunkbed conjecture turns into equality.

Proof.

Find the open component of y in $G \setminus T$ and switch the edges between the levels. $\square$

In the *alternative bunkbed percolation*, each edge e in G is either deleted while the corresponding hyperedge e' in G' is retained with probability $\frac{1}{2}$, or vice versa: edge e is retained and e' is deleted.

Theorem (Linusson, 2008)

If the BBC fails on some graph G for some probabilities p_e, then the alternative BBC fails on some minor H of G.

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Despite intuitiveness, proving this conjecture is not straightforward and is an active area of research in percolation theory.[6] It was proved for specific types of graphs, such as wheels,[7] complete graphs,[8] complete bipartite graphs and graphs with a local symmetry.[9] It was also proved in the limit $p \to 1$ for any graph[10][11].

Figure: Known cases of the BBC

Theorem (van den Berg–Haggström–Kahn)

$$\mathbf{P}(ab|cd)\mathbf{P}(a|d) \leq \mathbf{P}(ab|d)\mathbf{P}(a|cd).$$

In other words, conditional on a|d the events ab and cd are nonpositively correlated.

Proof sketch.

We run a Markov chain process with a stable distribution being the uniform measure on $a|d$. Then, we apply the Harris–Kleitman inequality to the events ab and cd, which turn out to be closed upward and downward in the new coordinates. □

Proposition

If there are only two transversal vertices v and w, the bunkbed conjecture is true.

Proof (G., Zimin).

Add together some Harris–Kleitman and van den Berg–Haggström–Kahn inequalities.

$$\begin{aligned}
&\mathbf{P}_{G_b}(xy) - \mathbf{P}_{G_b}(xy') = \\
&\qquad \mathbf{P}(xy|v|w) + \mathbf{P}(xy|vw) \\
&\qquad + \mathbf{P}((xv \cup xw) \cap (yv \cup yw)) - \mathbf{P}(xv \cup xw)\mathbf{P}(yv \cup yw) \\
&\qquad + \mathbf{P}(xv|w)\mathbf{P}(yw|v) - \mathbf{P}(xv|yw)\mathbf{P}(v|w) \\
&\qquad + \mathbf{P}(xw|v)\mathbf{P}(w|yv) - \mathbf{P}(xw|yv)\mathbf{P}(v|w).
\end{aligned}$$

$\square$

Question

What about 3 transversal vertices?

Theorem (Hollom, 2024)

For the following 3-regular hypergraph with 3 transversal vertices, the alternative hypergraph bunkbed conjecture is false.

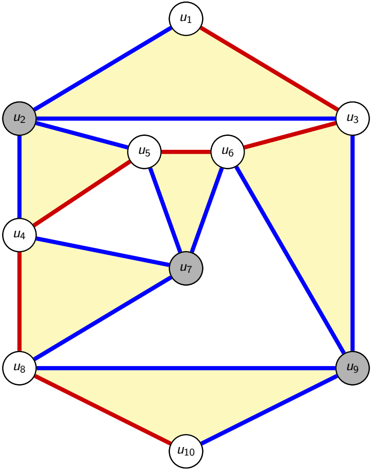

Figure: Hollom's 3-uniform hypergraph H.

Theorem (G., Pak, Zimin)

There exists a connected planar graph $G = (V, E)$ *with* $|V| = 7222$ *vertices and* $|E| = 14442$ *edges, a subset* $T \subset V$ *with three transversal vertices, and vertices* $u, v \in V$*, such that:*

$$\mathbf{P}^{\mathrm{bb}}_{\frac{1}{2}}[u \leftrightarrow v] < \mathbf{P}^{\mathrm{bb}}_{\frac{1}{2}}[u \leftrightarrow v'].$$

In particular, the bunkbed conjecture is false.

The counterexample

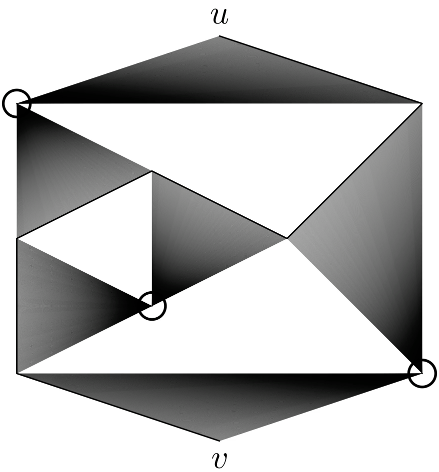

Figure: Graph counterexample with 7222 vertices and 14442 edges.

The counterexample

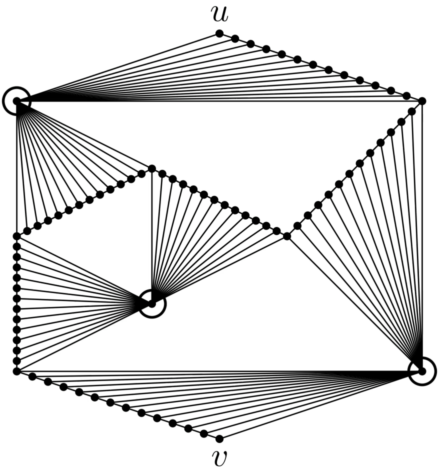

Figure: Smaller counterexample with 82 vertices.

Problem

An intern at a Boring Company has a computer with 2 desktop background options: light grey and dark grey. Initially, it was dark grey. On the first day, she becomes slightly bored and changes the background color with probability 10%. On the second day, she changes it with probability 20%, independent of what happened before. Every following day becomes more boring. On day n, the intern changes the color with probability $10n\%$*, independent of previous days. What is the probability that the screen will be dark grey at the end of the tenth day?*

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Solution: It is $\frac{1}{2}$, because changing the decision on day 5 is an involution preserving probabilities.
For the general case,
$\mathbf{P}(darkgrey) - \mathbf{P}(lightgrey) = (1 - 2p_1) \dots (1 - 2p_n)$.

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We use similar reasoning to conclude that in out counterexample after cancellations we are left with Hollom's counterexample and a negligible part.

Equality case of the Harris–Kleitman inequality:

$$\mathbf{P}(ab)\mathbf{P}(ac) = \mathbf{P}(abc)$$

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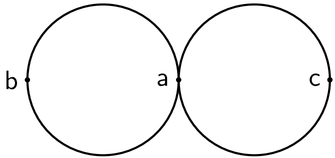


$$\Leftrightarrow$$

$$\mathbf{P}(a|bc) = 0$$

Theorem (G.)

$$\mathbf{P}(abc)\mathbf{P}(a|b|c) \geq \mathbf{P}(ab|c)\mathbf{P}(ac|b) + \mathbf{P}(ab|c)\mathbf{P}(a|bc) + \mathbf{P}(ac|b)\mathbf{P}(a|bc)$$

Corollary

The top-bottom direction is stable. If $\mathbf{P}(ab)\mathbf{P}(ac) \approx \mathbf{P}(abc)$*, then* $\mathbf{P}(a|bc) \approx 0$.

Conjecture

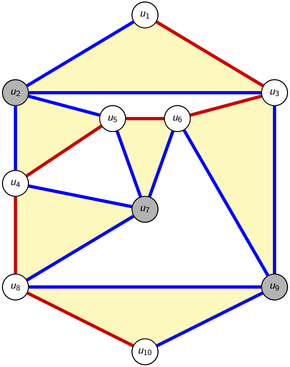
If $\mathbf{P}(a|bc) < \varepsilon$, then $\mathbf{P}(abc) - \mathbf{P}(ab)\mathbf{P}(ac) = O(\varepsilon \log\left(\frac{1}{\varepsilon}\right))$.

Lemma (G., Pak, Zimin)

Let H be Hollom's hypergraph with $T = \{u_2, u_7, u_9\}$. Consider the WZ hypergraph percolation where each hyperedge is replaced by a graph G with vertices a, b and c. Assume the connection probabilities satisfy

$$400\mathbf{P}(a|bc) \leq \mathbf{P}(abc)\mathbf{P}(a|b|c) - \mathbf{P}(ab|c)\mathbf{P}(ac|b).$$

Then we have $\mathbf{P}_{G_b}(u_1 u_{10}) < \mathbf{P}_{G_b}(u_1 u_{10}')$.



Lemma (G., Pak, Zimin)

Let $n \geq 3$ and $0 < p < 1$. Consider a weighted graph G_n on $(n+1)$ vertices given in Figure 6. Denote $b := v_1$ and $c := v_n$. Then $\mathbf{P}(ab|c) = \mathbf{P}(ac|b)$ *and*

$$\mathbf{P}(abc)\,\mathbf{P}(a|b|c) - \mathbf{P}(ab|c)\,\mathbf{P}(ac|b) > \left(n\,\tfrac{1-p}{1+p} - 1\right)\mathbf{P}(a|bc)\,,$$

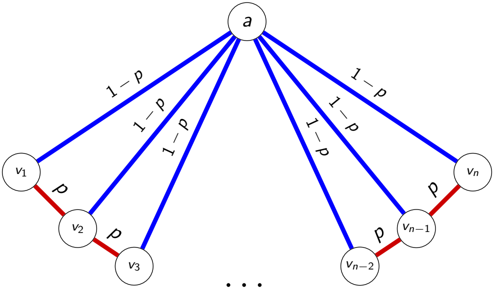
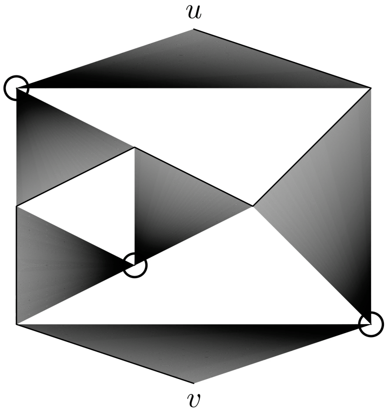


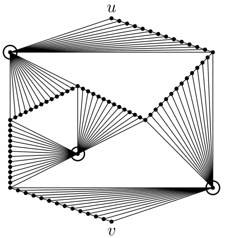
Figure: Graph G_n with $n+1$ vertices.

In the notation of Lemma, let $p = \frac{1}{2}$ and let $n := 3 \cdot 401 + 1 = 1204$. The resulting graph G_n is planar, has 1205 vertices and 2407 edges. Take Hollom's hypergraph H and substitute for each 3-hyperedge with a graph G_n from Lemma, placing it so a is a transversal vertex while $b = v_1$ and $c = v_n$ are the other two vertices. The resulting graph is still planar, has $10 + 6 \cdot 1202 = 7222$ vertices and $6 \cdot 2407 = 14442$ edges.



In the notation of Lemma, let $p = \frac{1}{2}$ and let $n := 3 \cdot 401 + 1 = 1204$. The resulting graph G_n is planar, has 1205 vertices and 2407 edges. Take Hollom's hypergraph H and substitute for each 3-hyperedge with a graph G_n from Lemma, placing it so a is a transversal vertex while $b = v_1$ and $c = v_n$ are the other two vertices. The resulting graph is still planar, has $10 + 6 \cdot 1202 = 7222$ vertices and $6 \cdot 2407 = 14442$ edges. Due to the multiple conditionings and the gadget structure, the difference of probabilities given by the counterexample is less than 10^{-4331}, out of reach computationally.

Computational experiments by Johann Beurich:
For G_{14}:



$$\mathbf{P}_{\frac{1}{2}}(u \leftrightarrow v) = \frac{14338618931856577717562832090209288500494566471890660882024612544451828239420648759937345337554592}{2^{324}}$$

$$\mathbf{P}_{\frac{1}{2}}(u \leftrightarrow v') = \frac{14338618931856577717562832090209288500494566473487998509172155267799513291159301135419470063068025}{2^{324}}$$

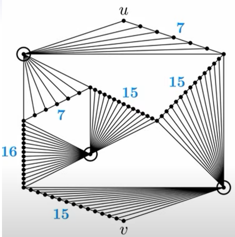


Figure: All edge probabilities are $\frac{1}{2}$, 73 vertices

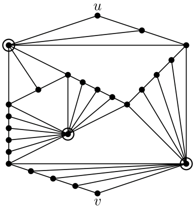


Figure: Edge probabilities are ε or $1 - \varepsilon$, 25 vertices

Hollom's example revisited

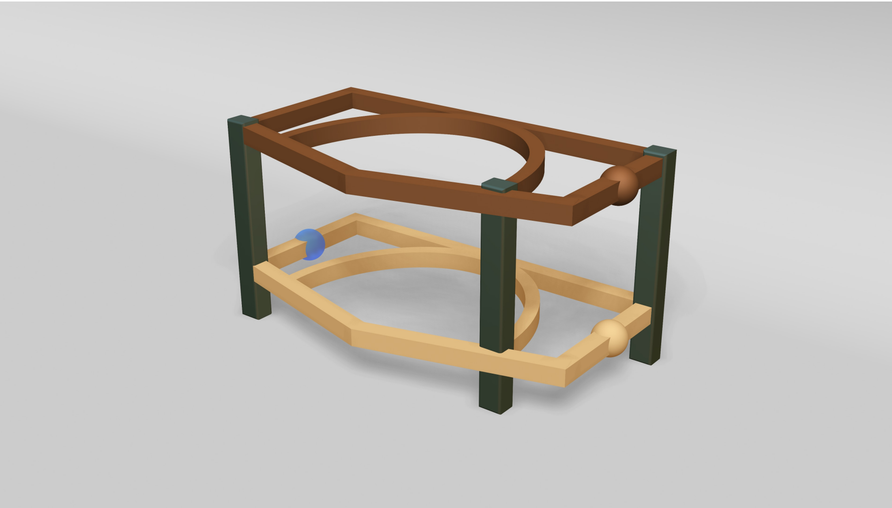


Figure: Visualization of Hollom's counterexample in 3D.

One can ask what happens in $\frac{1}{2}$-percolation on the product graph $G \times K_2$. To distinguish it from the BBC, we call this *Complete BBC*.
It turns out our counterexample extends to a counterexample to the Complete BBC since all nontransversal vertices lie on a single red path, but a counterexample is slightly larger due to the added gadgets at transversal vertices, similar to [Hollom, 2024].
The difference in probabilities is even smaller in this case, on the order of 10^{-6500}.

Thank you for your attention!