# Variants of the Monge-Kantorovich problem for the cost function $xyz$

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# Primal and Dual $(n, k)$-problem

Suppose $X_1, X_2, \dots, X_n$ are topological spaces with $\sigma$-algebras $\mathcal{B}_1$, $\mathcal{B}_2$, $\dots$, $\mathcal{B}_n$ respectively.
Let $Pr_{X_{i_1} \times \dots \times X_{i_k}}, Pr_I$ be the projection operator from $X = X_1 \times \dots \times X_n$ to the coordinate $k$-dimensional subspace $X_{i_1} \times \dots \times X_{i_k}$.

$$\mathcal{I}_k = \{(i_1, i_2, \dots, i_k) \mid 1 \le i_1 < i_2 < \dots < i_k \le n\}.$$

For any multi-index $I = (i_1, \dots, i_k) \in \mathcal{I}_k$, there is a given measure $\mu_I$ on the space $X_{i_1} \times \dots \times X_{i_k}$.

$$\mathcal{P}_\mu = \{\mu \mid Pr_I \mu = \mu_I \text{ for any } I \in \mathcal{I}_k\}$$

Also, assume $c : X \to \mathbb{R} \cup \{+\infty\}$ is a cost function.

# Primal and Dual $(n, k)$-problem

## Definition

The Primal $(n, k)$-problem is the problem of minimizing the functional

$$P(\pi) = \int_X c(x_1, \ldots, x_n)\ d\pi$$

over $\pi \in \mathcal{P}_\mu$.

## Definition

The Dual $(n, k)$-problem is the problem of maximizing the functional

$$D(\{f_I\}) = \sum_{I \in \mathcal{I}_k} \int f_I(x_{i_1}, \ldots, x_{i_k})\ d\mu_I$$

over (integrable) functions $\{f_I\}$ such that $\sum_I f_I(x_{i_1}, \ldots, x_{i_k}) \leq c(x_1, \ldots, x_n)$.

By the Riesz-Markov-Kakutani theorem and Fenchel-Rockafellar duality, the minimum in the primal problem is equal to the supremum in the dual problem. By the Komlós theorem and Fatou's lemma, the supremum in the dual problem can be achieved.

# Motivation for $(3, 2)-$problem.

Unlike the $(n, 1)$-problem, studied already in 1984 by Kellerer and later by Abbas Moameni, Brendan Pass, Simone di Marino, Augusto Gerolin, Luca Nenna and others, there is little research on the $(3, 2)$-problem.

Consider the following problem: We have types of furniture

$$F = \{chair, table, \dots\},$$

months

$$M = \{January, February, \dots\},$$

and institutions

$$I = \{university, hospital, \dots\}.$$

We need to furnish all the institutions within the year, spending as little money as possible. The cost function may vary on $F \times M \times I$ due to inflation and the geographical positions of the institutions. Natural constraints on this cost minimization problem are fixed marginal distributions $\pi(f, m)$, $\pi(f, i)$, and $\pi(m, i)$ corresponding to production limitations, institutional needs, and institutional constraints, respectively. This provides economic motivation for considering the $(3, 2)$-problem.

# Motivation for $(3,2)-$problem.

Any Latin square $n \times n$ corresponds to a measure on $\{1,\dots,n\}^3$ with 2-dimensional marginals uniform on $\{1,\dots,n\}^2$, in the same way a rearrangement matrix corresponds to a rearrangement. However, for the $(3,2)$ case, there is no analogue for the Birkhoff-von Neumann theorem [Linial, Luria, 2012], as illustrated by the cost function

$$c = \left[\begin{array}{ccc|ccc|ccc} 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \end{array}\right]$$

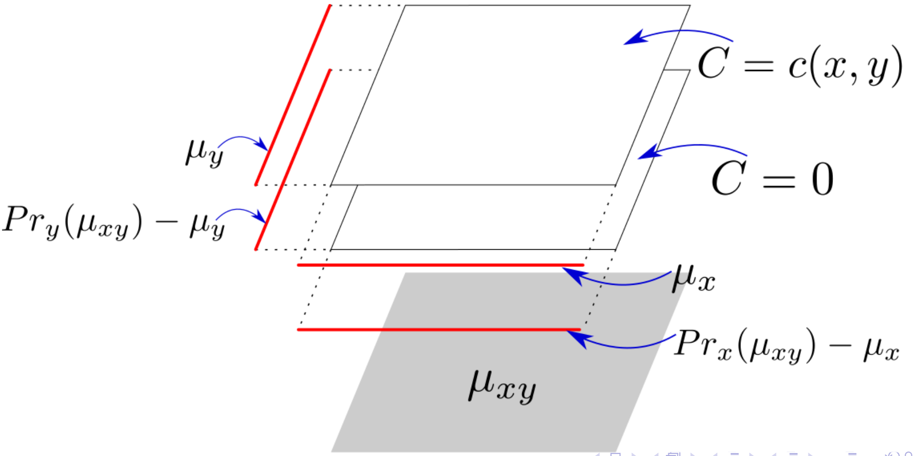
where the optimal solution is given by the measure

$$\left[\begin{array}{ccc|ccc|ccc} 0.5 & 0 & 0.5 & 0 & 0.5 & 0.5 & 0.5 & 0.5 & 0 \\ 0 & 1 & 0 & 0.5 & 0 & 0.5 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0.5 & 0.5 & 0 & 0 & 0.5 & 0.5 \end{array}\right].$$

Any Graeco-Latin square $n \times n$ corresponds to a measure on $\{1,\dots,n\}^4$ with fixed 2-dimensional marginals.

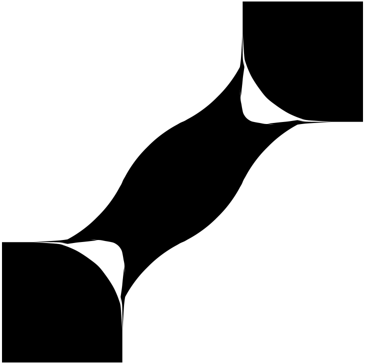
# Motivation for $(3,2)-$problem.

The Monge-Kantorovich problem with limited density was studied in (Korman, McCann) 2012. One needs to find a measure $\pi \in \Pi(\mu_x, \mu_y)$ not greater than $\mu_{xy}$ minimizing the integral of $c(x,y)$. It can be shown to be a particular case of the $(3,2)$-problem.



# Motivation for $(3, 2)-$problem.

For the base space $I = [0, 1]$ with the Lebesgue measure and cost function $c(x, y) = (x - y)^2$, with density not greater than 3, the solution is given by the following picture:



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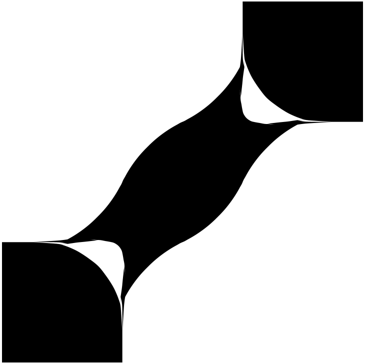


Minimization of $\int (x - y)^2 \, d\pi$ on the set of measures $\Pi(\mu, \nu)$ with fixed marginals $\mu$, $\nu$ is equivalent to the maximization of $\int xy \, d\pi$ on the same set. Therefore, the cost functions $-xyz$ and $xyz$ are natural analogues for $(x - y)^2$.

# Primal $(3,2)-$problem with the cost function $xyz$.

## Primal problem

Consider the $(3,2)$-problem on $I^3$ with Lebesgue measures on the coordinate planes. Our goal is to find a measure $\pi$ minimizing $P(\pi) = \int xyz\, d\pi$.

Let $T_x : \mathbb{R}^3 \to \mathbb{R}^3$ be an involution such that

$$T_x(x,y,z) = (1-x,y,z).$$

$T_y$ and $T_z$ are defined in the same way. Then for any measure $\mu$ with Lebesgue marginals,

$$P(\mu \circ T_x) = \int xyz\, d(\mu \circ T_x) = \int (1-x)yz\, d\mu = \int yz\, d\mu_{yz} - \int xyz\, d\mu = \frac{1}{4} - P(\mu).$$

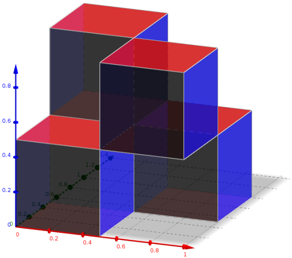
Thus, involutions $T_x \circ T_y$, $T_x \circ T_z$, and $T_y \circ T_z$ do not change $P$, and the primal solution $\pi$ can be assumed to be invariant under these involutions.

# Primal $(3,2)-$problem with the cost function $xyz$.

$$S_1 = \left[0, \frac{1}{2}\right]^3 \bigcup \left[\frac{1}{2}, 1\right]^2 \times \left[0, \frac{1}{2}\right] \bigcup \left[0, \frac{1}{2}\right] \times \left[\frac{1}{2}, 1\right]^2 \bigcup \left[\frac{1}{2}, 1\right] \times \left[0, \frac{1}{2}\right] \times \left[\frac{1}{2}, 1\right].$$

From the symmetries of $\pi$ under $T$'s, one can obtain $\int_{I^3 - S_1} xyz \; d\pi \geq \int_{I^3 - S_1} (1 - x)(1-y)(1-z) \; d\pi$, so the optimal $\pi$ is concentrated on $S_1$. By the same argument, $\mu(S_k) = 1$.

$$S_k = \bigcup_{\substack{a \oplus b \oplus c = 0 \\ 0 \leq a,b,c < 2^k}} \left[\frac{a}{2^k}, \frac{a+1}{2^k}\right] \times \left[\frac{b}{2^k}, \frac{b+1}{2^k}\right] \times \left[\frac{c}{2^k}, \frac{c+1}{2^k}\right]$$



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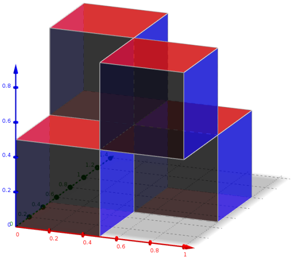


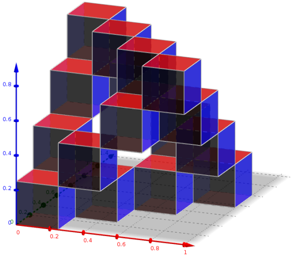

# Primal $(3,2)-$problem with the cost function $xyz$.

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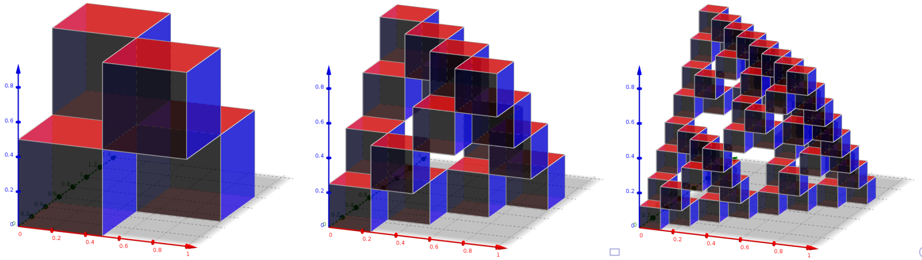
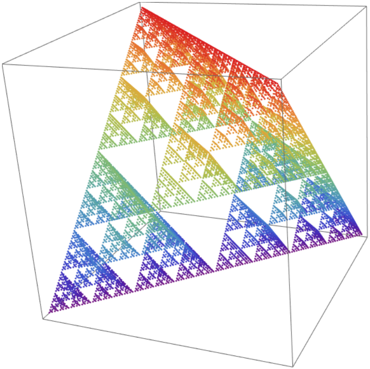


# Support of the optimal measure

The optimal measure $\pi$ is concentrated on $S = \bigcap S_k = \{(x, y, z) \in X \times Y \times Z \mid x \oplus y \oplus z = 0\}$, which forms the Sierpiński tetrahedron. Although $S$ is highly non-smooth, it is a graph of the function $z = x \oplus y$.



Sierpiński tetrahedron

# Dual $(3,2)-$problem with the cost function $xyz$.

## Dual problem

Find $f(x,y), g(x,z), h(y,z) \in L^1(\mu_{I^2})$ such that:

- $f(x,y)+g(x,z)+h(y,z) \le xyz$ for all $(x,y,z) \in X \times Y \times Z$,
- $D(f,g,h) = \int_{X\times Y} f \; d\mu_{xy} + \int_{X\times Z} g \; d\mu_{xz} + \int_{Y\times Z} h \; d\mu_{yz}$ is maximal.

**Remark:** Due to the symmetry of $c(x,y,z)$, one can assume $f=g=h$.

Assume $I(a,b) = \int_0^a \int_0^b x \oplus y \; dx \; dy$.

$$f(x,y) = I(x,y) - \frac{1}{4} I(x,x) - \frac{1}{4} I(y,y).$$

**How can one guess the answer?**
**Answer 1:** Recurrent relations;
**Answer 2:** Assume the existence of $\frac{\partial^2 f}{\partial x \partial y}$. By considering points $(x+\delta_1 x, y+\delta_1 y)$ such that $(x+\delta_1 x) \oplus (y+\delta_1 y) = x \oplus y$ and $(x-\delta_2 x, y+\delta_2 y)$ such that $(x-\delta_2 x) \oplus (y+\delta_2 y) = x \oplus y$, one can conclude that $\frac{\partial^2 f}{\partial x \partial y} = x \oplus y$.

# Dual $(3,2)-$problem with the cost function $xyz$.

The duality gives us some identities:

$$I(x,y)+I(x,z)+I(y,z)-\frac{1}{2}(I(x,x)+I(y,y)+I(z,z))\leq xyz,$$

where equality holds for $x\oplus y\oplus z=0$.

# Dual $(3,2)-$problem with the cost function $xyz$.

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$$I(x,y)+I(x,z)+I(y,z)-\frac{1}{2}(I(x,x)+I(y,y)+I(z,z))\leq xyz,$$

where equality holds for $x\oplus y\oplus z=0$.

By differentiating that with respect to $z$, we obtain

$$\int_0^x x\oplus y\oplus t\ dt+\int_0^y x\oplus y\oplus t\ dt-\int_0^{x\oplus y} x\oplus y\oplus t\ dt=xy.$$

# Dual $(4,3)-$problem with the cost function $xyzt$.

For the $(4,3)$-problem and the function

$$I(x,y,z)=\int_{[0,x]\times[0,y]\times[0,z]} u\oplus v\oplus w \; dudvdw,$$

there exists an identity:

$$\begin{aligned} xyzt = {} & I(x,y,z)+I(x,y,t)+I(x,z,t)+I(y,z,t) \\ & -\frac{1}{2}I(x,x,y)-\frac{1}{2}I(x,y,y)-\frac{1}{2}I(x,x,z)-\frac{1}{2}I(x,z,z) \\ & -\frac{1}{2}I(x,x,t)-\frac{1}{2}I(x,t,t)-\frac{1}{2}I(y,y,z)-\frac{1}{2}I(y,z,z) \\ & -\frac{1}{2}I(y,y,t)-\frac{1}{2}I(y,t,t)-\frac{1}{2}I(z,z,t)-\frac{1}{2}I(z,t,t) \\ & +\frac{1}{2}I(x,x,x)+\frac{1}{2}I(y,y,y)+\frac{1}{2}I(z,z,z)+\frac{1}{2}I(t,t,t) \\ & -\frac{1}{8}(x^4+y^4+z^4+t^4)+\frac{1}{4}(x^2y^2+x^2z^2+x^2t^2+y^2z^2+y^2t^2+z^2t^2) \end{aligned}$$

for $x\oplus y\oplus z\oplus t=0$.

# Dual $(4,3)-$problem with the cost function $xyzt$.

For the $(4,3)$-problem and the function

$$I(x,y,z) = \int_{[0,x]\times[0,y]\times[0,z]} u \oplus v \oplus w \ dudvdw,$$

there exists an identity:

$$\begin{aligned}
xyzt = {} & I(x,y,z) + I(x,y,t) + I(x,z,t) + I(y,z,t) \\
& - \frac{1}{2}I(x,x,y) - \frac{1}{2}I(x,y,y) - \frac{1}{2}I(x,x,z) - \frac{1}{2}I(x,z,z) \\
& - \frac{1}{2}I(x,x,t) - \frac{1}{2}I(x,t,t) - \frac{1}{2}I(y,y,z) - \frac{1}{2}I(y,z,z) \\
& - \frac{1}{2}I(y,y,t) - \frac{1}{2}I(y,t,t) - \frac{1}{2}I(z,z,t) - \frac{1}{2}I(z,t,t) \\
& + \frac{1}{2}I(x,x,x) + \frac{1}{2}I(y,y,y) + \frac{1}{2}I(z,z,z) + \frac{1}{2}I(t,t,t) \\
& - \frac{1}{8}(x^4 + y^4 + z^4 + t^4) + \frac{1}{4}(x^2y^2 + x^2z^2 + x^2t^2 + y^2z^2 + y^2t^2 + z^2t^2)
\end{aligned}$$

for $x \oplus y \oplus z \oplus t = 0$.

We don't know if there exist analogous identities in higher dimensions.

# $(3,1)-$problem with the cost function $-xyz$.

## Primal problem

Our goal is to find a measure $\pi$ with Lebesgue projections to the axes that minimizes $P(\pi) = \int -xyz\ d\pi$.

## Dual problem

Find $f(x), g(y), h(z) \in L^1(\mu_I)$ such that:

- $f(x) + g(y) + h(z) \leq xyz$ for all $(x, y, z) \in X \times Y \times Z$,
- $D(f, g, h) = \int_X f\ d\mu_x + \int_Y g\ d\mu_y + \int_Z h\ d\mu_z$ is maximal.

**Remark:** Due to the symmetry of $c(x, y, z)$, one can assume $f = g = h$.

This case is trivial. Let $\pi$ be the uniform probability measure on $\{(t, t, t)\}$. Let $f(t) = g(t) = h(t) = -\frac{t^3}{3}$.

It is easy to see that the projections of $\pi$ onto the axes are Lebesgue. Also, $f(x) + g(y) + h(z) \leq xyz$. Moreover, equality is achieved on the support of $\pi$. Then, by the *complementary slackness* condition, $\pi$ is the solution to the primal problem, and $(f, g, h)$ is the solution to the dual problem.

# $(3,1)-$problem with the cost function $xyz$.

## Primal problem

Our goal is to find a measure $\pi$ with Lebesgue projections to the axes that minimizes $P(\pi) = \int xyz \; d\pi$.

## Dual problem

Find $f(x) \in L^1(\mu_I)$ such that:

- $f(x) + f(y) + f(z) \leq -xyz$ for all $(x, y, z) \in X \times Y \times Z$,
- $D(f) = 3 \int_X f \; d\mu_x$ is maximal.

The key idea is the same as in the previous problem: One needs to guess the primal and dual solutions and then check the complementary slackness.

# Guess for dual problem

Let $l$ be the unique root of

$$9l + \ln(1-2l) - \ln l - 3 = 0,$$

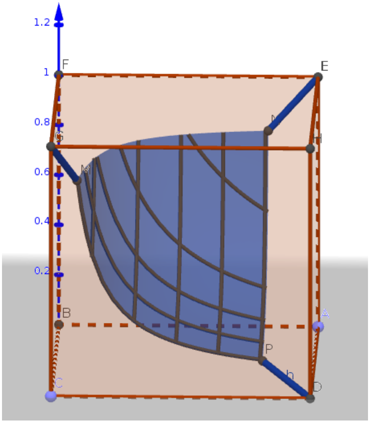
lying inside $\left(0, \frac{1}{6}\right)$. Denote $r = 1 - 2l$, $c = lr^2$. Define $f$ as follows:

$$f(x) = \begin{cases} c\ln l - \frac{1}{3}(c\ln c - c) + \frac{1}{6}((2x-1)^3 - (2l-1)^3), & \text{for } 0 \le x \le l, \\ c\ln x - \frac{1}{3}(c\ln c - c), & \text{for } l \le x \le r, \\ c\ln r - \frac{1}{3}(c\ln c - c) + \frac{1}{4}(x^2 - r^2) - \frac{1}{6}(x^3 - r^3), & \text{for } r \le x \le 1. \end{cases}$$

There holds $f(x) + f(y) + f(z) \le xyz$, with equality in $xyz = c$ and $l \le x, y, z \le r$, or in $y = z = 1 - 2x$ and $0 \le x \le l$, or in $x = z = 1 - 2y$ and $0 \le y \le l$, or in $x = y = 1 - 2z$ and $0 \le z \le l$.

# Support of the optimal primal solution

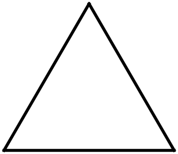
Let $M$ be the set of points where equality holds in the dual guess. For $\mu$ with Lebesgue projections to be the primal solution, it is sufficient to check that $\mu(M) = 1$. Here, we get the transport problem with fixed support [Zaev, 2014].

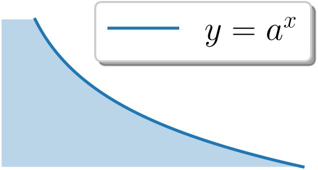



Points of equality / measure support

# Primal guess

$M$ is decomposed into 3 segments and a part of the surface $xyz = c$. The measure on the segments is uniform. After changing the coordinates, finding a measure on $xyz = c$ with uniform projections is reduced to the following problem: find a measure on $x + y + z = 2$, $0 \leq x, y, z \leq 1$, with projection onto each axis equal to $\alpha^x \, dx$, where $\alpha = \frac{r}{l}$.






The only $\alpha$ admitting such a measure is $\frac{1-2l}{l}$, where $9l + \ln(1 - 2l) - \ln l - 3 = 0$. For $\alpha = \frac{r}{l}$, there exist infinitely many such measures, but finding one is a difficult problem.

# Primal guess

One example of a measure on $x+y+z=2$, $0 \leq x, y, z \leq 1$ with exponential projections has density at the point $(x, y, z)$ equal to

$$\frac{\alpha^{1-t} - 4\alpha^{2t}}{1-3t} - 6\frac{2\alpha^{2t} + \alpha^{1-t}}{(1-3t)^2 \ln \alpha} - 18\frac{\alpha^{2t} - \alpha^{1-t}}{(1-3t)^3 \ln^2 \alpha},$$

where $t = \min(1-x, 1-y, x+y-1)$ and linear density in the points $(1-2t, 1-2t, 4t)$, $(1-2t, 4t, 1-2t)$, and $(4t, 1-2t, 1-2t)$ is equal to

$$\alpha^{2t} + 2\frac{2\alpha^{2t} + \alpha^{1-t}}{(1-3t)\ln \alpha} + 6\frac{\alpha^{2t} - \alpha^{1-t}}{(1-3t)^2 \ln^2 \alpha}.$$