A lower bound on forcing numbers based on height functions

Based on joint work with Fateh Aliyev

Nikita Gladkov, UCLA

ORMC, December 8, 2024

June handout on tilings

Tiling figures with polyominoes

ORMC

06/02/24

1 Intro to tilings

Problem 1.1. Can you cut a 10×10 board into 1×3 rectangles? In this and the following problems you can rotate the pieces as you want.

Diego problem

2 Olympiad problems

Problem 2.1 (Moscow 2020)**.** Ten cells have been cut out from a chessboard. It is known that among the cut-out cells, there are both black and white cells. What is the maximum number of two-cell rectangles that can still be guaranteed to be cut out from this board?

Problem 2.2 (Simplified from Peru 2012)**.** Diego wants to cover a 6×6 square with eighteen dominoes. Can he place 3 dominoes (without overlaps) so that the remaining part of the grid can be covered with the remaining dominoes in exactly one way?

Problem 2.3 (Tournament of Towns, 1994)**.** Consider an 8×8 square on a plane, divided into 1×1 cells. The square is covered with right isosceles triangles (two triangles cover one cell). There are 64 black and 64 white triangles. We consider "correct" coverings such that any two triangles sharing a side are of different colors. How many correct coverings are there?

Let's play a bit

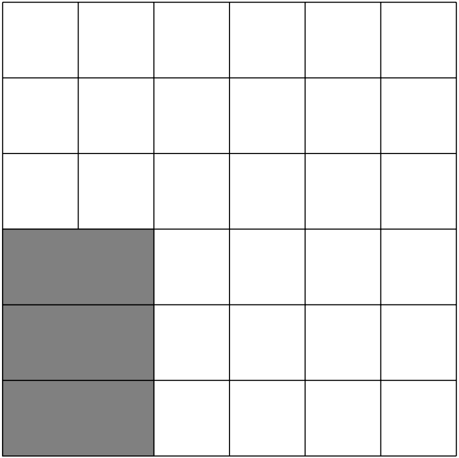

Figure: Too many ways to complete the grid.

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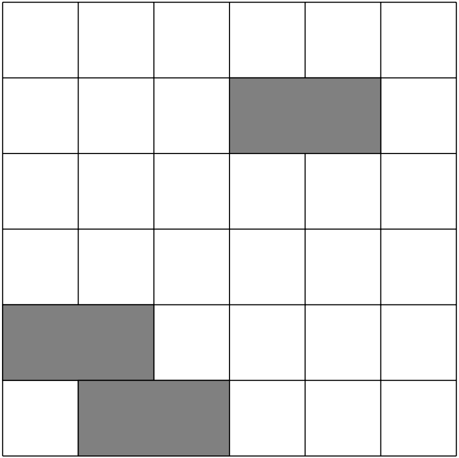

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Solution

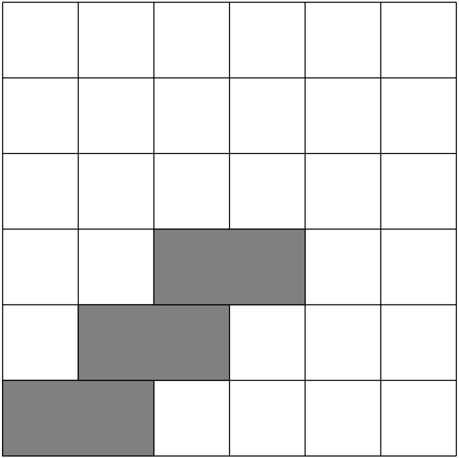
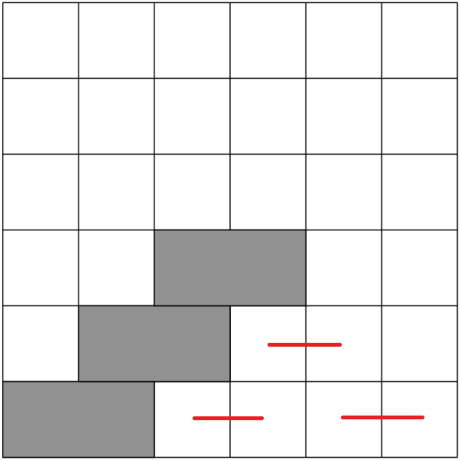


Figure: Just one way to complete the grid.

Solution



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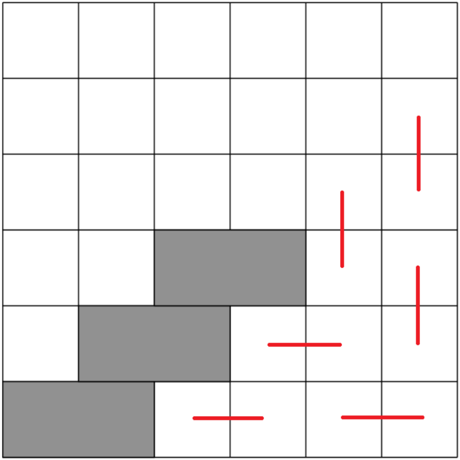

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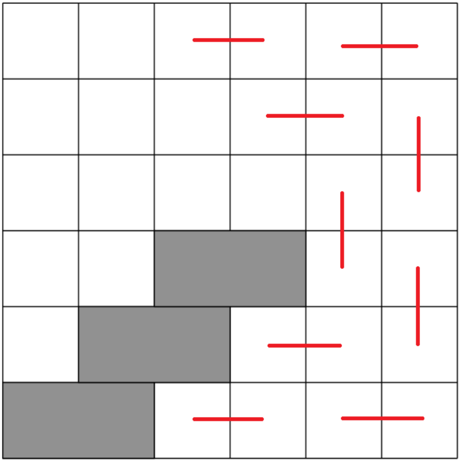

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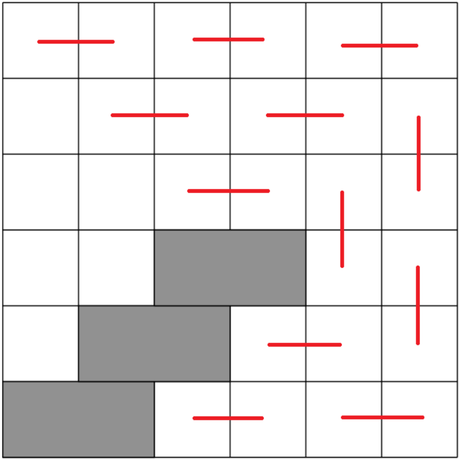

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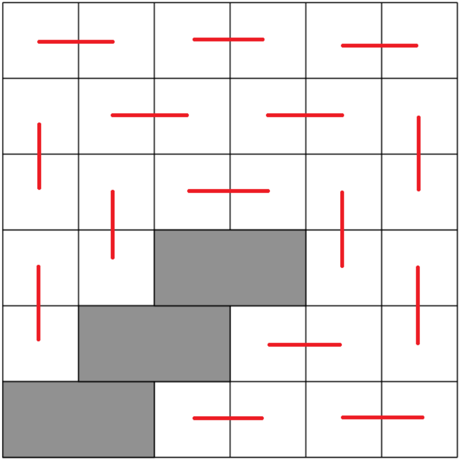

Figure: Just one way to complete the grid.

Email that started the chain

FATEH ALIYEV <faliyer0001@mymail.lausd.net>
to me

Hello Nikita,
On the final week's packet there was a problem about Diego placing 3 dominos on a 6x6 board so that the rest of domino tiling is completely determined (says it's simplified from Peru 2012). I was wondering if you could link me to the original problem. I looked online but I can't seem to find it.
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The original problem asked "What is the smallest positive integer k for which one can place down k dominoes on the board in such a way such that the remainder of the 6×6 board can be covered in a unique way?"

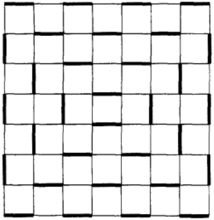

Fig. 2.

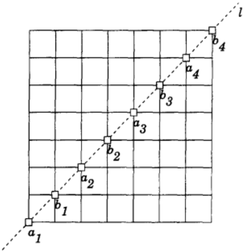

Fig. 3.

Pachter–Kim proof

Theorem
The forcing number of a $2n$ by $2n$ square is n.

Pachter–Kim proof

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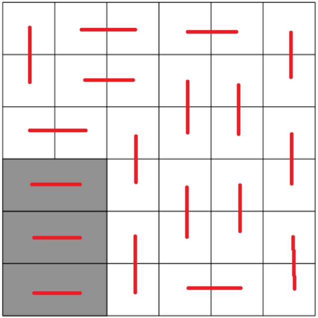
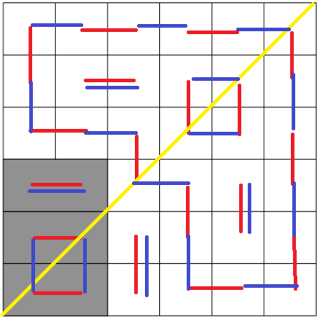
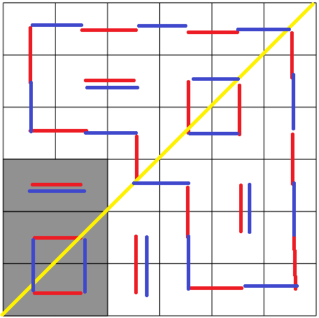


Pachter–Kim proof



If for the $2n \times 2n$ board less than n dominoes are fixed, there is another tiling obtained by a color switch in a cycle without a fixed domino.

Pachter–Kim proof

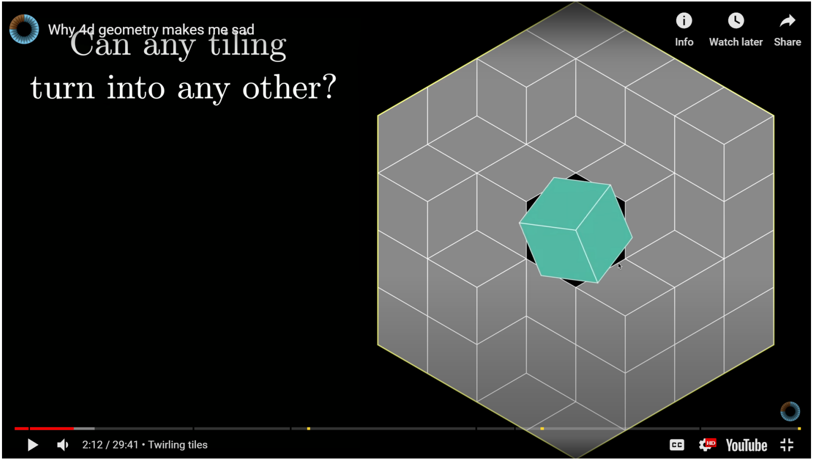


If for the $2n \times 2n$ board less than n dominoes are fixed, there is another tiling obtained by a color switch in a cycle without a fixed domino.

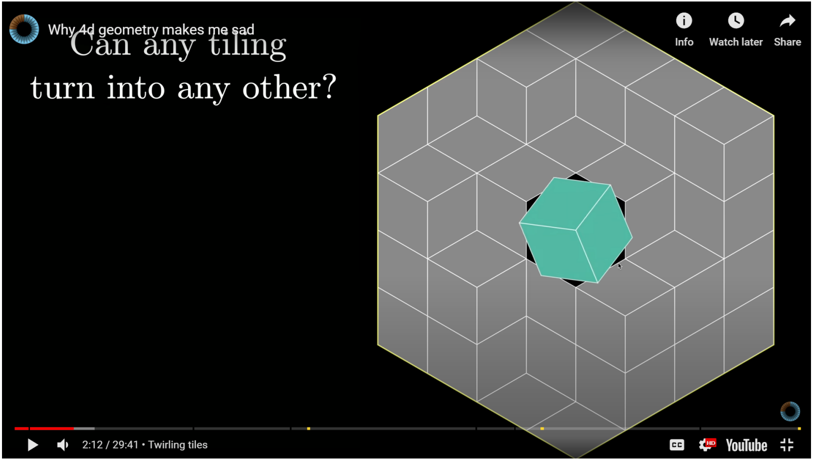
Remark

This argument can be used to show that the number of tilings of a $2n \times 2n$ board is divisible by 2^n and moreover after dividing it becomes a perfect square.

Tilings of a hexagon by lozenges

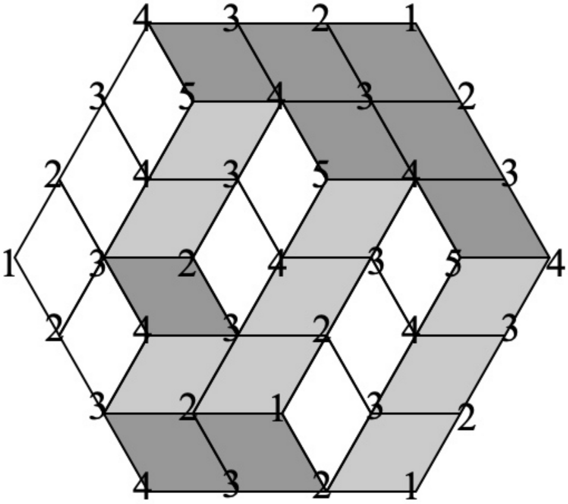


Tilings of a hexagon by lozenges



> I don't even have to squint my eyes here: it's harder for me not to see it as a cube stack.

Conway–Thurston height function



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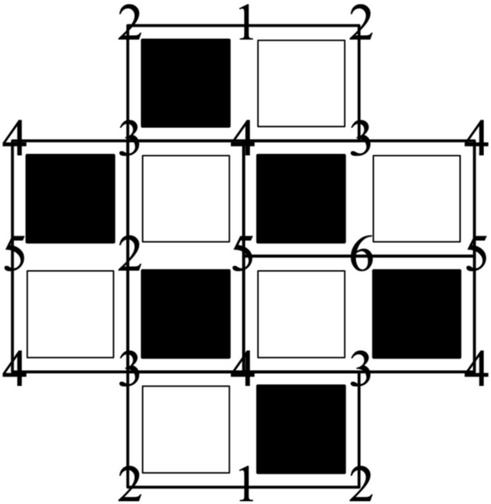

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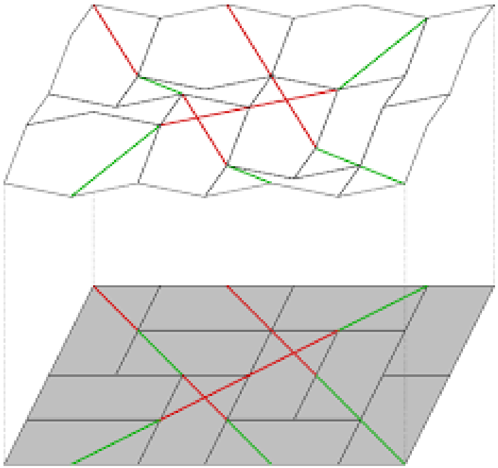


Our theorem

Theorem 1.1. The forcing number of a region R on the square lattice is bounded below by

$$\frac{1}{4}\max_{x\in R}\left(h_{\max}(x) - h_{\min}(x)\right).$$

We also derive a similar bound for the triangular lattice.

Theorem 1.2. The forcing number of a region R on the triangular lattice is bounded below by

$$\frac{1}{3}\max_{x\in R}\left(h_{\max}(x) - h_{\min}(x)\right).$$

Example

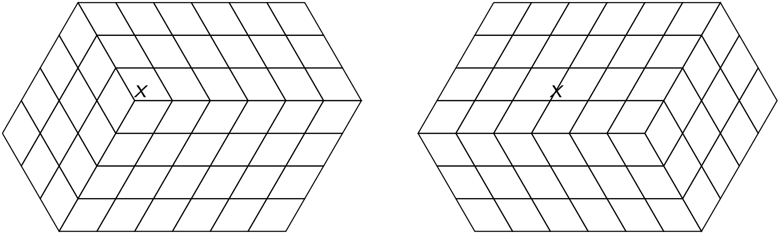
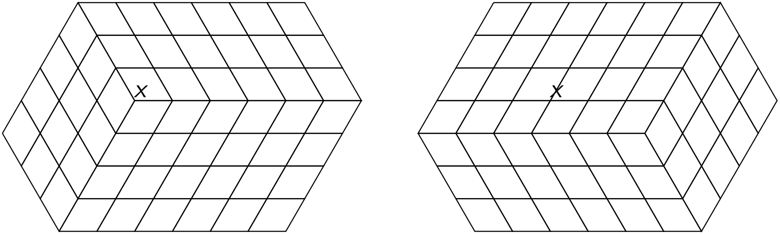


Figure: Minimal and maximal tilings of a hexagon with sides 3, 4, 6 with lozenges.

Example



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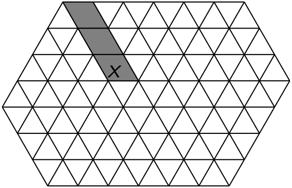

Figure: Forcing set for a hexagon with sides 3, 4, 6.

Thank you for attention!

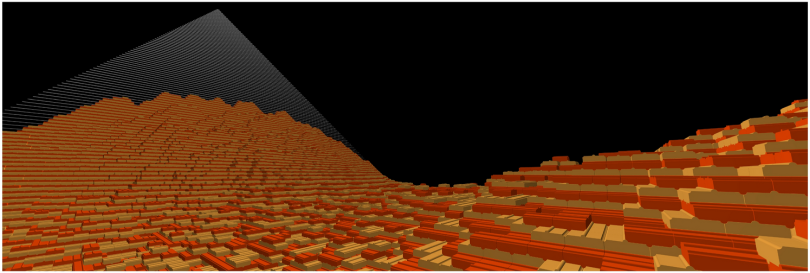

Figure: A postcard from an Aztec diamond world by Ben Young