RESEARCH STATEMENT

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Introduction and Definitions

Our research lies at the intersection of probability theory and combinatorics, focusing on percolation models and related inequalities. Percolation theory studies random subgraphs generated by retaining vertices or edges with certain probabilities and aims to analyze connectivity properties in these random structures. The main type of percolation we consider is Bernoulli *bond percolation*, where edges are randomly retained. There is also Bernoulli *site percolation*, where vertices are retained with a given probability. We study the relationship between them in [GZ24a].

For a given graph $G = (V, E)$, bond percolation assigns each edge $e \in E$ an independent probability p_e of being open (retained). The primary question of interest is the connectivity between different vertices in the resulting random subgraph.

Definition 1. We denote by "$v_{11}v_{12}\dots v_{1i_1}|v_{21}\dots v_{2i_2}|\dots|v_{n1}\dots v_{ni_n}$" the event that the vertices $v_{11}, \dots, v_{1i_1} \in V$ belong to the same cluster, vertices $v_{21}, \dots, v_{2i_2}$ belong to the same cluster, ..., vertices $v_{n1}, \dots, v_{ni_n}$ belong to the same cluster, and, moreover, these clusters are all different. By

$$\mathbf{P}(v_{11}v_{12}\dots v_{1i_1}|v_{21}\dots v_{2i_2}|\dots|v_{n1}\dots v_{ni_n})$$

we denote the probability of this event in the underlying bond percolation. In particular, $\mathbf{P}(abc)$ denotes the probability that vertices a, b, $c \in V$ lie in the same cluster, and $\mathbf{P}(a|b|c)$ is the probability that a, b and c belong to 3 different clusters.

Percolation models have applications across a variety of fields, including statistical physics, network theory, and computer science. A significant portion of our work focuses on deriving inequalities for connection probabilities and extending them to more complex models, such as hypergraph percolation.

Sources of Inequalities

Several techniques have been developed to prove inequalities concerning connection probabilities in percolation models. Below are the key sources of inequalities for this probability. In our research, we have improved on each of these methods.

Induction on the number of edges: The *Harris–Kleitman (HK)* inequality is a classical result used to prove correlation inequalities for percolation events. Inductive proofs of this inequality apply to both edge and bond percolation, as well as *full hypergraph percolation*, considered in [GZ24a]. In [G24a], we use this induction to obtain an inequality conjectured by Erik Aas. In an upcoming paper [GZ24b], we prove this inequality in a much more general form. In [GP24a], we introduce *colored percolation* and extend the inductive method to it, proving a generalization of the HK inequality.

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Extending events to $\prod_e \Omega_1(e) \cup \Omega_2(e)$ + induction: This technique first appeared to prove the *van den Berg-Kesten (vdBK)* inequality for increasing events.

Theorem 2 ([BK85])**.** *For any increasing events $A, B \subseteq H_n$, one has*

$$\mathbf{P}(A \,\square\, B) \le \mathbf{P}(A)\mathbf{P}(B), \tag{1}$$

where the disjoint occurrence of A and B, *denoted by $A \,\square\, B$, is*

$$A \,\square\, B := \{C \in \Omega, \ \text{s.t. there exist } I, J \subset E \\ \text{s.t. } I \text{ is a witness of } A \text{ in } C, \ J \text{ is a witness of } B \text{ in } C \text{ and } I \cap J = \varnothing\}.$$

The proof introduces additional events $A \square_k B$ on the probability space where the first k edges are sampled once, and the rest of the edges are sampled twice. The proof proceeds to show inductively that

$$\mathbf{P}(A)\mathbf{P}(B) = \mathbf{P}(A \,\square_0\, B) \ge \mathbf{P}(A \,\square_1\, B) \ge \cdots \ge \mathbf{P}(A \,\square_{|E|}\, B) = \mathbf{P}(A \,\square\, B).$$

The same proof shows that the vdBK inequality can be extended to handle union events:

$$\mathbf{P}(A_1 \,\square\, B_1 \cup \cdots \cup A_n \,\square\, B_n) \le \mathbf{P}(A_1 \times B_1 \cup \cdots \cup A_n \times B_n). \tag{2}$$

In [G24b, Section 7], we extend this method to:

- Work in the decision tree setup described below;
- Reprove the colored generalization of the HK inequality;
- Give a stronger version of inequalities (1) and (2) with $\square$ replaced by $\bowtie$, which is the same as $A \,\square\, B$, but there is an extra probability p^2 for an edge to turn into a double edge that can be used in both witnesses.

Linear Algebraic Method: The inequality (1) applies to arbitrary events in percolation. However, it stood for a long time as a conjecture until the work of Reimer [R00]. He proved the conjecture using a linear algebraic argument. His proof is not injective, as the key step involves an inequality between the dimensions of a vector space and its subspace. Since then, inequality (1) is sometimes called the *van den Berg-Kesten-Reimer (vdBKR)* inequality. This work led to the dual version of the inequality, which allowed the resolution of Rudich's conjecture [KSS00]. This method produces inequalities for connection events that do not lie in the convex hull of those obtained by other means. For example, it shows:

$$\mathbf{P}(ab|cd) \le \mathbf{P}(ab)\mathbf{P}(a|cd \cup b|cd).$$

We discovered several additional unexpected relationships between non-intersecting witnesses (see [G24+]) and used ideas from Reimer's proof to derive inequalities not implied by his result itself. For example:

$$\mathbf{P}(abc \cup abd \cup acd \cup bcd)\mathbf{P}(b|c \text{ and } a|d) \ge \mathbf{P}(ac|bd \cup ac|b|d \cup a|bd|c)\mathbf{P}(bc|d \cup ad|c \cup ad|b \cup a|bc).$$

Ahlswede–Daykin Inequality: The *Ahlswede-–Daykin (AD)* inequality is proved by induction, but the induction step involves division, which is hard to avoid. We show in [G24+] that the AD inequality doesn't admit a combinatorial proof in the sense of [IP22].

The AD inequality was used as a tool in the first proof of (3), but it can also derive correlation bounds for connection events inaccessible through other methods, such as

$$\mathbf{P}(a[bcd] \text{ and } b|c|d)\mathbf{P}(a|bcd) \le \mathbf{P}(abcd)\mathbf{P}(a|b|c|d).$$

Markov Chains and Uniform Measures: Using Markov chains, the paper [BHK06] (see also [H07]) shows a way to transition to uniform measures on conditional sets $S|T$. For instance, if we take $S = \{a\}$ and $T = \{d\}$, it derives inequalities such as:

$$\mathbf{P}(abc|d)\mathbf{P}(a|d) \geq \mathbf{P}(ab|d)\mathbf{P}(ac|d), \tag{3}$$

first obtained in [BK01], and:

$$\mathbf{P}(ab|cd)\mathbf{P}(a|d) \leq \mathbf{P}(ab|d)\mathbf{P}(ac|d). \tag{4}$$

In an upcoming work [GZ24b], we extend these inequalities using inductive techniques and show how inequality (4) implies the Bunkbed Conjecture with two transversal vertices – a goal stated but not realized in [BK01].

Decision Tree Techniques: The first proofs of the *RSW theorem* [R78, R81, SW78] and Smirnov's proof of it for critical site percolation on a triangular lattice [W07, Chapter 1.3] all, in one form or another, went through the following argument (quoting [T16]): "assuming that a left-right crossing exists in a square, one can first find the lowest one by exploring the region below it. Then the configuration can be sampled independently in the unexplored region (above the path)". In [GZ24a], we combined independent sampling in unexplored regions with the decision tree techniques used in the *OSSS inequality* [OSSS05] (see also [K20]). By itself, these inequalities only give useful estimates on structured graphs such as lattices or Cayley graphs, but we show in [GZ24a, G24b] that this combination is invaluable for proving inequalities for connection events in percolation.

Results

In this section, we present the main results of our recent works on developing inequalities for connectivity events in percolation theory.

A Strong FKG Inequality for Multiple Events In [G24a], we extend the classical FKG inequality to handle multiple events with equal pairwise intersections. This result has significant implications for correlation inequalities in random graphs and Bernoulli percolation. The main result is:

$$\mu(A)\mu(B) \geq e_2(\mu(C_1), \ldots, \mu(C_k)), \tag{5}$$

where

- Sets A, B and C_i's form a disjoint partition of a hypercube H_n;
- A is closed upwards;
- B is closed downwards;
- Any two elements from C_i and C_j for $i \neq j$ are incomparable. Alternatively, this condition means all sets $A \cup C_i$ are closed upwards.

This generalization resolves several open questions in the field, including a conjecture by Kahn [K22] and a conjectured inequality by Erik Aas:

$$\mathbf{P}(abc)\mathbf{P}(a|b|c) \geq \mathbf{P}(ab|c)\mathbf{P}(ac|b) + \mathbf{P}(ab|c)\mathbf{P}(a|bc) + \mathbf{P}(ac|b)\mathbf{P}(a|bc). \tag{6}$$

Positive Dependence for Colored Percolation In collaboration with I. Pak, in [GP24a], we study percolation on graphs where each edge is assigned a random color. We prove positive mutual dependence for specific combinations of colors and negative mutual dependence for others. These results extend the classical Harris-Kleitman inequality to colored percolation models. We introduce a new critical probability and conjecture that it is expressible through the critical probability for bond percolation.

Bond Percolation Does Not Simulate Site Percolation In collaboration with A. Zimin, in [GZ24a] we show that bond percolation cannot fully simulate site percolation, even approximately. In site percolation as well as full hypergraph percolation, it is easy to have three vertices a, b and c such that $\mathbf{P}(abc)$ and $\mathbf{P}(a|b|c)$ are both equal to 0.5. In the paper, we introduced the decision tree independence lemma with the following definition:

Definition 3. For two percolation configurations $C_1, C_2 \in \Omega$ and a set $S \subseteq E$, we denote by $C_1 \to_S C_2$ the configuration that coincides with C_1 on S and C_2 on its complement $\overline{S}$.

$$C_1 \underset{S}{\to} C_2(e) = \begin{cases} C_1(e), & \text{if } e \in S, \\ C_2(e), & \text{otherwise.} \end{cases} \tag{7}$$

With this definition:

Lemma 4 ([GZ24a, Lemma 4.2])**.** *Let graph G be finite and its edges assigned probabilities. Let T be a decision tree that opens edges in C_1 and C_2 for percolation configurations C_1 and C_2 on G. Assume that in the process, T builds a set $S(C_1, C_2)$ (see* [G24b, Definition 2.4, Algorithm 1] *for the formal definition). Then $C_1 \to_S C_2$ is independent of $C_2 \to_S C_1 = C_1 \to_{\bar{S}} C_2$, and both are distributed as bond percolation on G.*

Using this lemma, for bond percolation we obtain the inequality

$$\mathbf{P}(a|b \cap a|c)\mathbf{P}(ab \cup ac) \le \mathbf{P}(ab|c) + \mathbf{P}(ac|b) + \mathbf{P}(a|bc). \tag{8}$$

This inequality prohibits $\mathbf{P}(abc)$ and $\mathbf{P}(a|b|c)$ from being simultaneously greater than 0.37586. In the proof, we use not one decision tree, as in the proofs of the RSW theorem, but three, and consider the interplay between the configurations obtained from them.

Percolation Inequalities and Decision Trees The work [G24b] extends the previous paper and develops the decision tree techniques further. By analyzing how decision trees query edges and explore the graph, we derive new bounds for connection probabilities. We extend the decision tree techniques with the HK and the vdBK inequalities and obtain helpful inequalities on probabilities of events of the form $C_1 \to_S C_2 \in A$. We also show how to use the Cauchy–Schwarz inequality to bound the probabilities of events depending on multiple configurations. This has led to novel insights into connectivity and mutual dependence. The results obtained give bounds on the monochromatic arm exponents and the three-point exponent. The showcase inequality for the paper is

$$\mathbf{P}(abc)^2 \le 8\mathbf{P}(ab)\mathbf{P}(ac)\mathbf{P}(bc). \tag{9}$$

According to the conjectured integral formula from [DV11][1], for the critical regime on $\mathbb{Z}^2$, the quantity $\frac{\mathbf{P}(abc)}{\sqrt{\mathbf{P}(ab)\mathbf{P}(ac)\mathbf{P}(bc)}}$ converges to approximately 1.022 as a, b and c diverge from each other. This number is consistent with our upper bound of $\sqrt{8}$.

The Bunkbed Conjecture is False In collaboration with I. Pak and A. Zimin, in [GPZ24] we disprove the long-standing Bunkbed Conjecture. It is a celebrated open problem in probability, introduced by Kasteleyn in 1985 (see [BK01, Remark 5]). The conjecture is both natural and intuitively obvious but has defied repeated proof attempts.

Fix a finite connected graph $G = (V, E)$ and a subset $T \subseteq V$. A *bunkbed graph* $\overline{G} = (\overline{V}, \overline{E})$ is a subgraph of the graph product $G \times K_2$ defined as follows. Take two copies of G, denoted by G and $G' = (V', E')$, and add all edges of the form (w, w'), where $w \in T$ and w' is the corresponding vertex in T'. We denote this set of edges by $\overline{T}$. The resulting bunkbed graph has $\overline{V} = V \cup V'$ and $\overline{E} = E \cup E' \cup \overline{T}$.

[1] The proof of this formula for critical site percolation on a triangular lattice was recently announced by Morris Ang, Gefei Cai, Xin Sun, and Baojun Wu (personal communication, 10 Aug 2024)

In the *bunkbed percolation*, bond percolation is performed only on edges in G and G', while all edges in $\overline{T}$ are retained (i.e., not deleted).

Conjecture 5 (Bunkbed Conjecture)**.** Let $G = (V, E)$ be a connected graph, let $T \subseteq V$, and let $0 < p < 1$. Then, for all $u, v \in V$, we have:

$$\mathbf{P}_p[u \leftrightarrow v] \geq \mathbf{P}_p[u \leftrightarrow v'].$$

Our work constructs a counterexample to this conjecture, based on earlier work by L. Hollom [Hol24], showing that there exist graphs for which this inequality fails. This result sheds light on surprising and previously unknown behaviors of connectivity in random graph structures.

The preprint garnered significant attention from the mathematical community, including a tweet by Sir Timothy Gowers and a YouTube video by Dr. Trefor Bazett.

Directions for Future Work

Questions from the Previous Papers There are questions left unanswered in our previous papers. The most pressing one is the following conjecture, which is akin to the stability property of Harris–Kleitman inequality:

Conjecture 6 ([GZ24a, Conjecture 6.3])**.** For any $\varepsilon > 0$ there exists $\delta > 0$ such that if $\mathbf{P}(ab|c) < \delta$, then

$$\mathbf{P}(abc) - \mathbf{P}(ac)\mathbf{P}(bc) < \varepsilon.$$

I have a line of attack that suggests that $\varepsilon = O(\delta \log(\delta))$, at least for planar graphs. The sequence of graphs that achieves this bound was used in the gadget [GPZ24, Section 4.1] that played the key role in the disproof the Bunkbed Conjecture.

Another question is the gap in the Bunkbed Conjecture. The best gap between $\mathbf{P}_p[u \leftrightarrow v]$ and $\mathbf{P}_p[u \leftrightarrow v']$ we could achieve is of the order 10^{-40}. With three posts, we can prove that this gap is at most 0.06. For the general case, we are interested in a question by T. Hutchcroft and M. Aizenman, whether there is an example with more posts where this gap can be arbitrarily close to 1 or where the ratio of the probabilities can be arbitrarily large.

Finally, the monotonicity of connection probability for the bond percolation of $\mathbb{Z}^d$ that motivated the Bunkbed Conjecture is still an open question. We have a counterexample for the Bunkbed Conjecture, but it is not clear if for the square lattice $\mathbf{P}_p[(0,0) \leftrightarrow (x,y)] \geq \mathbf{P}_p[(0,0) \leftrightarrow (x+1,y)]$ for all $x \geq 0$.

Kozma–Nitzan Conjecture For future research on percolation on general graphs, the most natural candidate is the Kozma–Nitzan conjecture:

Conjecture 7 ([KN24])**.** In bond percolation on a graph having vertices a, b, c_1, $\ldots$, c_n, one has

$$\mathbf{P}(ab) \geq \mathbf{P}(ac_1 \cup ac_2 \cup \cdots \cup ac_n) \min_i \mathbf{P}(c_i b).$$

Conjecture 7 was introduced recently by G. Kozma and S. Nitzan. They show that, if true, this conjecture would imply that there is almost surely no infinite cluster in percolation on $\mathbb{Z}^d$ at a critical probability.

This conjecture shares some structure with the Bunkbed Conjecture. If we let the parameter n denote the size of T in Conjecture 5, then for $n = 1$, both conjectures reduce to the same special case of the Harris–Kleitman inequality. For $n = 2$, both conjectures turn out to be true by inequality (4). The case $n = 2$ and the case $n = 3$ with various additional conditions are already proved in [KN24]. We intend to either prove the case $n = 3$ using the techniques from [G24b] or disprove it using the techniques from [GPZ24].

Complex Generalizations of [G24b] The independence result of Lemma 4 is likely generalizable from decision trees to quantum circuits. If this is the case, it provides a natural way to extend inequalities to different complex observables.

Complex observables played a crucial role in establishing the conformal invariance of the critical regime of site percolation on a triangular lattice [S01] and the conformal invariance of the Ising model [CS12]. We intend to extend the complex observables from these papers to pairs of independent configurations and use insights from general graphs to study statistical models on lattices.

Investigations of RSW Recent years marked progress in RSW theory. In [T16] and [KT23], the RSW theorem is proved without the "independent sampling above the path" trick, which allowed its generalization to other statistical physics models. For the papers [GZ24a, G24b], we wrote Julia code[2] to check which events are compatible for the tuples of configurations

$$(C_1, C_2, C_1 \underset{S_1}{\to} C_2, C_1 \underset{\overline{S_1}}{\to} C_2, C_1 \underset{S_2}{\to} C_2, C_1 \underset{\overline{S_2}}{\to} C_2, C_1 \underset{S_3}{\to} C_2, C_1 \underset{\overline{S_3}}{\to} C_2).$$

Linear programming then gives a set of accessible probabilities for these events. In particular, inequalities (6) and (8) are obtained from the considerations of this set.

Lattice structure offers more possibilities for decision trees and events such as "segment $\{0\} \times [0, n]$ is connected to the segment $[0, n] \times \{2n\}$ inside the square $[0, 2n] \times [0, 2n]$. Advances in LLM capabilities allow for easier computer experimentation, so our plan is to extend our code to include such events. In particular, the current proofs of RSW provide very crude bounds for rectangle crossing probabilities. For the critical regime on a square lattice, the probability of a square annulus crossing with the internal ring being a square with side n and the external ring being a square with side $2n$ is currently bounded below by a minuscule value of 7.88×10^{-32} as estimated by Vincent Tassion.[3] This probability is far from the value that one obtains from the conjectured conformal invariance. We intend to improve this bound, which would lead to the improvement of the known bounds on critical exponents.

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[2] See https://github.com/Kroneckera/bunkbed

[3] Talk slides available at https://www.nccr-swissmap.ch/application/files/9616/0206/1564/Vincent2.pdf

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