A strong FKG inequality for multiple events

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Abstract

We extend the FKG inequality to cover multiple events with equal pairwise intersections. We then apply this inequality to resolve Kahn's question on positive associated (PA) measures, as well as prove new inequalities concerning random graphs and probabilities of connection in Bernoulli percolation.

1 Introduction

The Fortuin–Kasteleyn–Ginibre (FKG) inequality [FKG71] is an inequality with numerous applications ranging from percolation to graph theory, from poset theory to probability theory [AS16, Ch. 6], [G83]. It generalizes the Harris–Kleitman inequality to a large class of measures on a hypercube.

Denote by H_n the *n*-dimensional discrete hypercube. It is conceptualized as a distributive lattice, where \lor is a coordinatewise maximum and \land is a coordinatewise minimum. Let μ be a probability measure on H_n . Assume μ satisfies the following *FKG property*:

$$\mu(a \vee b)\mu(a \wedge b) \ge \mu(a)\mu(b) \text{ for all } a, b \in H_n.$$

The FKG inequality guarantees nonnegative correlations of events that are closed upwards:

$$P(E_1 \cap E_2) \ge P(E_1)P(E_2) \tag{1.1}$$

The Harris-Kleitman inequality is the special case of the FKG inequality for product measures on H_n . Measures for which all closed upwards events correlate nonnegatively are said to have *positive associations* (PA). In other words, the FKG inequality says that all measures with the FKG property are PA. The FKG inequality is used to show that measures arising from random cluster model are PA [G06].

In Theorem 2.1, we prove a strong version of the Harris–Kleitman inequality for closed upwards events E_1, \ldots, E_k with equal pairwise intersections (i.e. $E_i \cap E_j = A$, where event A is the same for every pair $i \neq j$). We then use the approach from [K22] to generalize it to measures with the FKG property (Theorem 3.2). There are generalizations of the Harris–Kleitman and FKG inequalities as well as the more general AD inequality (four functions theorem) [AD78] to multiple sets [AK96, RS92]. Richards [R04] claimed to prove another generalization of FKG to n sets. Sahi [S08] noticed that the proof has essential gaps and conjectured another generalization, which agrees with Richards' for $n \leq 5$. He also proved two special cases, for "strongly increasing" functions, and for all

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increasing functions on the 2-dim Boolean square. The paper [LS22] proves additional special cases in the continuous setting. As far as we know, our generalization is different from all the others.

We use our inequality to prove a conjecture of Kahn (Theorem 3.4). Roughly speaking, we establish that the FKG inequality can be extended beyond the random variables which are monotonically determined by underlying independent variables.

2 Strong Harris–Kleitman inequality

We say that measure μ on H_n is a *product measure* if there exist probability measures $\mu_1, \mu_2, \ldots, \mu_n$ on $\{0, 1\}$, such that μ coincides with the direct product $\mu_1 \times \mu_2 \times \cdots \times \mu_n$. Recall the notation $e_2(x_1, \ldots, x_k) := \sum_{1 \le i < j \le k} x_i x_j$ for the second symmetric polynomial. We say that a subset of H_n is *closed upwards* if with each vector $v \in H_n$ it also contains all vectors greater than v in the natural partial order. Here is our main result:

Theorem 2.1. Let μ be a probability product measure on H_n , and

 $H_n = A \sqcup C_1 \sqcup C_2 \sqcup \cdots \sqcup C_k \sqcup B \text{ for } k \ge 2$

such that A is closed upwards, B is closed downwards and the C_i 's are pairwise incomparable (where, for $i \neq j$, C_i and C_j are called incomparable if there is no $v \in C_i$ and $w \in C_j$ such that $v \leq w$ or $w \geq v$). Then:

$$\mu(A)\mu(B) \ge e_2(\mu(C_1), \dots, \mu(C_k)).$$
(2.1)

Remark 2.2. For k = 2 we obtain the Harris–Kleitman inequality (1.1).

Indeed, let E_1 and E_2 be two closed-upwards events. Take $A = E_1 \cap E_2$, $B = (E_1 \cup E_2)^c$, $C_1 = A \setminus B$, and $C_2 = B \setminus A$. It is easy to see that these A, B, and C_i 's satisfy the conditions of Theorem 2.1. Applying the theorem gives

$$\mu(A)\mu(B) \ge \mu(C_1)\mu(C_2),$$

which after simple rewriting/manipulations gives

$$\mu(A)\big(\mu(A) + \mu(C_1) + \mu(C_2) + \mu(B)\big) \ge \big(\mu(A) + \mu(C_1)\big)\big(\mu(A) + \mu(C_2)\big)$$

or $\mu(E_1 \cap E_2) \ge \mu(E_1)\mu(E_2)$.

For k > 2 the partition of the hypercube from the statement may be also viewed as the selection of closed upwards events $E_i = A \cup C_i$ such that E_i 's form a *sunflower*, i.e. the intersection of every two different E_i 's is the same set A.

Proof of Theorem 2.1. Fix k. We proceed by induction on n. The case n = 0 is trivial. Let $\mathbf{v} \in H_{n-1}$, we can identify it with the corresponding vector in H_n , which has n-th coordinate equal to 0. Denote by $\mathbf{v} \uparrow$ the corresponding vector in H_n , which has n-th coordinate equal to 1. Define

$$A_{0} := \{ \mathbf{v} \in H_{n-1} : \mathbf{v} \in A, \mathbf{v} \uparrow \in A \}, \\ B_{0} := \{ \mathbf{v} \in H_{n-1} : \mathbf{v} \in B, \mathbf{v} \uparrow \in B \}, \\ C_{i}^{+} := \{ \mathbf{v} \in H_{n-1} : \mathbf{v} \in C_{i}, \mathbf{v} \uparrow \in A \}, \\ C_{i}^{\circ} := \{ \mathbf{v} \in H_{n-1} : \mathbf{v} \in C_{i}, \mathbf{v} \uparrow \in C_{i} \}, \\ C_{i}^{-} := \{ \mathbf{v} \in H_{n-1} : \mathbf{v} \in B, \mathbf{v} \uparrow \in C_{i} \}, \\ D := \{ \mathbf{v} \in H_{n-1} : \mathbf{v} \in B, \mathbf{v} \uparrow \in A \}.$$

Using the assumptions in the theorem, we have:

$$H_{n-1} = A_0 \sqcup B_0 \sqcup_{i=1}^k C_i^+ \sqcup_{i=1}^k C_i^\circ \sqcup_{i=1}^k C_i^- \sqcup D \text{ (see Fig. 1)}.$$



Figure 1: Subdivision of $\{0,1\}^{n-1}$

Note that the projection of product measure μ to H_{n-1} along the *n*-th coordinate is also a product measure. Denote it by μ' . Also, denote

$$a_0 := \mu'(A_0), \quad b_0 := \mu'(B_0), \quad c_i^+ := \mu'(C_i^+), \quad c_i^\circ := \mu'(C_i^\circ), \quad c_i^- := \mu'(C_i^-), \quad d := \mu'(D).$$

By the induction hypothesis we have (see first subdivision in Fig. 1):

$$\left(a_0 + d + \sum_{i=1}^k c_i^+\right) b_0 \geq e_2(c_1^\circ + c_1^-, \dots, c_k^\circ + c_k^-)$$

and (see second subdivision in Fig. 1)

$$a_0\left(b_0+d+\sum_{i=1}^k c_i^-\right) \geq e_2(c_1^\circ+c_1^+,\ldots,c_k^\circ+c_k^+).$$

Let $p := \mu(H_{n-1})$. We need to show that

$$\left(a_{0} + p\left(d + \sum_{i=1}^{k} c_{i}^{+}\right)\right) \left(b_{0} + (1-p)\left(d + \sum_{i=1}^{k} c_{i}^{-}\right)\right) \\ \geq e_{2}\left(c_{1}^{\circ} + pc_{1}^{-} + (1-p)c_{1}^{+}, \ldots, c_{k}^{\circ} + pc_{k}^{-} + (1-p)c_{k}^{+}\right).$$

Note that this inequality is quadratic in p and holds for p = 0 and p = 1. Thus it suffices to prove that the coefficient in p^2 in the LHS is less than that of the RHS:

$$-\left(d + \sum_{i=1}^{k} c_{i}^{+}\right)\left(d + \sum_{i=1}^{k} c_{i}^{-}\right) \leq e_{2}\left(c_{1}^{-} - c_{1}^{+}, \dots, c_{k}^{-} - c_{k}^{+}\right).$$

And after canceling the terms this can be rewritten as

$$-d\left(2d + \sum_{i=1}^{k} c_{i}^{+} + \sum_{i=1}^{k} c_{i}^{-}\right) - \sum_{i=1}^{k} c_{i}^{-} c_{i}^{+} \leq e_{2}\left(c_{1}^{-}, \dots, c_{k}^{-}\right) + e_{2}\left(c_{1}^{+}, \dots, c_{k}^{+}\right).$$

This follows since the LHS is nonpositive and the RHS is nonnegative. This completes the induction step, and implies inequality (2.1) for all n.

3 Applications

3.1 UI and FUI measures

Recall the definition of UI and FUI measures introduced in [K22]. Suppose X_1, \ldots, X_n are (dependent) Bernoulli random variables and μ is their joint distribution. Measure μ on H_n is called *FUI* (which stands for *finitely many underlying independents*), if there is a realization of X_i 's as increasing functions of independent Bernoulli random variables Y_1, \ldots, Y_m for some m. Measure μ is called UI, if it is a limit of FUI measures on the same hypercube.

Notice that the FUI and UI properties are weaker than the FKG property:

Proposition 3.1. [K22, Footnote 1] All measures μ with the FKG property are FUI (and therefore UI).

Proof. Let Z_1, \ldots, Z_n be i.i.d. U(0, 1) random variables. For all $i \in \{1, \ldots, n\}$ we recursively define X_i as functions of Z's as follows. Assume that at the *i*-th step, we have

$$X_j := f_j(Z_1, \ldots, Z_j)$$
 for all $j < i$.

Define

$$X_i := \begin{cases} 0, \text{ if } Z_i < \mu(\mathbf{v}_i = 0 \mid \mathbf{v}_j = X_j \text{ for all } 1 \le j < i); \\ 1, \text{ otherwise.} \end{cases}$$
(3.1)

It is easy to see that μ is the law of (X_1, \ldots, X_n) . Moreover, the FKG property implies that X_i 's are non-decreasing in the Z_j 's.

To prove that μ is FUI we need to represent X_i 's as functions of independent Bernoulli variables. Notice that (3.1) depends monotonically on a finite (though, exponential in n) number of events of form

$$A(i, \mathbf{v}_1, \dots, \mathbf{v}_{i-1}) := \{ Z_i < \mu(\mathbf{v}_i = 0 \mid \mathbf{v}_j = X_j \text{ for all } 1 \le j < i) \}.$$

It is possible to realize indicators of $A(\cdot)$'s as non-decreasing functions of independent, but possibly differently distributed, Bernoulli variables $Y(i, \mathbf{v}_1, \ldots, \mathbf{v}_{i-1})$.

Proposition 3.1 allows us to generalize Theorem 2.1 to all UI-measures. In particular, it holds for all measures with the FKG property.

Theorem 3.2. Let μ be a UI measure on H_n , and

$$H_n = A \sqcup C_1 \sqcup C_2 \sqcup \cdots \sqcup C_k \sqcup B \text{ for } k \ge 2$$

such that A is closed upwards, B is closed downwards and the C_i 's are pairwise incomparable. Then

$$\mu(A)\mu(B) \ge e_2(\mu(C_1), \dots, \mu(C_k)).$$
(3.2)

Proof. Suppose μ is an FUI measure on (X_1, \ldots, X_n) . Then we can assume X_i 's are binary non-decreasing functions of the auxiliary independent m Bernoulli variables Y_1, \ldots, Y_m as in the proof of Proposition 3.1. Then events A, B and C_i 's will be determined on the hypercube generated by Y_i 's. One can easily check that in these new coordinates A is still closed upwards, B is closed downwards and C_i 's are pairwise incomparable. Moreover, in these coordinates μ is a product measure, so by Theorem 2.1 we have inequality (3.2). For UI measures, inequality (3.2) is obtained as a limit of inequalities for FUI measures. \Box

The following is the main result of Kahn [K22] and a basis for our main application:

Theorem 3.3. [K22, Corollary 4] There are measures on H_n with positive associations which are not FUI.

We identify subsets of $\{1, 2, ..., n\}$ with points in H_n . Consider the law μ_n of the set of fixed points of a uniform permutation $\sigma \in S_n$. It was shown in [FDS88] that μ_n has positive associations. Kahn uses the measure μ_3 to prove Theorem 3.3. He writes: "it seems surprisingly hard to say anything about the law of a UI μ that uses more than positive association".

It turns out that Theorem 3.2 helps us to extend Kahn's theorem to UI measures. In fact, we use the same measure μ_3 . This answers a question dating back to at least 2002 [K22, Question 1].

Theorem 3.4. There are measures on H_n with positive associations which are not UI.

Proof. Note that

$$\mu_3(\{1\}) = \mu_3(\{2\}) = \mu_3(\{3\}) = \mu_3(\{1, 2, 3\}) = \frac{1}{6}$$

and $\mu_3(\emptyset) = \frac{1}{3}$. Consider $A = \{|S| \ge 2\}, B = \{S = \emptyset\}, C_i = \{S = \{i\}\}$ for $1 \le i \le 3$. Suppose μ_3 is UI. Then by Theorem 3.2 we have

$$\frac{1}{18} \ge e_2\left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right) = \frac{1}{12},$$

a contradiction. Thus μ_3 is not UI, as desired.

Unfortunately, we can not say anything on whether classes UI and FUI coinside. In particular, we don't know if Theorem 2 from [K22] generalizes to UI measures. Analysis of the proof of Theorem 2.1 reveals that in every equality case at most two of C_i 's have nonzero probability. So this is also true for FUI measures in Theorem 3.2. But we don't know if there are UI measures with different equality cases.

3.2 Counting graphs

We give here an application in the style of [AS16, Problem 6.5.3].

Corollary 3.5. Let G be a uniform random graph on 2n labeled vertices and denote by S its set of vertices with degree $\geq n$. Then for every k

$$\frac{\binom{2n}{k} - 1}{2\binom{2n}{k}} \mathbf{P}(|S| = k) \le \mathbf{P}(|S| > k) \mathbf{P}(|S| < k)$$

Proof. Random graphs on 2n vertices form a hypercube $H = \{0, 1\}^d$ by inclusion, where $d = \binom{2n}{2}$. We can consider events A and B in this hypercube equal to $\{|S| > k\}$ and $\{|S| < k\}$ and events $C_T = \{S = T\}$ indexed by all k-subsets T of $\{1, 2, \ldots, 2n\}$. All C_T share a probability equal to $\frac{\mathbf{P}(|S|=k)}{\binom{2n}{k}}$, so applying Theorem 2.1, we get

$$\binom{\binom{2n}{k}}{2} \frac{\mathbf{P}(|S|=k)^2}{\binom{2n}{k}^2} \le \mathbf{P}(|S|>k)\mathbf{P}(|S|$$

Note that by using just the Harris–Kleitman inequality, the best we can achieve is

$$\left\lfloor \frac{\binom{2n}{k}}{2} \right\rfloor \left\lceil \frac{\binom{2n}{k}}{2} \right\rceil \frac{\mathbf{P}(|S|=k)^2}{\binom{2n}{k}^2} \le \mathbf{P}(|S|>k)\mathbf{P}(|S|$$

which is worse by a factor approaching 2 as $n \to \infty$. In particular, for k = n, we have

$$\mathbf{P}(|S|=n) \le \mathbf{P}(|S|>n) \frac{\binom{2n}{n}}{\sqrt{\binom{\binom{2n}{n}}{2}}}$$

This implies

$$\mathbf{P}(|S|=n) \le \frac{\binom{2n}{n}}{\binom{2n}{n} + 2\sqrt{\binom{\binom{2n}{n}}{2}}} \to \sqrt{2} - 1 \quad \text{as } n \to \infty.$$

This is an improvement over $\frac{1}{2}$ which follows from the Harris–Kleitman inequality.¹

3.3 Percolation

Theorem 2.1 allows us to say more about connectedness events in percolation than the Harris–Kleitman inequality. Consider a graph G = (V, E), where $V = \{1, 2, ..., n\}$. Consider the percolation on G, where each edge $e \in E$ has probability $p_e \in (0, 1)$ of surviving, independent of other edges. This gives a spanning subgraph $H \subseteq G$ with probability

$$\prod_{e \in H} p_e \prod_{e \notin H} (1 - p_e)$$

Consider three vertices $1, 2, 3 \in V$. Denote by $\mathbf{P}(123)$ the probability that vertices 1, 2 and 3 lie in the same connected component of H. Denote by $\mathbf{P}(12|3)$ the probability that 1 and 2 lie in the same connected component, different from the component of 3. Define $\mathbf{P}(13|2)$ and $\mathbf{P}(1|23)$ analogously. Finally, denote by $\mathbf{P}(1|2|3)$ the probability that all three vertices lie in different connected components.

Corollary 3.6. In the notation above, we have:

$$\mathbf{P}(123)\mathbf{P}(1|2|3) \ge \mathbf{P}(12|3)\mathbf{P}(13|2) + \mathbf{P}(12|3)\mathbf{P}(1|23) + \mathbf{P}(13|2)\mathbf{P}(1|23).$$
(3.3)

Proof. Note that events A = (123), B = (1|2|3), $C_1 = (1|23)$, $C_2 = (13|2)$, $C_3 = (12|3)$ satisfy the conditions of Theorem 2.1. The inequality (3.3) follows.

¹In reality, this number goes to zero, see this Mathoverflow answer. So the inequality is of interest for relatively small n.

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