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AN EXPLICIT SOLUTION FOR A MULTIMARGINAL MASS TRANSPORTATION PROBLEM*

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Abstract. We construct an explicit solution for the multimarginal transportation problem on the unit cube $[0,1]^3$ with the cost function xyz and one-dimensional uniform projections. We show that the primal problem is concentrated on a set with a nonconstant local dimension and admits many solutions, whereas the solution to the corresponding dual problem is unique (up to addition of constants).

Key words. optimal transport, Monge–Kantorovich, multidimensional, multistochastic, explicit solution

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1. Introduction.

1.1. Notation. Assume we are given n Polish spaces $X_1, X_2, \ldots, X_n$, equipped with probability measures μ_i on X_i and a cost function $c : X_1 \times \cdots \times X_n \to \mathbb{R}$.

In the multimarginal Monge–Kantorovich problem (called the primal problem throughout this paper) we seek to minimize

$$\int_{X_1\times\cdots\times X_n} c(x_1, x_2, \ldots, x_n)\ d\mu(x_1, x_2, \ldots, x_n)$$

over the set $\Pi(\mu_1, \mu_2, \ldots, \mu_n)$ of positive joint measures μ on the product space $X_1 \times \cdots \times X_n$ whose marginals are the μ_i. See [19, 2] for an account of the optimal transportation problem with two marginals and [18].

The dual formulation of the multimarginal optimal transport problem is defined by the supremum of

$$\sum_{i=1}^{n} \int_{X_i} f_i(x_i)\ d\mu_i,$$

where the supremum is taken over all sets of functions $\{f_i\}$ such that $\sum_{i=1}^n f_i(x_i) \le c(x_1, \ldots, x_n)$ for any $x_i \in X_i$.

It is easy to show that the minimum in the primal problem is greater than or equal to the supremum in the dual problem. Under some conditions it is true that these numbers are equal [19, 18, 11].

We do not need a full power of duality here. This paper relies on the following easy fact.

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LEMMA 1.1 (complementary slackness condition). *Let $\mu \in \Pi(\mu_1, \dots, \mu_n)$ be a joint measure, and let $f_1, f_2, \dots, f_n$ be a tuple of functions such that $\sum_{i=1}^n f_i(x_i) \le c(x_1, \dots, x_n)$. If there is a set $M \subset X_1 \times X_2 \times \dots \times X_n$ such that on M one has $\sum_{i=1}^n f_i(x_i) = c(x_1, \dots, x_n)$ with the additional property $\mu(M) = 1$, then μ is a primal solution, and f_i is a dual solution.*

The aim of this paper is to describe an example of an explicit solution to the mass transportation problem on $[0,1]^3$ ($X_1 = X_2 = X_3 = [0,1]$) with one-dimensional Lebesgue measure projections and the cost function $c(x,y,z) = xyz$. In this paper we call the measures on $[0,1]^3$ with Lebesgue projections onto the axes $(3,1)$-stochastic measures.

In fact, we will construct the primal and dual solutions for any cost function $c(x,y,z) = C(xyz)$ for some continuously differentiable function $C : [0,1] \to \mathbb{R}$ such that the function $tC'(t)$ strictly increases on the segment $[0,1]$.

1.2. Motivation. Our problem appears to be the simplest generalization of the classical Monge–Kantorovich problem with one-dimensional marginals and quadratic cost function. It seems to never have been considered in the literature, though other generalizations mentioned in subsection 1.4 received some attention. Note that the particular cost function $(x-y)^2$ (equivalently, $-xy$) is mostly used in the classical Monge–Kantorovich theory. A natural replacement of $-xy$ for the case of three variables is $-xyz$. For the cost function $-xyz$ the solution to the primal problem with the same marginals admits a simple structure: it is concentrated on the main diagonal of $[0,1]^3$ (this can be viewed as a "continuous rearrangement inequality" or "Hardy–Littlewood inequality"). Unlike this, solutions for xyz are nontrivial and that is why we are interested in the cost function xyz.

1.3. Main results. In this paper we construct the set M which is c-monotone for the cost function $c(x,y,z) = xyz$. The set M is the union of three segments and one two-dimensional part as follows:

$$
\begin{aligned}
M_x &= \{(t, 1-2t, 1-2t) \mid 0 \le t \le l\}, \\
M_y &= \{(1-2t, t, 1-2t) \mid 0 \le t \le l\}, \\
M_z &= \{(1-2t, 1-2t, t) \mid 0 \le t \le l\}, \\
M_2 &= \{(x,y,z) \mid l \le x, y, z \le r = 1-2l, xyz = lr^2\}, \\
M &= M_x \cup M_y \cup M_z \cup M_2,
\end{aligned}
$$

where $l \approx 0.0945$, $r \approx 0.8119$ are some transcendent constants (see Figure 1).

Initially, we got an explicit construction of this set from heuristic considerations (see section 2). In section 3 we see that the integral $\int xyz \, d\mu$ is the same for any $(3,1)$-stochastic measure μ such that $\operatorname{supp}(\mu) \subset M$ (see Proposition 3.1). After that we explicitly construct a $(3,1)$-stochastic measure π concentrated on the set M (see the proof of Theorem 3.14). The proof contains nontrivial construction and technical computations. The constructed measure is the primal solution of the related transport problem.

To prove that the measure π is the primal solution, in section 4 we solve a related dual problem for a cost function $c(x,y,z) = C(xyz)$. Our proof works for $C : [0,1] \to \mathbb{R}$ such that the function $tC'(t)$ strictly increases on the segment $[0,1]$. Naturally, that means $C(xyz) = \widehat{C}(\ln x + \ln y + \ln z)$, where $\widehat{C}$ is a bounded continuously differentiable convex function on $(-\infty, 0]$.

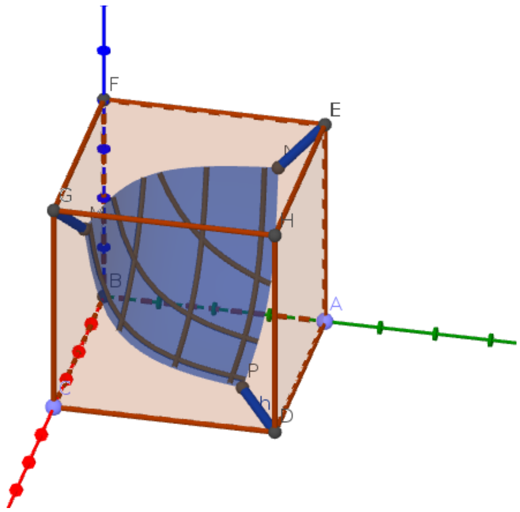

FIG. 1. *Set M.*

The following theorem gives an explicit construction for the dual solution (also see Theorem 4.6). Thus, together with Theorem 3.14, it gives a characterization of both primal and dual solutions.

THEOREM 1.2 (main result). *Suppose that $c(x, y, z) = C(xyz)$ for some continuously differentiable function $C : [0, 1] \to \mathbb{R}$ and that the function $tC'(t)$ strictly increases on the segment $[0, 1]$. Set*

$$\hat{f}(s) = \int_0^s \lambda(t)C'(t\lambda(t))\, dt,$$

where the function λ is as in Definition 4.1. Then for any constants C_x, C_y, C_z such that

$$C_x + C_y + C_z = C(0) - 2\int_0^1 \lambda(t)C'(t\lambda(t))\, dt,$$

the inequality

$$(\hat{f}(x) + C_x) + (\hat{f}(y) + C_y) + (\hat{f}(z) + C_z) \le c(x, y, z)$$

holds with equality on M.

Using the complementary slackness conditions (see Lemma 1.1), we conclude that for any cost function $C(xyz)$, for which the conditions above are satisfied, any $(3, 1)$-stochastic measure π with $\mathrm{supp}(\pi) \subset M$ is a primal solution for the multimarginal mass transportation problem, and the functions $\hat{f}$ defined in Theorem 1.2 are a dual solution.

In subsection 4.3 the explicit form of the dual solution for the cost function $c(x, y, z) = xyz$ is specified. It has the following form (see Proposition 4.7):

$$\hat{f}(x) = \begin{cases} c\ln l - \frac{1}{3}(c\ln c - c) + \frac{1}{6}((2x-1)^3 - (2l-1)^3) & \text{if } 0 \le x \le l, \\ c\ln x - \frac{1}{3}(c\ln c - c) & \text{if } l \le x \le r, \\ c\ln r - \frac{1}{3}(c\ln c - c) + \frac{1}{4}(x^2 - r^2) - \frac{1}{6}(x^3 - r^3) & \text{if } r \le x \le 1, \end{cases}$$

$$f(t) = g(t) = h(t) = \hat{f}(t)$$

for constants l, r, c.

In section 5 we prove that for any cost function $c(x, y, z) = C(xyz)$ the dual solution is unique up to adding constants and measure zero.

Structural results (see [17, 18]) allow us to estimate the local dimension d of M. We apply this result in section 6 to see that d is bounded above by 2. The dimension of the support is important for computations and was studied in detail in [8]. It is interesting that the local dimension of M is not constant, as M admits one-dimensional parts along with a two-dimensional part.

This two-dimensional part is a source of nonuniqueness for the primal problem. After the logarithmic change of coordinates the cost function $C(xyz)$ becomes convex in the sum of coordinates, the Lebesgue measure on the axis becomes an exponential distribution, and the two-dimensional part of M becomes a triangle on a plane $x+y+z = \text{const}$. This resembles the situation in [7, Lemma 4.3], where the authors consider the multimarginal problem with the same cost function and Lebesgue marginals. They prove that the plan is optimal if and only if it is concentrated on a plane $x + y + z = \text{const}$.

The cost function xyz violates the standard uniqueness assumption, the so-called twist condition (see [12, 16, 18]). The primal problem admits many solutions. In particular, we show that there exist solutions which are singular with respect to the Hausdorff measure on M. We also propose the following.

Conjecture 1.3. *There exists a solution that is concentrated on a set which has Hausdorff dimension less than* 2*.*

This conjecture is motivated by [7, Theorem 4.6], where the authors construct a primal solution with a fractal support.

1.4. Related problems. Our example contributes to the list of several known explicit examples and to the list of cost functions for which the structure of solutions is investigated in detail. In the following we list some additional examples:

1. Cost function
$$-\sum_{i\neq j} x_i x_j.$$
This cost function is related to the geodesic barycenter problem (see [3, 1]).
2. Determinantal cost [4].
3. Coulomb cost [6] (see [5] for generalizations). The motivation for this problem comes from mathematical physics.
4. $\min(x_1, \dots, x_n)$ (more generally, minimum of affine functions) [13].
5. Convex function of $x_1 + \cdots + x_n$ (see [7]).

Other examples can be found in [18].

Also, our problem is closely related to the $(3, 2)$-problem, studied in [9]. In particular, our example can be considered as a solution to the primal $(3, 2)$-problem with the same cost function xyz and the corresponding two-dimensional projections. In the $(3, 2)$-problem, we consider a modification of the transportation problem. Namely, we deal with the space of measures with fixed projections onto

$$X_1 \times X_2,\ X_2 \times X_3,\ X_1 \times X_3.$$

The main result of [9] describes a solution to the $(3, 2)$-problem on $[0, 1]^3$ with the cost function xyz $(-xyz)$ and two-dimensional Lebesgue measure projections. It turns out that in strong contrast with the classical transportation problem, the solution is supported by the fractal set (Sierpiński tetrahedron)

$$z = x \oplus y,$$

where $\oplus$ is bitwise addition. Let us also mention another related important modification, the Monge–Kantorovich problem with linear constraints, which has been introduced and studied in [20].

2. A heuristic description of M.

In this subsection we collect some informal observations related to our main construction. In particular, we briefly analyze the cyclical monotonicity property of the support set of our primal solution and describe how to approach the problem numerically.

Let M be a full measure set for the primal solution. Since all of the marginals and the cost function are symmetric with respect to the coordinate axes interchange, we may assume without loss of generality that M is also symmetric in this sense.

The set M can be chosen to be c-cyclically monotone. This is well known for two marginals; for many marginals, we refer the reader to the work [10]. In particular, that means that for any $(x_1, y_1, z_1), (x_2, y_2, z_2) \in M$, one has

$$\begin{aligned} &c(x_1, y_1, z_1) + c(x_2, y_2, z_2) \le c(x_2, y_1, z_1) + c(x_1, y_2, z_2),\\ &c(x_1, y_1, z_1) + c(x_2, y_2, z_2) \le c(x_1, y_2, z_1) + c(x_2, y_1, z_2),\\ &c(x_1, y_1, z_1) + c(x_2, y_2, z_2) \le c(x_1, y_1, z_2) + c(x_2, y_2, z_1). \end{aligned} \tag{2.1}$$

Algorithm 2.1 constructs an approximation to a primal solution and is based on the inequality above.

Algorithm 2.1. Primal solution approximation (general version).

Generate three samples: $x_1, x_2, \dots, x_n$ from μ_1, $y_1, y_2, \dots, y_n$ from μ_2, $z_1, z_2, \dots, z_n$ from μ_3; n is a parameter, μ_i are the marginals in the primal problem.
Define $S := \{(x_k, y_k, z_k) \text{ for } 1 \le k \le n\}$.
while S does not satisfy (2.1) **do**
  Take two points (a_1, b_1, c_1) and (a_2, b_2, c_2) from S.
  if $c(a_1, b_1, c_1) + c(a_2, b_2, c_2) > c(a_2, b_1, c_1) + c(a_1, b_2, c_2)$ **then**
    replace (a_1, b_1, c_1) and (a_2, b_2, c_2) with (a_2, b_1, c_1) and (a_1, b_2, c_2) in S
  else if $c(a_1, b_1, c_1) + c(a_2, b_2, c_2) > c(a_1, b_2, c_1) + c(a_2, b_1, c_2)$ **then**
    replace (a_1, b_1, c_1) and (a_2, b_2, c_2) with (a_1, b_2, c_1) and (a_2, b_1, c_2) in S
  else if $c(a_1, b_1, c_1) + c(a_2, b_2, c_2) > c(a_1, b_1, c_2) + c(a_2, b_2, c_1)$ **then**
    replace (a_1, b_1, c_1) and (a_2, b_2, c_2) with (a_1, b_1, c_2) and (a_2, b_2, c_1) in S
  end if
end while
S is an approximation of the primal solution.

In our case, $c(x, y, z) = xyz$, so

$$\begin{aligned} &x_1 y_1 z_1 + x_2 y_2 z_2 \le x_1 y_1 z_2 + x_2 y_2 z_1,\\ &(x_1 y_1 - x_2 y_2)(z_1 - z_2) \le 0. \end{aligned}$$

It follows that if $(x_1, y_1, z_1), (x_2, y_2, z_2) \in M$, then

$$\begin{aligned} &z_1 < z_2 \Rightarrow x_1 y_1 \ge x_2 y_2 \text{ and by the symmetry}\\ &y_1 < y_2 \Rightarrow x_1 z_1 \ge x_2 z_2,\\ &x_1 < x_2 \Rightarrow y_1 z_1 \ge y_2 z_2. \end{aligned} \tag{2.2}$$

Algorithm 2.2. Primal solution approximation (faster version).

1: Generate three samples: $x_1, x_2, \dots, x_n$ from μ_1, $y_1, y_2, \dots, y_n$ from μ_2, $z_1, z_2, \dots, z_n$ from μ_3; n is a parameter, μ_i are the marginals in the primal problem.
2: Define $S := [(x_k, y_k, z_k)$ for $1 \leq k \leq n]$.
3: **while** S does not satisfy (2.2) **do**
4: Sort S by the first coordinate in ascending order. Denote by (a_k, b_k, c_k) the kth item of S after sorting, $1 \leq k \leq n$.
5: Update $S := [(a_k, b_{\sigma(k)}, c_{\sigma(k)})$ for $1 \leq k \leq n]$ where $\sigma \in S_n$ and the sequence $b_{\sigma(k)} c_{\sigma(k)}$ is descending.
6: Repeat Line 4 and Line 5 for the second and third coordinates.
7: **end while**
8: S is an approximation of the primal solution.

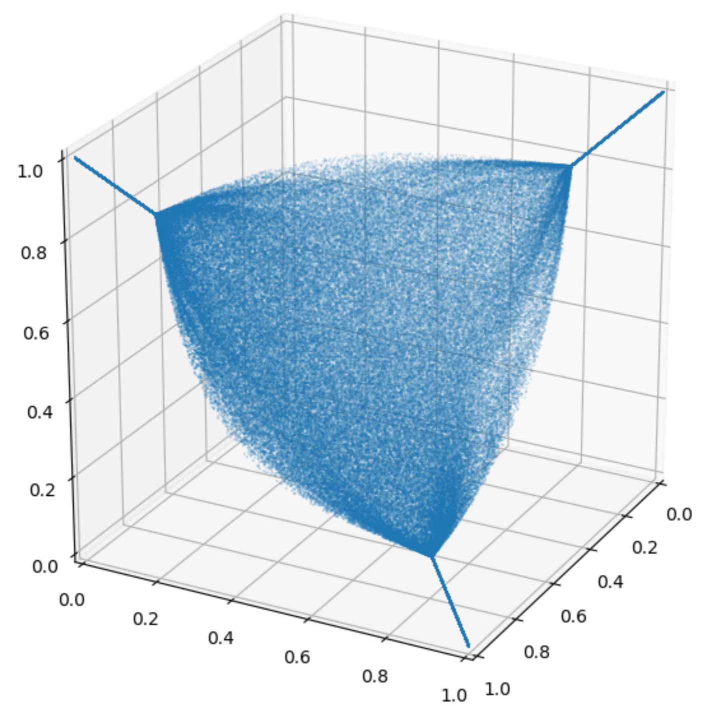

FIG. 2. *The final set S for $n =$ 200,000.*

This allows us to improve the performance of Algorithm 2.1 using sortings. That leads us to a much faster version, namely Algorithm 2.2. We were able to run Algorithm 2.2 for $n = 2 \times 10^5$.

Despite the fact that this algorithm does not necessarily converge to the solution for all admissible data, our numerical experiments demonstrate that the algorithm works well in many cases. A proof of convergence for a suitable set of admissible data must be investigated.

Figure 2 shows a scatter plot of S after the completion of the algorithm. As one can see on this graph, the set M consists of four parts. There exist real values $0 < l < r < 1$ such that if $l \leq x, y, z \leq r$ and $(x, y, z) \in M$, then (x, y, z) lies on the two-dimensional part M_2. If $0 \le x \le l$ and $(x, y, z) \in M$, then (x, y, z) lies on a one-dimensional curve $M_x = (p(t), p_y(t), p_z(t))$, $0 \le t \le 1$. By virtue of the symmetry, $p_y(t) = p_z(t) = q(t)$. Also, $q(0) = 1$, $q(1) = r$, and $q(t)$ strictly decrease; $p(0) = 0$, $p(1) = l$, and $p(t)$ strictly increase.

By virtue of the symmetry, if $(x, y, z) \in M$ and $0 \le y \le l$, then this point lies on a curve $M_y = (q(t), p(t), q(t))$, and if $0 \le z \le l$, then (x, y, z) lies on a curve $M_z = (q(t), q(t), p(t))$.

Let ν be a primal solution, and let ν_x, ν_y, ν_z be the restrictions of ν to M_x, M_y, and M_z accordingly. Suppose $F_x(a) = \nu_x(\{(p(t), q(t), q(t)) \mid 0 \le t \le a\})$. Define F_y and F_z in a similar way. By virtue of the symmetry, we can assume that $F_x(a) = F_y(a) = F_z(a) = F(a)$ for any $0 \le a \le 1$.

For any $0 \le a \le 1$, one has

$$\begin{aligned}\nu(\{0 \le x \le p(a)\}) = \nu_x(\{(p(t), q(t), q(t)) \mid 0 \le t \le a\}) = F_x(a)\\ = \frac{1}{2}(F_y(a) + F_z(a)) = \frac{1}{2}\nu(\{q(a) \le x \le 1\}).\end{aligned}$$

Since all of the marginals are Lebesgue measures on the segments $[0, 1]$, one has $2p(a) = 1 - q(a)$ for any $0 \le a \le 1$.

Thus,

$$M_x = (t, 1 - 2t, 1 - 2t),$$

$$M_y = (1 - 2t, t, 1 - 2t),$$

and

$$M_z = (1 - 2t, 1 - 2t, t),$$

for $0 \le t \le l$; $r = 1 - 2l$. That means that one-dimensional parts of the set M are segments.

The set M_x is c-cyclically monotone. In particular, if $1 - 2t_1 < 1 - 2t_2$, then $t_1(1 - 2t_1) \ge t_2(1 - 2t_2)$ or, equivalently, the function $t(1 - 2t)$ increases on the set $[0, l]$. The derivative of this function is $1 - 4t \ge 0$ for any $0 \le t \le l$. That means that $0 < l \le \frac{1}{4}$.

Let us describe the set M_2. As we know from the general duality theory, there exist functions

$$f, g, h : [l, r] \to \mathbb{R}$$

satisfying $f(x) + g(y) + h(z) \le xyz$, and the equality holds provided $(x, y, z) \in M_2$. Again by symmetry we can assume that $f(x) = g(x) = h(x)$ for any $l \le x \le r$.

Suppose that f is continuously differentiable. Let $(x, y, z) \in M_2$ and $l \le x, y, z \le r$. Then that point is an inner maximum point of the function

$$F(x, y, z) = f(x) + f(y) + f(z) - xyz.$$

This means that

$$\nabla F = (f'(x) - yz, f'(y) - xz, f'(z) - xy)^T = \vec{0}.$$

So if $(x, y, z) \in M_2$, then $xf'(x) = yf'(y) = zf'(z) = xyz$. From Figure 2 we see that if we fix $x = l$, then for any $l \le y \le r$ there exists $l \le z \le r$ such that $(x, y, z) \in M_2$. Then the function $tf'(t)$ is equal to a constant $C = lf'(l)$ for any $l \le t \le r$. In this case if $(x, y, z) \in M_2$, then $xyz = xf'(x) = C$. Since $(l, r, r) \in M_2$, the constant C is equal to lr^2.

3. Solving the primal problem. Summarizing the facts about the set M, which supports the primal solutions, we realize that one can try to find M in the following form:

$$\begin{aligned}
M_x &= \{(t, 1-2t, 1-2t) \mid 0 \le t \le l\},\\
M_y &= \{(1-2t, t, 1-2t) \mid 0 \le t \le l\},\\
M_z &= \{(1-2t, 1-2t, t) \mid 0 \le t \le l\},\\
M_2 &= \{(x, y, z) \mid l \le x, y, z \le r = 1-2l, xyz = lr^2\},
\end{aligned}$$

where l is an unknown parameter; $0 \le l < \frac{1}{4}$.

PROPOSITION 3.1. *An integral* $\int xyz\ d\nu(x,y,z)$ *is the same for any probability measure* ν *such that* $\mathrm{Pr}_x(\nu) = \mathrm{Pr}_y(\nu) = \mathrm{Pr}_z(\nu) = \lambda$, *where* λ *is the Lebesgue measure on the segment* $[0,1]$, *and* $\mathrm{supp}(\nu) \subset M$.

Proof. Let ν_x, ν_y, ν_z, and ν_2 be restrictions of ν to M_x, M_y, M_z, and M_2, respectively. Since the projection of ν_x on the first marginal is a restriction of λ to the segment $[0,l]$, one has

$$\int_{M_x} xyz\ d\nu_x(x,y,z) = \int x(1-2x)(1-2x)\ d\nu_x(x,y,z) = \int_0^l x(1-2x)^2\ dx.$$

Similarly,

$$\int_{M_y} xyz\ d\nu_y(x,y,z) = \int_{M_z} xyz\ d\nu_z(x,y,z) = \int_0^l x(1-2x)^2\ dx.$$

Finally, the projection of ν_2 on the first marginal is a restriction of λ to the segment $[l,r]$. So,

$$\int_{M_2} xyz\ d\nu_2 = lr^2 \cdot \nu_2(\{l \le x \le r\}) = lr^2(r-l).$$

Consequently, $\int xyz\ d\nu(x,y,z) = 3\int_0^l x(1-2x)^2\ dx + lr^2(r-l)$, and this integral does not depend on ν. ∎

We only have to find any measure with desired projections such that its support is contained in M. In Theorem 4.6 we find an appropriate triple of functions, and by Lemma 1.1 we rigorously prove that any $(3,1)$-stochastic measure on M is indeed a primal solution.

First, we define a measure on the three one-dimensional segments. Let $L = \sqrt{l^2 + 2(1-r)^2}$ be the lengths of these segments. On every segment, we set a uniform measure with density $\frac{l}{L}$. Clearly, projections of two segments coincide with $[r,1]$, and the densities are equal to $\frac{L}{1-r} \cdot \frac{l}{L} = \frac{1}{2}$. Their sum is the Lebesgue measure on $[r,1]$. The projection of the third interval is a measure on $[0,l]$, and its density equals $\frac{L}{l} \cdot \frac{l}{L} = 1$.

After this, it remains to determine the measure on the remaining two-dimensional set such that its projection on each of the axes is uniform.

Let us make the following change of coordinates:

$$u := \frac{\ln x - \ln l}{\ln r - \ln l}, \quad v := \frac{\ln y - \ln l}{\ln r - \ln l}, \quad w := \frac{\ln z - \ln l}{\ln r - \ln l}.$$

The two-dimensional set

$$xyz = c, l \le x, y, z \le r$$

admits the following parametrization:

$$u + v + w = 2, 0 \leq u, v, w \leq 1.$$

One has the following relations:

$$\begin{aligned} dx &= de^{u(\ln r - \ln l) + \ln l} = l \ln\left(\frac{r}{l}\right)\left(\frac{r}{l}\right)^u du = l\ln(\alpha)\alpha^u du, \\ dy &= l\ln(\alpha)\alpha^v dv, \\ dz &= l\ln(\alpha)\alpha^w dw, \end{aligned}$$

where $\alpha = \frac{r}{l}$.

Clearly, the problem is reduced to the following problem: find a measure on the triangle $u + v + w = 2, 0 \leq u, v, w \leq 1$ with exponential projections onto the axes.

3.1. Necessary conditions for existence of a measure on the triangle with given projections. One can put the problem into a more general setting. When does there exist a measure μ on the triangle

$$\Delta = \{x + y + z = 2,\ 0 \leq x, y, z \leq 1\}$$

with given projections μ_x, μ_y, μ_z?

In what follows we are only interested in the case $\mu_x = \mu_y = \mu_z = \pi$. A necessary condition is given in the following lemma.

Lemma 3.2. *Let function $f : [0, 1] \to \mathbb{R}$ satisfy $f(x)+f(y)+f(z) \leq 0$ for $x+y+z = 2$. Suppose there exists a measure μ on Δ, whose projections onto the axes are equal to π. Then $\int_0^1 f(x)d\pi \leq 0$.*

Proof. We compute $\int_\Delta (f(x)+f(y)+f(z))\, d\mu$. On the one hand, it is nonpositive, since at each point, $f(x) + f(y) + f(z) \leq 0$. On the other hand,

$$\int_\Delta (f(x) + f(y) + f(z))d\mu = 3\int_0^1 f(x)d\pi(x) \leq 0.$$

In particular, for the function $f(x) = x - \frac{2}{3}$ one has $f(x) + f(y) + f(z) = 0$ for $x + y + z = 2$. So we get

$$\int_0^1 \left(x - \frac{2}{3}\right) d\pi(x) = 0. \tag{3.1}$$

Check this for $d\pi = \alpha^x dx$:

$$\int_0^1 \left(x - \frac{2}{3}\right) d\pi = \int_0^1 \left(x - \frac{2}{3}\right)\alpha^x dx = \frac{\alpha(\ln\alpha - 3) + 3 + 2\ln\alpha}{3\ln^2\alpha}.$$

Thus, α must satisfy

$$\alpha(\ln\alpha - 3) + 3 + 2\ln\alpha = 0. \tag{3.2}$$

Apply the relation $\alpha = \frac{1-2l}{l}$:

$$\alpha(\ln\alpha - 3) + 3 + 2\ln\alpha = \frac{\ln(1 - 2l) - \ln l - 3 + 9l}{l} = 0.$$

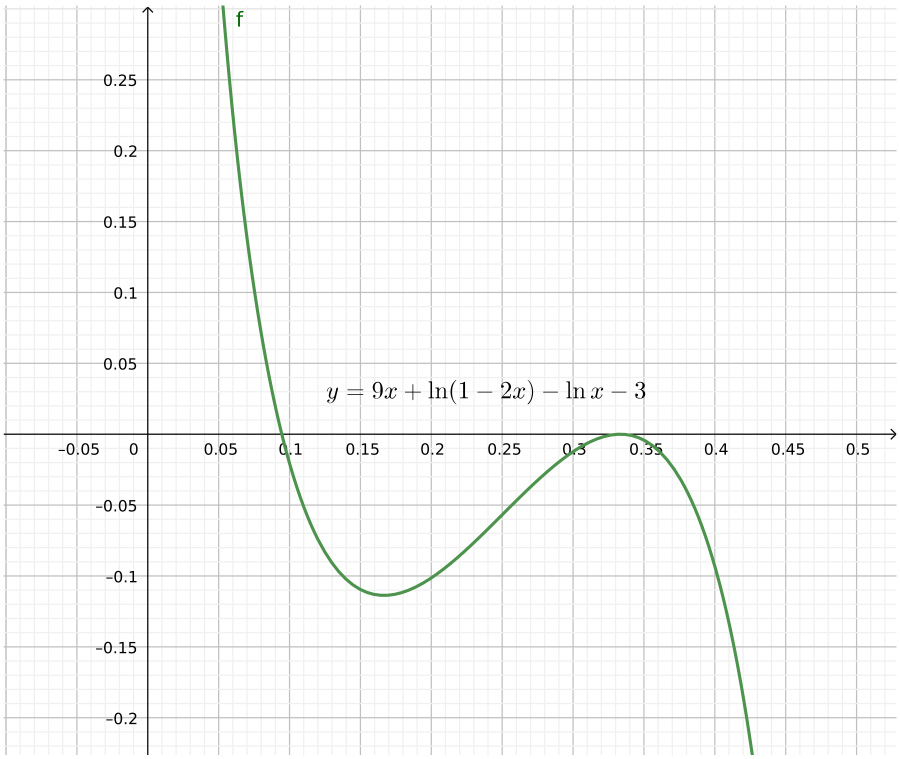

FIG. 3. *A graph of a function* $y(x) = 9x + \ln(1-2x) - \ln x - 3$.

It is seen from Figure 3 that the function $\ln(1-2l) - \ln l - 3 + 9l$ has exactly one root lying in the interval $\left(0, \frac{1}{4}\right)$, namely $l \approx 0.0945$. So $r \approx 0.8109$ and $\alpha \approx 8.577$.

Let us prove that there is a unique root of h lying inside $\left(0, \frac{1}{3}\right)$. To this end, we find the derivative of $h(l) = 9l + \ln(1-2l) - \ln l - 3$ and show it is negative for $l < \frac{1}{6}$ and positive for $\frac{1}{6} < l < \frac{1}{3}$. Indeed,

$$h'(l) = (9l + \ln(1-2l) - \ln l - 3)' = -\frac{(3l-1)(6l-1)}{l(1-2l)},$$

and it is easy to check the signs.

For $l \to +0$ one has

$$h(l) \to \infty.$$

For $l = \frac{1}{6}$ there holds

$$h\left(\frac{1}{6}\right) = 2\ln 2 - \frac{3}{2} < 0,$$

since $\ln 2 \approx 0.69 < \frac{3}{4}$. For $l = \frac{1}{3}$ there holds

$$h\left(\frac{1}{3}\right) = 3 + \ln\left(1 - \frac{2}{3}\right) - \ln\left(\frac{1}{3}\right) - 3 = 0.$$

It follows that on the interval $\left(0, \frac{1}{4}\right)$, function $h(l)$ has exactly one root, and this root is less than $\frac{1}{6}$.

The assumption of Lemma 3.2 is satisfied for the following broad class of functions.

LEMMA 3.3. *Let $f(x) : [0,1] \to \mathbb{R}$ be convex on $\left[0, \frac{2}{3}\right]$, and let $f(2x)+2f(1-x) = 0$ for $0 \le x \le \frac{1}{3}$. Then $f(x) + f(y) + f(z) \le 0$ for $x + y + z = 2$.*

Proof. Assume that $f(x)+f(y)+f(z) > 0$ for some x, y, and z satisfying $x+y+z = 2$. Among x, y, z let there be at least two numbers (say, $x \le y$) less than $\frac{2}{3}$. Replace these numbers by x', y' in such a way that $x + y = x' + y'$, $[x', y'] \subset \left[0, \frac{2}{3}\right]$, and either $x' = 0$ or $y' = \frac{2}{3}$. By convexity, $f(x') + f(y') + f(z) \ge f(x) + f(y) + f(z) > 0$. If $x' = 0$, then $y' \le \frac{2}{3}$ and $z \le 1$; thus $x' + y' + z < 2$. Hence $y' = \frac{2}{3}$.

Thus from the very beginning one can assume that $x \le \frac{2}{3}$ and $y, z \ge \frac{2}{3}$. If $x = y = z = \frac{2}{3}$, then $f(x) + f(y) + f(z) = f\left(2 \cdot \frac{1}{3}\right) + 2f\left(1 - \frac{1}{3}\right) = 0$. Repeating the same trick and using concavity of f on $\left[\frac{2}{3}, 1\right]$ one can reduce the problem to the case $y = z$. But for any triple $x, y = z = 1 - \frac{x}{2}$ there holds $f(x) + f(y) + f(z) = 0$, which contradicts the assumption $f(x) + f(y) + f(z) > 0$. ∎

3.2. Description of projections of measure classes on the triangle. We will consider special classes of measures on Δ and describe their projections onto the axes.

First, consider the Lebesgue measure on Δ. It can be normalized in such a way that the measure of the whole triangle is equal to $\frac{1}{2}$. We denote the normalized measure by λ_Δ. Projecting it to any hyperplane $\{x = 0\}$, $\{y = 0\}$, $\{z = 0\}$, we get a triangle with the usual Lebesgue measure. In what follows, we consider the densities with respect to this normalized measure.

DEFINITION 3.4. *Let μ be a measure on Δ absolutely continuous with respect to λ_Δ. For any point $(x, y, z) \in \Delta$ define $M(x, y, z) = \min(1 - x, 1 - y, 1 - z)$. We call a measure μ* layered *if for any t the density of μ is constant on a set $M(x, y, z) = t$, that is, density depends only on $M(x, y, z)$.*

It is easy to see that M is proportional to the distance from the point to the nearest side of Δ. Therefore, points with constant M form a triangle homothetic to the original one, with the same center. It is also easy to see that due to the symmetry of the layered measure, its projections on all three axes will be the same. Also note that M takes values only in $\left[0, \frac{1}{3}\right]$.

DEFINITION 3.5. *We say that a function $p : \left[0, \frac{1}{3}\right] \to \mathbb{R}$* generates *a layered measure μ if $\frac{d\mu}{d\lambda_\Delta}(x, y, z) = p\,(M(x, y, z))$.*

Let us find the projections of a layered measure μ generated by p to the coordinate axes.

PROPOSITION 3.6. *Let μ be a layered measure generated by a function p. Let $p_* : [0, 1] \to \mathbb{R}_+$ be the density of the projection of this measure onto an axis. Then*

$$p_*(x) = \begin{cases} 2\int_0^{\frac{x}{2}} p(t)dt & \text{if } x \le \frac{2}{3}, \\ (3x - 2)p(1 - x) + 2\int_0^{1-x} p(t)dt & \text{if } x \ge \frac{2}{3}. \end{cases}$$

Proof. Denote the projection of μ onto the hyperplane xy by μ_{xy}. It is concentrated on the triangle T with vertices $(0, 1)$, $(1, 0)$, and $(1, 1)$. Its density with respect to the Lebesgue measure on the plane at the point (x, y) lying inside T is

$$p(M(x, y, 2 - x - y)) = p(\min(1 - x, 1 - y, x + y - 1)).$$

Define μ_x as the projection of μ onto x or, what is the same, the projection of the measure μ_{xy} onto x. Then the measure of $[0, x_0]$ on the one hand is $\int_0^{x_0} p_*(x)dx$, and on the other hand is equal to the measure of the part of the triangle T where the x coordinate belongs to $[0, x_0]$. Thus, we have established the equality $\int_0^{x_0} p_*(x)dx = \int_0^{x_0} \int_{1-x}^1 p(M(x, y, 2-x-y))dxdy$. Differentiating both sides of this equality with respect to x_0, we obtain $p_*(x) = \int_{1-x}^1 p(\min(1-x, 1-y, x+y-1))dy$.

Assume $x \leq \frac{2}{3}$. Then

$$\min(1-x, 1-y, x+y-1) = \begin{cases} x+y-1 & \text{for } y \in \left[1-x, 1-\frac{x}{2}\right], \\ 1-y & \text{for } y \in \left[1-\frac{x}{2}, 1\right]. \end{cases}$$

From here we get

$$\begin{aligned} p_*(x) &= \int_{1-x}^1 p(\min(1-x, 1-y, x+y-1))dy \\ &= \int_{1-\frac{x}{2}}^1 p(1-y)dy + \int_{1-x}^{1-\frac{x}{2}} p(x+y-1)dy = 2\int_0^{\frac{x}{2}} p(t)\ dt. \end{aligned}$$

Analogously for $x \geq \frac{2}{3}$,

$$\min(1-x, 1-y, x+y-1) = \begin{cases} x+y-1 & \text{for } y \in [1-x, 2-2x], \\ 1-x & \text{for } y \in [2-2x, x], \\ 1-y & \text{for } y \in [x, 1]. \end{cases}$$

After this we calculate $p_*(x)$:

$$\begin{aligned} p_*(x) &= \int_{1-x}^1 p(\min(1-x, 1-y, x+y-1))dy \\ &= \int_{1-x}^{2-2x} p(x+y-1)dy + \int_{2-2x}^{x} p(1-x)dy + \int_x^1 p(1-y)dy \\ &= 2\int_0^{1-x} p(t)dt + (3x-2)p(1-x). \end{aligned}$$ ◻

Next we define *median measure.*

DEFINITION 3.7. *The median subset of* Δ *is the set*

$$\{(x, y, z) \in \Delta \mid x = y \geq z\} \cup \{(x, y, z) \in \Delta \mid y = z \geq x\} \cup \{(x, y, z) \in \Delta \mid x = z \geq y\}.$$

From a geometric point of view, this is a union of three segments in Δ *from the vertices to the center of the triangle* Δ.

Projections of any segment from the median set are $\left[0, \frac{2}{3}\right]$ and $\left[\frac{2}{3}, 1\right]$. On these segments one can define a measure proportional to the Lebesgue measure such that the measure of each segment is $\frac{2}{3}$. In what follows, we consider all of the densities on the median set with respect to this measure.

DEFINITION 3.8. *Median measure* μ, *generated by a density function* $q : [0, \frac{2}{3}] \to \mathbb{R}_+$, *is a measure concentrated on the median set such that its density on each of the segments is equal to* $q(t)$ *at the points* $(t, t, 2-2t)$, $(t, 2-2t, t)$, $(2-2t, t, t)$ *with respect to the reference measure described above.*

It is easy to verify the following assertion.

PROPOSITION 3.9. *Let μ be a median measure generated by q. Let $q_*(x)$ be the density of the projection of this measure onto an arbitrary axis. Then*

$$q_*(x) = \begin{cases} q(x) & \text{for } x < \frac{2}{3}, \\ 4q\,(2-2x) & \text{for } x > \frac{2}{3}. \end{cases}$$

This implies, in particular, the identity

$$4q_*(2x) = q_*(1-x),\ x < \frac{1}{3}. \tag{3.3}$$

The converse is also true: if nonnegative q_* satisfies (3.3), then there is a median measure whose projection onto the arbitrary axis coincides with q_*.

3.3. Combining measures. Let π be a measure on the segment $[0,1]$ with density f. We are concerned with $f(x) = \alpha^x$, but we will only use the fact that $f(x)$ is continuously differentiable, increasing, and convex and satisfies $\int_0^1 \left(t - \frac{2}{3}\right) f(t)dt = 0$. The last means that the measure with density f satisfies (3.1).

We want to find a measure μ that is the sum of the layered measure μ_p generated by a function p and the median measure μ_q generated by a function q, whose projection on each of the axes coincides with π.

We subtract μ_p from μ and look at the projection of $\mu - \mu_p$ onto the axes with the density $q_*(x)$. By Proposition 3.6, the projection is equal to

$$q_*(x) = \begin{cases} f(x) - 2\int_0^{\frac{x}{2}} p(t)dt & \text{for } x \le \frac{2}{3}, \\ f(x) - (3x-2)p(1-x) - 2\int_0^{1-x} p(t)dt & \text{for } x \ge \frac{2}{3}. \end{cases}$$

In order for $q_*(x)$ to be a density of the projection of a median measure, it suffices that $q_*(x) \ge 0$ and $4q_*(2x) = q_*(1-x)$ for $x \le \frac{1}{3}$. Using the identities on $q_*(x)$ given above, we obtain the equivalent equation,

$$4\left(f(2x) - 2\int_0^x p(t)dt\right) = f(1-x) - (1-3x)p(x) - 2\int_0^x p(t)dt. \tag{3.4}$$

Assuming $P(x) = \int_0^x p(t)dt$, we get the equation

$$4\left(f(2x) - 2P(x)\right) = f(1-x) - (1-3x)P'(x) - 2P(x).$$

This is a differential equation of the first degree, and its solutions have the form

$$P(x) = \frac{c_1 + \int_0^x (1-3t)(f(1-t) - 4f(2t))dt}{(1-3x)^2}.$$

Using $P(0) = 0$ we get $c_1 = 0$, and therefore

$$P(x) = \frac{1}{(1-3x)^2}\int_0^x (1-3t)(f(1-t) - 4f(2t))dt.$$

Now suppose that f is continuously differentiable. We find $p(x)$ using integration by parts:

$$\begin{aligned} p(x) &= P'(x) \\ &= \frac{(f(1-x) - 4f(2x))(1-3x)^2 + 6\int_0^x (1-3t)(f(1-t) - 4f(2t))dt}{(1-3x)^3} \\ &= \frac{1}{(1-3x)^3}\left(f(1) - 4f(0) - \int_0^x (1-3t)^2(f'(1-t) + 8f'(2t))dt\right). \end{aligned}$$

To prove that $p(x)$ and corresponding $q(x)$ generate a nonnegative density, we need to check that $p(x) \geq 0$ for $x \in \left[0, \frac{1}{3}\right]$, $p\left(\frac{1}{3}\right)$ is well-defined, and $f(2x) - 2P(x) = q_*(2x) = q(2x) \geq 0$ for $x \in \left[0, \frac{1}{3}\right]$, where $q(x)$ generates the median measure.

LEMMA 3.10. *Suppose that $f : [0,1] \to \mathbb{R}$ is a continuously differentiable monotonically increasing function and that $\int_0^1 \left(t - \frac{2}{3}\right) f(t)dt = 0$. Then the function*

$$I(x) = f(1) - 4f(0) - \int_0^x (1-3t)^2 (f'(1-t) + 8f'(2t))dt$$

is nonnegative on $\left[0, \frac{1}{3}\right]$ and $I\left(\frac{1}{3}\right) = 0$.

Proof. Since f is increasing, $f' \geq 0$, and the integrand $(1-3t)^2(f'(1-t)+8f'(2t))$ is nonnegative. So the integral increases, and $I(x)$ monotonically decreases to $I\left(\frac{1}{3}\right)$. Integrating by parts we get

$$\begin{aligned} I\left(\frac{1}{3}\right) &= f(1) - 4f(0) - \int_0^{\frac{1}{3}} (1-3t)^2(f'(1-t) + 8f'(2t))dt \\ &= \int_0^{\frac{1}{3}} (4f(2t) - f(1-t))d(1-3t)^2 \\ &= 6\int_0^{\frac{1}{3}} (1-3t)(f(1-t) - 4f(2t))dt \\ &= 18\int_0^1 \left(t - \frac{2}{3}\right) f(t)dt \\ &= 0. \end{aligned}$$ ∎

Using this lemma one can check that $p(x)$ is nonnegative and well-defined.

PROPOSITION 3.11. *Suppose that $f(x)$ satisfies the conditions of Lemma* 3.10. *Then the function*

$$p(x) = \frac{1}{(1-3x)^3}\left(f(1) - 4f(0) - \int_0^x (1-3t)^2(f'(1-t) + 8f'(2t))dt\right)$$

is nonnegative, and $\lim_{x \to \frac{1}{3}} p(x) = f'\left(\frac{2}{3}\right)$.

Proof. Using the function $I(x)$ from Lemma 3.10 we can rewrite the function $p(x)$ as follows:

$$p(x) = \frac{I(x)}{(1-3x)^3}.$$

$I(x)$ is nonnegative, as is $p(x)$. Let us check that $p\left(\frac{1}{3}\right)$ is well-defined.

Since $I\left(\frac{1}{3}\right) = 0$, one can apply the l'Hopital rule to $p(x)$:

$$\begin{aligned} \lim_{x \to \frac{1}{3}} p(x) &= \lim_{x \to \frac{1}{3}} \frac{I(x)}{(1-3x)^3} = \lim_{x \to \frac{1}{3}} -\frac{I'(x)}{9(1-3x)^2} \\ &= \lim_{x \to \frac{1}{3}} \frac{(1-3x)^2(f'(1-x) + 8f'(2x))}{9(1-3x)^2} \\ &= \frac{1}{9}\lim_{x \to \frac{1}{3}} (f'(1-x) + 8f'(2x)) = f'\left(\frac{2}{3}\right). \end{aligned}$$ ∎

Now we will check that the function q is nonnegative as well, so it generates the measure with nonnegative density.

Proposition 3.12. *Suppose that $f(x)$ satisfies the conditions of Lemma 3.10 and that $f(x)$ is convex on $[0,1]$. Then the function*

$$q(2x) = f(2x) - 2P(x) = f(2x) - \frac{2}{(1-3x)^2}\int_0^x (1-3t)(f(1-t) - 4f(2t))dt$$

is nonnegative.

Proof. Write the function q in the following form:

$$\begin{aligned}
q(2x) &= f(2x) - \frac{2}{(1-3x)^2}\int_0^x (1-3t)(f(1-t) - 4f(2t))dt \\
&= f(2x) + \frac{1}{3(1-3x)^2}\int_0^x (f(1-t) - 4f(2t))d(1-3t)^2 \\
&= f(2x) + \frac{(1-3t)^2(f(1-t)-4f(2t))|_0^x - \int_0^x (1-3t)^2(f'(1-t)+8f'(2t))dt}{3(1-3x)^2} \\
&= \frac{(1-3x)^2(f(1-x)-f(2x)) - I(x)}{3(1-3x)^2}.
\end{aligned}$$

To check that $q \geq 0$, it suffices to check that the numerator $n(x) = (1-3x)^2(f(1-x) - f(2x)) - I(x)$ is nonnegative. From Lemma 3.10, $n\left(\frac{1}{3}\right) = 0$. So we check that n is decreasing:

$$n'(x) = 6(1-3x)(f'(2x)(1-3x) - (f(1-x) - f(2x))),$$
$$n'(x) \leq 0 \Leftrightarrow f'(2x) \leq \frac{f(1-x) - f(2x)}{1-3x}.$$

The last equality holds since f is convex. ◻

Summarizing the last two propositions, we obtain the following theorem.

Theorem 3.13. *For any continuously differentiable, increasing, and convex function $f : [0,1]$ satisfying $\int_0^1 \left(t - \frac{2}{3}\right) f(t)dt = 0$, there exists a measure on Δ with projections onto the axes having densities $f(x)$.*

All of the assumptions can be applied to $f(x) = \alpha^x$, where α is a solution of (3.2). Also, we find $p(x)$ and $q(x)$ explicitly:

$$\begin{aligned}
p(x) &= -\frac{1}{(1-3x)^3}\int_0^x (1-3t)^2(f'(1-t) + 8f'(2t))dt \\
&= \frac{\alpha^{1-x} - 4\alpha^{2x}}{1-3x} - 6\frac{2\alpha^{2x} + \alpha^{1-x}}{(1-3x)^2\ln\alpha} - 6\frac{3\alpha^{2x} + 3\alpha - \alpha\ln\alpha - 3 - 2\ln\alpha - 3\alpha^{1-x}}{(1-3x)^3\ln^2\alpha} \\
&= \frac{\alpha^{1-x} - 4\alpha^{2x}}{1-3x} - 6\frac{2\alpha^{2x} + \alpha^{1-x}}{(1-3x)^2\ln\alpha} - 18\frac{\alpha^{2x} - \alpha^{1-x}}{(1-3x)^3\ln^2\alpha},
\end{aligned}$$

$$\begin{aligned}
q(2x) &= f(2x) - 2P(x) = f(2x) - \frac{2}{(1-3x)^2}\int_0^x (1-3t)(f(1-t) - 4f(2t))dt \\
&= \alpha^{2x} + 2\frac{2\alpha^{2x} + \alpha^{1-x}}{(1-3x)\ln\alpha} + 2\frac{3\alpha^{2x} + 3\alpha - \alpha\ln\alpha - 3 - 2\ln\alpha - 3\alpha^{1-x}}{(1-3x)^2\ln^2\alpha} \\
&= \alpha^{2x} + 2\frac{2\alpha^{2x} + \alpha^{1-x}}{(1-3x)\ln\alpha} + 6\frac{\alpha^{2x} - \alpha^{1-x}}{(1-3x)^2\ln^2\alpha}.
\end{aligned}$$

The last identities follow from (3.2).

Now we are ready to present the main theorem of this section.

THEOREM 3.14. *There exists a* $(3,1)$*-stochastic measure concentrated on the set* M.

Proof. Let us collect all details of the proof and describe our measure explicitly. Set M contains segments connecting points $(0,1,1)$ and (l,r,r), $(1,0,1)$ and (r,l,r), and $(1,1,0)$ and (r,r,l). These segments have length $L=\sqrt{l^2+2(1-r)^2}$. Define measure μ_{lin} as a sum of Lebesgue measures on these segments divided by $\frac{l}{L}$.

The projections of two segments coincide with $[r,1]$, and the densities are equal to $\frac{L}{1-r}\cdot\frac{l}{L}=\frac{1}{2}$. Their sum is the Lebesgue measure on $[r,1]$. The projection of the third interval is a measure on $[0,l]$, and its density equals $\frac{L}{l}\cdot\frac{l}{L}=1$.

The mapping

$$u=\frac{\ln x-\ln l}{\ln r-\ln l},\quad v=\frac{\ln y-\ln l}{\ln r-\ln l},\quad w=\frac{\ln z-\ln l}{\ln r-\ln l}$$

transforms the two-dimensional part of M into triangle Δ. We equip Δ with the layered measure μ_p generated by

$$p(x)=\frac{\alpha^{1-x}-4\alpha^{2x}}{1-3x}-6\frac{2\alpha^{2x}+\alpha^{1-x}}{(1-3x)^2\ln\alpha}-18\frac{\alpha^{2x}-\alpha^{1-x}}{(1-3x)^3\ln^2\alpha},$$

and the median measure μ_q generated by

$$q(2x)=\alpha^{2x}+2\frac{2\alpha^{2x}+\alpha^{1-x}}{(1-3x)\ln\alpha}+6\frac{\alpha^{2x}-\alpha^{1-x}}{(1-3x)^2\ln^2\alpha}.$$

Then by Proposition 3.6, the projection of μ_p coincides with

$$\begin{cases}2\int_0^{\frac{x}{2}}p(t)dt & \text{for } x\le\frac{2}{3},\\ (3x-2)p(1-x)+2\int_0^{1-x}p(t)dt & \text{for } x\ge\frac{2}{3}.\end{cases}$$

Since p is a solution of (3.4) for $f(x)=\alpha^x$, we can conclude that for

$$q_*(x)=\begin{cases}f(x)-2\int_0^{\frac{x}{2}}p(t)dt & \text{for } x\le\frac{2}{3},\\ f(x)-(3x-2)p(1-x)-2\int_0^{1-x}p(t)dt & \text{for } x\ge\frac{2}{3},\end{cases}$$

$4q_*(2x)=q_*(1-x)$ holds. Thus by Proposition 3.9, $q_*(x)$ is the projection of μ_q generated by $q(2x)=f(2x)-2\int_0^{\frac{x}{p}}(t)dt=\alpha^{2x}+2\frac{2\alpha^{2x}+\alpha^{1-x}}{(1-3x)\ln\alpha}+6\frac{\alpha^{2x}-\alpha^{1-x}}{(1-3x)^2\ln^2\alpha}$.

By Propositions 3.11 and 3.12 this construction is well-defined. Projections of $\mu_p+\mu_q$ onto the axes coincide with α^x in coordinates u,v,w and with the uniform measure on $[l,r]$ in initial coordinates.

Thus the projections of $\mu=\mu_p+\mu_q+\mu_{lin}$ coincide with the Lebesgue measure on $[0,1]$. ∎

4. The dual solution construction. To prove that the measure μ from Theorem 3.14 is the primal solution, it is enough to find a triple of functions $f,g,h:[0,1]\to\mathbb{R}$ such that $f(x)+g(y)+h(z)\le c(x,y,z)$, and equality holds on the set M by Lemma 1.1. In this case the triple (f,g,h) will be a dual solution of the related problem. In this section we will construct the dual solution for a wide class of cost functions.

We will construct the dual solution for $c(x, y, z) = \widehat{C}(\ln x + \ln y + \ln z)$, where $\widehat{C}$ is a bounded continuously differentiable strictly convex function on $(-\infty, 0]$. Our function $c(x, y, z) = xyz$ is a partial case for $\widehat{C}(t) = \exp(t)$. At the same time we will use the more convenient equivalent description. Namely, $c(x, y, z) = C(xyz)$ for some continuously differentiable function $C : [0, 1] \to \mathbb{R}$, and the function $tC'(t)$ strictly increases on the segment $[0, 1]$.

4.1. Another description of the support of primal solutions.

DEFINITION 4.1. *Set $c = lr^2$. Define a function $\lambda : [0, 1] \to \mathbb{R}$ as follows:*

$$\lambda(x) = \begin{cases} (1-2x)^2 & \text{if } x \in [0, l), \\ \frac{c}{x} & \text{if } x \in [l, r), \\ \frac{1}{2}x(1-x) & \text{if } x \in [r, 1]. \end{cases}$$

LEMMA 4.2. *The function λ defined above is continuous and strictly decreases.*

Proof. It suffices to check the continuity at points l and r. For this it suffices to check that $(1-2l)^2 = \frac{c}{l}$ and $\frac{c}{r} = \frac{1}{2}r(1-r)$. All of these equalities are trivial.

Let us check that the derivative of λ is negative everywhere except for the points l and r: in these points, λ has no derivatives.

If $x \in (0, l)$, then $\lambda'(x) = 2(2x-1) < 0$, since $x < l < \frac{1}{2}$. If $x \in (l, r)$, then $\lambda'(x) = -\frac{c}{x^2} < 0$ since $c > 0$. If $x \in (r, 1)$, then $\lambda'(x) = 1 - \frac{1}{2}x < 0$ since $x > r > \frac{1}{2}$.

It follows from this that λ strictly decreases. ∎

PROPOSITION 4.3. *Suppose that M is the (hypothetical) primal solution support as in the previous sections. Then a point (x, y, z) is contained in M if and only if the following equalities hold: $\lambda(x) = yz$, $\lambda(y) = xz$, $\lambda(z) = xy$.*

Proof. $\Leftarrow$ Suppose that $\kappa(x) = x\lambda(x)$. If $\lambda(x) = yz$, $\lambda(y) = xz$, and $\lambda(z) = xy$, then $\kappa(x) = \kappa(y) = \kappa(z) = xyz$.

The function $\kappa(x)$ is continuous and has a continuous derivative on intervals $(0, l)$, (l, r), and $(r, 1)$. If $x \in (0, l)$, then $\kappa'(x) = (1-2x)^2 - 2x(1-2x) = (1-2x)(1-4x) > 0$ since $x < l < \frac{1}{4}$. On the segment $[l, r]$, κ is constant: $\kappa(x) = lr^2 = c$. If $x \in (r, 1)$, then $\kappa'(x) = x(1-x) - \frac{1}{2}x^2 = x\left(1 - \frac{3}{2}x\right) < 0$ since $x > r > \frac{2}{3}$. So $\kappa(x)$ strictly increases on the segment $[0, l]$, is constant on $[l, r]$, and strictly decreases on $[r, 1]$.

Note, in addition, that $\kappa(0) = \kappa(1) = 0$. Thus, the equation $\kappa(x) = c_0$ for $0 \le x \le 1$

1. has no root if $c_0 < 0$ or $c_0 > c$;
2. has exactly two roots if $0 \le c_0 < c$: one of them lies on the interval $[0, l)$, and the other lies on the interval $(r, 1]$;
3. holds on the whole segment $[l, r]$ if $c_0 = c$.

If $\lambda(x) = yz$, $\lambda(y) = xz$, and $\lambda(z) = xy$, then $\kappa(x) = \kappa(y) = \kappa(z) = xyz$, and one of the following cases occurs:

1. $x, y, z \in [l, r]$. In this case, $c = \kappa(x) = \kappa(y) = \kappa(z) = xyz$ so $(x, y, z) \in M$.
2. $x = y = z \in [0, l)$. Then $\lambda(x) = x^2$. On the other hand, if $x \in [0, l)$, then $\lambda(x) = (1-2x)^2$. The equation $(1-2x)^2 = x^2$ has two solutions $x = 1$ and $x = \frac{1}{3}$. But these values are not feasible because $x \in [0, l)$ and $l < \frac{1}{6}$. So, this case is not possible.
3. $x = y = z \in (r, 1]$. Similarly, in this case $\lambda(x) = x^2$. On the other hand, if $x \in (r, 1]$, then $\lambda(x) = \frac{1}{2}x(1-x)$. Equation $\frac{1}{2}x(1-x) = x^2$ has two solutions $x = 0$ and $x = \frac{1}{3}$, but they do not belong to $(r, 1]$ for any $r > \frac{1}{2}$. So, this case is not possible.

4. $x = y \in [0, l)$, $z \in (r, 1]$, *and similar cases obtained by permutations of coordinates.* One has $x(1-2x)^2 = \kappa(x) = \kappa(z) = \frac{1}{2}z^2(1-z)$. The function $\kappa(z)$ strictly decreases on the interval $(r, 1]$, and hence for a fixed x there exists at most one z satisfying this equality. But $z = 1-2x \in (r, 1]$ and $\kappa(z) = \frac{1}{2}z^2(1-z) = \frac{1}{2}(1-2x)^2 \cdot 2x = \kappa(x)$. This means that $z = 1-2x$. In this case, $x(1-2x) = \frac{1}{2}z(1-z) = \lambda(z) = xy = x^2$. Hence $x = 0$ or $x = \frac{1}{3}$. But for $x = 0$, one has $1 = \lambda(x) = yz = xz = 0$. The value $x = \frac{1}{3}$ is not suitable because $x \in [0, l)$ and $l < \frac{1}{6}$. So, this case is not possible.
5. $x \in [0, l)$, $y = z \in (r, 1]$, *and similar cases obtained by permutations of coordinates.* Arguing as above, we get $\kappa(x) = \kappa(z)$, $x \in [0, l)$, $z \in (r, 1]$, and so $y = z = 1-2x$. The points $(x, 1-2x, 1-2x)$ are contained in M for any $x \in [0, l)$.

So, only cases 1 and 5 are possible. In these cases, $(x, y, z) \in M$.

$\Rightarrow$ The set M consists of four parts: $M = M_x \cup M_y \cup M_z \cup M_2$. If $(x, y, z) \in M_x$, then $y = z = 1-2x$. Hence $\lambda(x) = (1-2x)^2$ and $yz = (1-2x)^2$. In addition, $\lambda(y) = \lambda(1-2x) = \frac{1}{2}(1-2x) \cdot 2x = x(1-2x) = xz$ since $r \le 1-2x \le 1$. Similarly, $\lambda(z) = xy$.

Hence if $(x, y, z) \in M_x$, then $\lambda(x) = yz$, $\lambda(y) = xz$, and $\lambda(z) = xy$. By symmetry, these conditions hold for any $(x, y, z) \in M_y$ and for any $(x, y, z) \in M_z$.

If $(x, y, z) \in M_2$, then $l \le x, y, z \le r$ and $xyz = c$. This means that $\lambda(x) = \frac{c}{x} = yz$, $\lambda(y) = \frac{c}{y} = xz$, $\lambda(z) = \frac{c}{z} = xy$. ∎

4.2. The construction of the dual solution. If M is indeed a support of the primal solution and if f, g, h is a dual solution, then by complementary slackness, $f(x) + g(y) + h(z)$ is equal to $c(x, y, z)$ on almost all points of M. This will help us to guess the form of f, g, h.

LEMMA 4.4. *Assume that $c(x, y, z) = C(xyz)$ for some continuously differentiable function $C : [0, 1] \to \mathbb{R}$ and that the triple of functions*

$$f, g, h : [0, 1] \to \mathbb{R}$$

satisfies inequality $f(x) + g(y) + h(z) \le c(x, y, z)$, and $f(x) + g(y) + h(z) = c(x, y, z)$ for all $(x, y, z) \in M$. Then the functions f, g, h are continuously differentiable, and we have $f'(x) = \lambda(x)C'(x\lambda(x))$, $g'(y) = \lambda(y)C'(y\lambda(y))$, $h'(z) = \lambda(z)C'(z\lambda(z))$.

Proof. For any x_0 there exist y_0 and z_0 such that $(x_0, y_0, z_0) \in M$. This means that $f(x_0) + g(y_0) + h(z_0) = c(x_0, y_0, z_0) = C(x_0\lambda(x_0))$. In addition, for any x one has

$$f(x) + g(y_0) + h(z_0) \le c(x, y_0, z_0) = C(x\lambda(x_0)).$$

Hence, for any $x_0, x \in [0, 1]$ one has $f(x) - f(x_0) \le C(x\lambda(x_0)) - C(x_0\lambda(x_0))$.

Passing to the limit $x \to x_0$ one gets

$$C(x\lambda(x_0)) - C(x_0\lambda(x_0)) = (x - x_0) \cdot \lambda(x_0)C'(x_0\lambda(x_0)) + o(|x - x_0|).$$

Interchanging x_0 and x one gets $f(x_0) - f(x) \le C(x_0\lambda(x)) - C(x\lambda(x))$. By the mean value theorem, $C(x_0\lambda(x)) - C(x\lambda(x)) = (x_0 - x)\lambda(x)C'(\xi(x))$, where $\xi(x) \in [x_0\lambda(x), x\lambda(x)]$. If $x \to x_0$, then $\xi(x) \to x_0\lambda(x_0)$ and

$$\begin{aligned} C(x_0\lambda(x)) - C(x\lambda(x)) &= (x_0 - x)\lambda(x)C'(x_0\lambda(x_0)) + o(|(x_0 - x)\lambda(x)|) \\ &= (x_0 - x)\lambda(x)C'(x_0\lambda(x_0)) + o(|x - x_0|) \\ &= (x_0 - x)\lambda(x_0)C'(x_0\lambda(x_0)) + o(|x - x_0|). \end{aligned}$$

This means that

$$\begin{aligned}\lambda(x_0)C'(x_0\lambda(x_0))\cdot(x-x_0)+o(|x-x_0|)&\\ \le f(x)-f(x_0)&\\ \le \lambda(x_0)C'(x_0\lambda(x_0))\cdot(x-x_0)+o(|x-x_0|).&\end{aligned}$$

Hence, $f(x)$ has a derivative at the point $x=x_0$, and it is equal to $\lambda(x_0)C'(x_0\lambda(x_0))$. This function is continuous since λ and C' are continuous.

One can check in the same way the statements of the theorem for the functions g and h. ∎

THEOREM 4.5. *Suppose that $c(x,y,z)=C(xyz)$ for some continuously differentiable function $C:[0,1]\to\mathbb{R}$ and the function $U(t)=tC'(t)$ strictly increases on the segment $[0,1]$. Suppose that $\hat f(s)=\int_0^s\lambda(t)C'(t\lambda(t))\,dt$. Then the* arg max *of the function $\hat f(x)+\hat f(y)+\hat f(z)-c(x,y,z)$ contains the set M.*

Proof. Assume that $T(x,y,z)=\hat f(x)+\hat f(y)+\hat f(z)-c(x,y,z)=\hat f(x)+\hat f(y)+\hat f(z)-C(xyz)$. If $(x,y,z)\in M$, then

$$\nabla T(x,y,z)=\begin{pmatrix}\lambda(x)C'(x\lambda(x))-yzC'(xyz)\\ \lambda(y)C'(y\lambda(y))-xzC'(xyz)\\ \lambda(z)C'(z\lambda(z))-xyC'(xyz)\end{pmatrix}=\vec{0}.$$

Hence, all values of T on the set M are the same since M is path-connected.

The function T is continuous on the compact set $[0,1]^3$, so the function T reaches its maximum at some point (x_0,y_0,z_0). Then either x_0 lies on the boundary of the segment $[0,1]$ or $\frac{\partial T}{\partial x}(x_0,y_0,z_0)=0$.

For any $x>0$ the following equality holds:

$$\frac{\partial T}{\partial x}(x,y_0,z_0)=\lambda(x)C'(x\lambda(x))-y_0z_0C'(xy_0z_0)=\frac{U(x\lambda(x))-U(xy_0z_0)}{x}.$$

Assume that $x_0=0$. By the mean value theorem, for any $x>0$ there exists $0<\xi(x)<x$ such that

$$\begin{aligned}T(x,y_0,z_0)-T(x_0,y_0,z_0)&=x\frac{\partial T}{\partial x}(\xi(x),y_0,z_0)\\ &=\frac{x}{\xi(x)}\left(U[\xi(x)\lambda(\xi(x))]-U[\xi(x)y_0z_0]\right).\end{aligned}$$

One has $T(x,y_0,z_0)\le T(x_0,y_0,z_0)$ since (x_0,y_0,z_0) is a maximum point of T. Hence, $U[\xi(x)\lambda(\xi(x))]\le U[\xi(x)y_0z_0]$ and $\xi(x)\lambda(\xi(x))\le\xi(x)y_0z_0$ since U strictly increases. This means that $\lambda(\xi(x))\le y_0z_0$ for all $x>0$. If $x\to 0$, then $\lambda(\xi(x))\to\lambda(0)=1$. Thus $y_0z_0\ge 1\Rightarrow\lambda(x_0)=1=y_0z_0$.

Suppose that $x_0=1$. In this case $\frac{\partial T}{\partial x}(x_0,y_0,z_0)$ must be nonnegative. But $\frac{\partial T}{\partial x}(x_0,y_0,z_0)=\frac{U(x_0\lambda(x_0))-U(x_0y_0z_0)}{x_0}=U(0)-U(y_0z_0)$. The function $U(t)$ strictly increases; hence $y_0z_0=0$. This implies $0=\lambda(x_0)=y_0z_0$.

Otherwise, one has $\frac{\partial T}{\partial x}(x_0,y_0,z_0)=\frac{1}{x_0}(U(x_0\lambda(x_0))-U(x_0y_0z_0))=0$. The function $U(t)$ strictly increases. Hence, $x_0\lambda(x_0)=x_0y_0z_0$ and $\lambda(x_0)=y_0z_0$.

Consequently, if the function T has a maximum at the point (x_0,y_0,z_0), one gets $\lambda(x_0)=y_0z_0$. Similarly, one can prove that $\lambda(y_0)=x_0z_0$ and $\lambda(z_0)=x_0y_0$. Hence by Proposition 4.3, $(x_0,y_0,z_0)\in M$. Since T is constant on M, one has $M\subset\arg\max T$. ∎

Summarizing the results from the last two sections, we get the following.

THEOREM 4.6. *Suppose that* $c(x, y, z) = C(xyz)$ *for some continuously differentiable function* $C : [0, 1] \to \mathbb{R}$ *and that the function* $tC'(t)$ *strictly increases on the segment* $[0, 1]$. *Set*

$$\hat{f}(s) = \int_0^s \lambda(t)C'(t\lambda(t))\ dt.$$

Then for any constants C_x, C_y, C_z *such that*

$$C_x + C_y + C_z = C(0) - 2\int_0^1 \lambda(t)C'(t\lambda(t))\ dt$$

the inequality

$$(\hat{f}(x) + C_x) + (\hat{f}(y) + C_y) + (\hat{f}(z) + C_z) \le c(x, y, z)$$

holds with equality on M.

This means that by Lemma 1.1 *the triple* $(\hat{f} + C_x,\ \hat{f} + C_y,\ \hat{f} + C_z)$ *is the dual solution for the cost function* $c(x, y, z)$, *and any probability measure* μ *such that* $\mathrm{Pr}_X(\mu) = \mathrm{Pr}_Y(\mu) = \mathrm{Pr}_Z(\mu) = \lambda$ *and* $\mathrm{supp}(\mu) \subset M$ *is the primal solution to the related problem.*

Moreover, such a measure μ *exists by Theorem* 3.14.

We note that any primal solution is universal in the sense that it is the same for the cost functions of type $C(xyz)$, where $tC'(t)$ is strictly increasing on $[0, 1]$. It is important for the proof that M is path-connected. Numerical experiments for other marginals show that sometimes the support of a primal solution is not necessarily path-connected. For example, for a measure SF on $[0, 5]$ given by a density

$$\rho_{SF}(t) = \begin{cases} \frac{1}{15} \text{ if } t \in [0, 1] \cup [2, 3] \cup [4, 5], \\ \frac{2}{5} \text{ if } t \in (1, 2) \cup (3, 4), \end{cases}$$

the primal solution (more precisely, the result of Algorithm 2.2) for the cost function $c(x, y, z) = xyz$ is pictured in Figure 4.

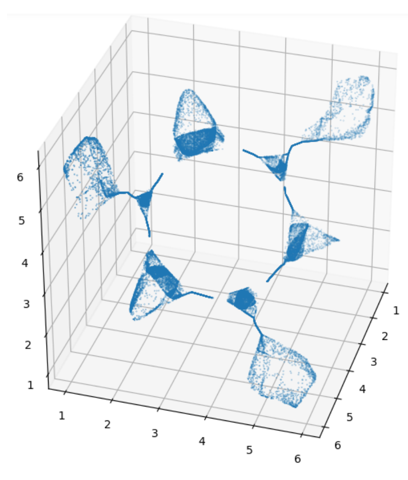

FIG. 4. *Primal solution for marginals* SF.

4.3. Construction for the cost function $c(x, y, z) = xyz$. Suppose that

$$c(x, y, z) = xyz = C(xyz),$$

where $C(t) = t$, $0 \leq t \leq 1$. The function $C(t)$ is continuously differentiable, and $tC'(t) = t$ strictly increases. Theorem 4.6 implies that any probability measure μ with projections $\mathrm{Pr}_X(\mu) = \mathrm{Pr}_Y(\mu) = \mathrm{Pr}_Z(\mu) = \lambda$ and $\mathrm{supp}(\mu) \subset M$ is the primal solution to the related problem; in particular, the probability measure from Theorem 3.14 is the primal solution. Also, we can construct explicitly the dual solution in this case.

Consider the following functions:

$$\begin{aligned}
f_1(x) &= c \ln l - \frac{1}{3}(c \ln c - c) + \frac{1}{6}((2x-1)^3 - (2l-1)^3), \\
f_2(x) &= c \ln x - \frac{1}{3}(c \ln c - c), \\
f_3(x) &= c \ln r - \frac{1}{3}(c \ln c - c) + \frac{1}{4}(x^2 - r^2) - \frac{1}{6}(x^3 - r^3).
\end{aligned}$$

These functions satisfy the following identities:

$$\begin{aligned}
f_1(l) &= f_2(l), \\
f_2(r) &= f_3(r), \\
f_1'(l) &= f_2'(l), \\
f_2'(r) &= f_3'(r).
\end{aligned}$$

The first and second equalities are easy to check directly. For the third and the fourth, compute $f_1'(x) = (2x-1)^2$, $f_2'(x) = \frac{c}{x}$, $f_3'(x) = \frac{1}{2}(x - x^2)$. $f_1'(l) = (2l-1)^2 = r^2 = \frac{c}{l} = f_2'(l)$, $f_2'(r) = lr = \frac{1}{2}r(1-r) = f_3'(r)$.

Define

$$f(x) = g(x) = h(x) = \begin{cases} f_1(x) \text{ if } 0 \leq x \leq l, \\ f_2(x) \text{ if } l \leq x \leq r, \\ f_3(x) \text{ if } r \leq x \leq 1. \end{cases}$$

It follows immediately from the properties checked above of the functions f_1, f_2, f_3 that f is continuous and continuously differentiable on $[0, 1]$ and that $f'(x) = \lambda(x)$.

Proposition 4.7. *The triple of functions (f, g, h) defined above is a dual solution of the related problem for the cost function $c(x, y, z) = xyz$.*

Proof. Since $f'(x) = \lambda(x)$, it follows that

$$f(x) = g(x) = h(x) = \int_0^x \lambda(x)\, dx + C_f = \int_0^x \lambda(x) C'(x\lambda(x))\, dx + C_f$$

for some constant C_f. By Theorem 4.6 it is enough to check that $f(0) + f(1) + f(1) = c(0, 1, 1) = 0$:

$$\begin{aligned}
f(0) = f_1(0) &= c \ln l - \frac{1}{3}(c \ln c - c) - \frac{1}{6}(2l-1)^3 - \frac{1}{6}, \\
f(1) = f_3(1) &= c \ln r - \frac{1}{3}(c \ln c - c) - \frac{1}{4}r^2 + \frac{1}{6}r^3 + \frac{1}{12}, \\
f(0) + 2f(1) &= c \ln(lr^2) - (c \ln c - c) + 2 \cdot \frac{1}{12} - \frac{1}{6} - \frac{1}{2}r^2 + \frac{1}{3}r^3 - \frac{1}{6}(2l-1)^3 \\
&= c - \frac{1}{2}r^2 + \frac{1}{2}r^3 = c - \frac{1-r}{2}r^2 = c - lr^2 = 0.
\end{aligned}$$

So, the triple (f, g, h) is the dual solution for the cost function $c(x, y, z) = xyz$. ∎

5. Uniqueness of the dual solution.

THEOREM 5.1. *Suppose that $c(x, y, z) = C(xyz)$ for some continuously differentiable function $C : [0, 1] \to \mathbb{R}$, and the function $tC'(t)$ strictly increases on the segment $[0, 1]$. Then the triple (f, g, h) is a dual solution if and only if there exist constants C_f, C_g, C_h such that*

$$C_f + C_g + C_h = C(0) - 2\int_0^1 \lambda(t)C'(t\lambda(t))\ dt$$

and

$$f(x) \le \int_0^x \lambda(t)C'(t\lambda(t))\ dt + C_f,$$

$$g(y) \le \int_0^y \lambda(t)C'(t\lambda(t))\ dt + C_g,$$

$$h(z) \le \int_0^z \lambda(t)C'(t\lambda(t))\ dt + C_h,$$

where equality is achieved almost everywhere.

Proof. $\Leftarrow$ Suppose that $\widetilde{f}(x) = \int_0^x \lambda(t)C'(t\lambda(t))\ dt + C_f$, $\widetilde{g}(y) = \int_0^y \lambda(t)C'(t\lambda(t))\ dt + C_g$ and $\widetilde{h}(z) = \int_0^z t\lambda(t)C'(t\lambda(t))\ dt + C_h$. Then the triple $(\widetilde{f}, \widetilde{g}, \widetilde{h})$ is the dual solution by Theorem 4.6. Also, $f(x) + g(y) + h(z) \le \widetilde{f} + \widetilde{g} + \widetilde{h} \le c(x, y, z)$ and $\int_0^1 f(x) + g(x) + h(x)\ dx = \int_0^1 \widetilde{f} + \widetilde{g} + \widetilde{h}\ dx$, so the triple (f, g, h) is the dual solution.

$\Rightarrow$ For any dual solution (f, g, h) there exists a triple $(\widetilde{f}, \widetilde{g}, \widetilde{h})$ such that $f \le \widetilde{f}$, $g \le \widetilde{g}$, $h \le \widetilde{h}$ and $\widetilde{f}(x) = \inf_{y,z}(c(x, y, z) - \widetilde{g}(y) - \widetilde{h}(z))$, $\widetilde{g}(y) = \inf_{x,z}(c(x, y, z) - \widetilde{f}(x) - \widetilde{h}(z))$, $\widetilde{h}(z) = \inf_{x,y}(c(x, y, z) - \widetilde{f}(x) - \widetilde{g}(y))$. One can prove this by applying the Legendre transformation subsequently to f, g, h.

For any x, y, z inequality $\widetilde{f}(x) + \widetilde{g}(y) + \widetilde{h}(z) \le c(x, y, z)$ holds since $\widetilde{f}(x) = \inf_{y,z}(c(x, y, z) - \widetilde{g}(y) - \widetilde{h}(z))$. Also,

$$\int_0^1 \widetilde{f}(x)\ dx + \int_0^1 \widetilde{g}(y)\ dy + \int_0^1 \widetilde{h}(z)\ dz \ge \int_0^1 f(x)\ dx + \int_0^1 g(y)\ dy + \int_0^1 h(z)\ dz$$

since $f \le \widetilde{f}$, $g \le \widetilde{g}$, and $h \le \widetilde{h}$. This means that the triple $(\widetilde{f}, \widetilde{g}, \widetilde{h})$ is a dual solution and that $\widetilde{f} = f$, $\widetilde{g} = g$, $\widetilde{h} = h$ almost everywhere.

A function $F[y, z] : [0, 1] \to \mathbb{R}$, $F[y, z](x) = c(x, y, z) - \widetilde{g}(y) - \widetilde{h}(z)$ is a Lipschitz continuous function since $\frac{\partial}{\partial x} c(x, y, z)$ is a well-defined continuous function on the cube $[0, 1]^3$. This means that $\widetilde{f}(x)$ is a Lipschitz continuous function since $\widetilde{f}$ is an infimum of the family of Lipschitz continuous functions $F[y, z]$ with common constant $\max_{x,y,z} \frac{\partial}{\partial x} c(x, y, z)$. In particular, this means that $\widetilde{f}$ is continuous on the segment $[0, 1]$. Similarly, the functions $\widetilde{g}$ and $\widetilde{h}$ are continuous.

For any primal solution μ equality $\widetilde{f}(x) + \widetilde{g}(y) + \widetilde{h}(z) = c(x, y, z)$ holds μ-almost everywhere. The set of equality points is closed, because f, g, and h are continuous. This means that $\widetilde{f}(x) + \widetilde{g}(y) + \widetilde{h}(z) = c(x, y, z)$ on the support of μ. For the primal solution μ from section 3 $\operatorname{supp}(\mu) = M$. So the equality $\widetilde{f}(x) + \widetilde{g}(y) + \widetilde{h}(z) = c(x, y, z)$ holds on the set M.

By Lemma 4.4 the functions $\widetilde{f}$, $\widetilde{g}$, and $\widetilde{h}$ are continuously differentiable, and $\widetilde{f}'(x) = \lambda(x)C'(x\lambda(x))$, $\widetilde{g}'(y) = \lambda(y)C'(y\lambda(y))$, $\widetilde{h}'(z) = \lambda(z)C'(z\lambda(z))$. This means that $\widetilde{f}(x) = \hat{f}(x) + C_f$, $\widetilde{g}(y) = \hat{f}(y) + C_g$, and $\widetilde{h}(z) = \hat{f}(z) + C_h$ for some constants C_f, C_g, and C_h. Since $(0,1,1) \in M$, the equality $C_f + C_g + C_h = c(0,1,1) - \hat{f}(0) - \hat{f}(1) - \hat{f}(1) = C(0) - 2\int_0^1 \lambda(t)C'(t\lambda(t))\,dt$ holds. ∎

6. A priori estimates for the dimension. Following [18] we introduce the following sets of matrices:

$$g_{\{x\}} = g_{\{y,z\}} = \begin{pmatrix} 0 & z & y \\ z & 0 & 0 \\ y & 0 & 0 \end{pmatrix},$$

$$g_{\{y\}} = g_{\{x,z\}} = \begin{pmatrix} 0 & z & 0 \\ z & 0 & x \\ 0 & x & 0 \end{pmatrix},$$

$$g_{\{z\}} = g_{\{x,y\}} = \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & x \\ y & x & 0 \end{pmatrix}.$$

Further, G is a linear combination of g_p with nonnegative coefficients:

$$G = \left\{ \begin{pmatrix} 0 & (\alpha+\beta)z & (\alpha+\gamma)y \\ (\alpha+\beta)z & 0 & (\beta+\gamma)x \\ (\alpha+\gamma)y & (\beta+\gamma)x & 0 \end{pmatrix} \middle|\, \alpha,\beta,\gamma \geq 0 \right\}.$$

By Theorem 2.1.2 from [18] the supports of solutions to the primal problem are locally contained inside a manifold of dimension

$$d = 3 - \text{positive index of inertia of } g$$

for any $g \in G$. This index is computed below.

PROPOSITION 6.1. *The quadratic form given by*

$$g = \begin{pmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{pmatrix}$$

with nonnegative a, b, and c has positive index of inertia at most 1.

Proof. Consider two cases.

Case 1. Let $a, b, c > 0$. Then the principal upper left minors are $\Delta_0 = 1$, $\Delta_1 = 0$, $\Delta_2 = -a^2 < 0$, and $\Delta_3 = 2abc > 0$. So the number of sign changes in the sequence of principal upper left minors is 2, and the negative index of inertia is 2. This means that the positive index of inertia is at most 1.

Case 2. Without loss of generality, $c = 0$. Then g has the form $2axy + 2bxz = \frac{1}{2}(x + (ay + bz))^2 - \frac{1}{2}(x - (ay + bz))^2$. Thus, the positive index of inertia is at most 1. ∎

We see that the local dimension of our solution is indeed not bigger than 2, but unfortunately this bound does not help us to determine the local dimension of our solution without solving the problem explicitly.

7. Extreme points. We show in this section that the extreme points of the primal solutions are singular to the surface (Hausdorff) measure on M. Applying logarithmic transformation from the proof of Theorem 3.14 and noticing that this is a (locally) bi-Lipschitz transformation, one can easily verify that it is sufficient to prove the claim for the triangle Δ. Further, projecting Δ onto the xy-hyperplane, we reduce the proof of the statement to the proof of the following fact.

THEOREM 7.1. *Let μ_x, μ_y, and μ_{x+y} be one-dimensional probability measures on the axes x, y and on the line $l_{x+y} = \{(x,y) \in \mathbb{R}^2 \colon x = y\}$, respectively. We assume that μ_x, μ_y and μ_{x+y} are compactly supported. Let Π be the set of probability measures with projections*

$$\mu_x = \mathrm{Pr}_x(\pi), \quad \mu_y = \mathrm{Pr}_y(\pi), \quad \mu_{x+y} = \mathrm{Pr}_{x+y}(\pi),$$

where Pr_x, Pr_y are projections onto x, y, and Pr_{x+y} is the projection onto l_{x+y}: $\mathrm{Pr}_{x+y}(x,y) = x+y$.

Assume that Π is nonempty and that $\pi \in \Pi$ is an extreme point. Then there exists a set S of zero Lebesgue measure such that $\pi(S) = 1$.

Proof. Without loss of generality, let us assume that π is supported by $X = [0,1]^2$. Let us consider the set of tuples of 6 points,

$$N = \Big\{ \Big((x_1,y_2),(x_1,y_3),(x_2,y_1),(x_2,y_3),(x_3,y_2),(x_3,y_1) \Big) \colon\ x_1 < x_2 < x_3, \\ y_1 < y_2 < y_3, x_1+y_2 = x_2+y_1, x_1+y_3 = x_3+y_1, x_2+y_3 = x_3+y_2 \Big\} \subset X^6.$$

For arbitrary $\Gamma \in N$ let us set

$$\Gamma_+ = \{(x_1,y_2),(x_2,y_3),(x_3,y_1)\},\ \Gamma_- = \{(x_1,y_3),(x_2,y_1),(x_3,y_2)\}.$$

Note that $\Gamma = \Gamma_- \sqcup \Gamma_+$ and that uniform distributions on the sets Γ_+ and Γ_- have the same projections onto both axes and l_{x+y}.

Let us show that there exists a set $S \subset X$ with the following properties: $\pi(S) = 1$, and S does not contain any subset of 6 points in N. According to Kellerer's result (see [11]) exactly one of the following holds:

- There exists a measure γ on X^6 with the property $\gamma(N) > 0$, such that $\mathrm{Pr}_i \gamma \le \pi$, $1 \le i \le 6$.
- For $1 \le i \le 6$ there exists a set $N_i \subset [0,1]^2 = X$ with the properties $\pi(N_i) = 0$ and
$$N \subset \cup_{i=1}^6 X \times \cdots \times N_i \times \cdots \times X.$$

In the second case by Kellerer,

$$S = X \setminus \cup_{i=1}^6 N_i$$

will be a desired set. We will prove it later.

First, we prove that the first case by Kellerer is impossible. We can assume that $\pi(X^6 \backslash N) = 0$ and π is still nonzero.

Suppose that $\Gamma = \Big((x_1,y_2),(x_1,y_3),(x_2,y_1),(x_2,y_3),(x_3,y_2),(x_3,y_1) \Big)$ is an arbitrary point of N and that $B_\Gamma \subset X^6$ is a ball with a center at Γ and a radius of $\varepsilon < \frac{1}{2}\min(x_2 - x_1, x_3 - x_2, y_2 - y_1, y_3 - y_2)$. Also suppose that $\tilde{\gamma} = \gamma|_{B_\Gamma}$ is a (possibly zero) measure on X^6 and that $\gamma_i = \mathrm{Pr}_i \tilde{\gamma}$ are measures on X. If $\gamma(B_\Gamma) > 0$, then full measure sets for γ_i are pairwise disjoint. In this case measures $\delta_- = \frac{1}{3}(\gamma_1 + \gamma_4 + \gamma_6)$ and $\delta_+ = \frac{1}{3}(\gamma_2 + \gamma_3 + \gamma_5)$ are distinct and have the same projections onto the axes and diagonal l_{x+y}.

LEMMA 7.2. *$\delta_- = \frac{1}{3}(\gamma_1 + \gamma_4 + \gamma_6)$ and $\delta_+ = \frac{1}{3}(\gamma_2 + \gamma_3 + \gamma_5)$ have the same projections onto the axes x, y and the line l_{x+y}.*

Proof. The functions $\mathrm{Pr}_x \circ \mathrm{Pr}_1$ and $\mathrm{Pr}_x \circ \mathrm{Pr}_2$, $\mathrm{Pr}_x \circ \mathrm{Pr}_3$ and $\mathrm{Pr}_x \circ \mathrm{Pr}_4$, and $\mathrm{Pr}_x \circ \mathrm{Pr}_5$ and $\mathrm{Pr}_x \circ \mathrm{Pr}_6$ coincide on N. So the images of π under this projections coincide. That means that $\mathrm{Pr}_x(\gamma_1) = \mathrm{Pr}_x(\gamma_2)$, $\mathrm{Pr}_x(\gamma_3) = \mathrm{Pr}_x(\gamma_4)$, $\mathrm{Pr}_x(\gamma_5) = \mathrm{Pr}_x(\gamma_6)$.

Analogously $\mathrm{Pr}_y(\gamma_1) = \mathrm{Pr}_y(\gamma_5)$, $\mathrm{Pr}_y(\gamma_2) = \mathrm{Pr}_y(\gamma_4)$, $\mathrm{Pr}_y(\gamma_3) = \mathrm{Pr}_y(\gamma_6)$ and $\mathrm{Pr}_{x+y}(\gamma_1) = \mathrm{Pr}_{x+y}(\gamma_3)$, $\mathrm{Pr}_{x+y}(\gamma_2) = \mathrm{Pr}_{x+y}(\gamma_6)$, $\mathrm{Pr}_{x+y}(\gamma_4) = \mathrm{Pr}_{x+y}(\gamma_5)$. ◻

Also, $\delta_- \le \pi$ and $\delta_+ \le \pi$ since $\gamma_i = \mathrm{Pr}_i\tilde{\gamma} \le \mathrm{Pr}_i\gamma \le \pi$.

Hence, $\pi_1 = \pi + \delta_+ - \delta_-$ and $\pi_2 = \pi - \delta_+ + \delta_-$ are nonnegative measures and have the same projections as π. So, $\pi = \frac{1}{2}(\pi_1 + \pi_2)$ is not an extreme point.

That means that for any $\Gamma \in N$ the measure of B_Γ with respect to γ is 0. Hence $\gamma(N) = 0$, which contradicts the assumption.

Thus, we get that there exists a set S with $\pi(S) = 1$ such that S does not contain sets of the type

$$\Big\{(x_1, y_2),\ (x_1, y_3),\ (x_2, y_1),\ (x_2, y_3),\ (x_3, y_2),\ (x_3, y_1),\ \ x_1 < x_2 < x_3,\ y_1 < y_2 < y_3,$$
$$x_1 + y_2 = x_2 + y_1,\ x_1 + y_3 = x_3 + y_1,\ x_2 + y_3 = x_3 + y_2\Big\}.$$

Let us show that S has Lebesgue measure zero. Assuming the contrary, let us apply the Lebesgue density theorem. According to this theorem for almost all $(x, y) \in S$ and every $\varepsilon > 0$ there exists an r-neighborhood U of (x, y) such that $\lambda(U \cap S) > (1 - \varepsilon)\lambda(U)$.

On the other hand, for all α and β the tuple of points

$$\Big\{(x+\alpha, y+\beta), \Big(x+\alpha, y+\frac{r}{10}+\beta\Big), \Big(x+\frac{r}{10}+\alpha, y+\beta\Big), \Big(x+\frac{r}{10}+\alpha, y+\frac{2r}{10}+\beta\Big),$$
$$\Big(x+\frac{2r}{10}+\alpha, y+\frac{r}{10}+\beta\Big), \Big(x+\frac{2r}{10}+\alpha, y+\frac{2r}{10}+\beta\Big)\Big\}$$

belongs to M. Hence, at least one of these points does not belong to S. If $0 \le \alpha, \beta \le \frac{r}{10}$, all of these points belong to the r-neighborhood of (x, y); hence the measure of the set $U \setminus S)$ is at least $\frac{r^2}{100}$. Choosing $\varepsilon < \frac{1}{100\pi}$ one gets a contradiction with the Lebesgue density theorem. ◻

Remark 7.3. Conjecture 1.3 says that there exist extreme measures with Hausdorff dimension less than 2. Numerical experiments reveal certain empirical evidence of this. Nevertheless, we were not able to verify this conjecture. In general, it is not true that sets which do not contain given configurations of points have dimension strictly less than the ambient space (see [14, 15]). An example of a low-dimensional solution is given in [7, Theorem 4.6].

Appendix A. Discrete case. Consider the following problem.

Problem A.1. We are given three copies A, B, C of the set $\{1, \dots, n\}$. Divide these $3n$ numbers into n groups of triples (a, b, c), where $a \in A, b \in B, c \in C$. We want to minimize the sum

$$S(n) = \sum_{(a,b,c)} abc.$$

Here the sum is taken over all the triples.

The main result of this appendix is as follows.

THEOREM A.2. *The minimum $F_D(n)$ of $S(n)$ over all partitions satisfies*

$$F_D(n) \sim C_P n^4,$$

where C_P is the value of the integral in the primal problem.

A.1. Connection with rearrangement inequality. The rearrangement inequality can be formulated as follows.

THEOREM A.3 (rearrangement inequality). *Assume that*

$$x_1 \le x_2 \le \cdots \le x_n,$$

$$y_1 \le y_2 \le \cdots \le y_n$$

are two ordered sets of real numbers, and σ is a permutation (rearrangement) of $\{1, 2, \ldots, n\}$. Then the following inequality holds:

$$x_1y_1+x_2y_2+\cdots+x_ny_n \ge x_1y_{\sigma(1)}+x_2y_{\sigma(2)}+\cdots+x_ny_{\sigma(n)} \ge x_1y_n+x_2y_{n-1}+\cdots+x_ny_1.$$

In other words, for the expression $x_1y_{\sigma(1)} + x_2y_{\sigma(2)} + \cdots + x_ny_{\sigma(n)}$ the maximum is attained at the identity permutation σ, and the minimum is attained at the permutation

$$\begin{pmatrix} 1 & 2 & \ldots & n \\ n & n-1 & \ldots & 1 \end{pmatrix}.$$

There exists a generalization of the rearrangement inequality for the case of several sets of variables.

THEOREM A.4. *Assume we are given s ordered sequences $x_1^{(i)} \le x_2^{(i)} \le \cdots \le x_n^{(i)}, i = 1, \ldots, s$. Consider the following functions of permutations:*

$$V(\sigma_1, \ldots, \sigma_s) = x^{(1)}_{\sigma_1(1)} x^{(2)}_{\sigma_2(1)} \cdots x^{(n)}_{\sigma_s(1)} + \cdots + x^{(1)}_{\sigma_1(n)} x^{(2)}_{\sigma_2(n)} \cdots x^{(n)}_{\sigma_s(n)}.$$

Let σ_0 be the identity permutation. Then for any permutation set $\sigma_1, \ldots, \sigma_n$ the inequality $V(\sigma_0, \ldots, \sigma_0) \ge V(\sigma_1, \sigma_2, \ldots, \sigma_s)$ holds.

Unfortunately, we do not know for which set of permutations $\sigma_1, \ldots, \sigma_s$ the value of $V(\sigma_1, \sigma_2, \ldots, \sigma_s)$ is minimal.

The permutations in the generalized rearrangement inequality correspond to a Monge solution for the multimarginal Monge–Kantorovich problem with cost function $x_1x_2 \ldots x_s$ and the marginals equal to counting measures on $x_j^{(i)}$. We remark that the generalized rearrangement inequality for 3 variables corresponds to the maximization problem $\int xyz d\pi \to \max$.

A.2. Approximation of a partition by measures. Let us introduce some notation. For every partition

$$Sp = \{(x_i, y_i, z_i) \mid 1 \le i \le n\}$$

of

$$A = B = C = \{1, 2, \ldots, n\}$$

into triples define

$$S_0(Sp) = \sum_{(x_i,y_i,z_i)\in Sp} x_iy_iz_i.$$

Denote by $|Sp| = n$ the *size* of partition Sp.

Let us try to reduce our problem to the transportation problem with the cost function xyz. For this purpose we construct the corresponding measure $\mu_1(Sp)$ on $[0,1]^3$ which is concentrated at points with denominator n; namely, every point $(\frac{x_i}{n}, \frac{y_i}{n}, \frac{z_i}{n})$ carries the mass $\frac{1}{n}$. Set $S_1(Sp) = \int_{[0,1]^3} xyz \, d\mu_1(Sp)$. It is easy to check that

$$n^4 S_1(Sp) = S_0(Sp).$$

Projections of $\mu_1(Sp)$ on axes are discrete, namely measures of points $\frac{i}{n}, 1 \leq i \leq n$, are equal to $\frac{1}{n}$. Thus, measure $\mu_1(Sp)$ is not $(3,1)$-stochastic in our sense, since its projections are not Lebesgue measures. This can be easily fixed. To this end, let us introduce another measure $\mu_2(Sp)$ on $[0,1]^3$: for all $1 \leq k \leq n$ there exists the uniform measure on

$$I_k = \left[\frac{x_k - 1}{n}, \frac{x_k}{n}\right] \times \left[\frac{y_k - 1}{n}, \frac{y_k}{n}\right] \times \left[\frac{z_k - 1}{n}, \frac{z_k}{n}\right]$$

with density n^2 (it is chosen in such a way that the measure of the whole given small cube equals $\frac{1}{n}$). This measure is $(3,1)$-stochastic.

Set

$$S_2(Sp) = \int_{[0,1]^3} xyz \, d\mu_2(Sp).$$

Let us estimate $S_2(Sp)$. For this, we set

$$\varepsilon(n) = \sup\left(|x_1 y_1 z_1 - x_2 y_2 z_2| \text{ subject to } \max(|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|) \leq \frac{1}{n}\right).$$

Function xyz is continuous on $[0,1]^3$, and then it is uniformly continuous on the given cube. It immediately follows that $\varepsilon(n) \to 0$ for $n \to \infty$. Then we can estimate $|S_1(Sp) - S_2(Sp)|$:

$$\begin{aligned}
|S_1(Sp) - S_2(Sp)| &= \left| \sum_{(x_k, y_k, z_k) \in Sp} \int_{I_k} (xyz - x_k y_k z_k) \, d\mu_2(Sp) \right| \\
&\leq \sum_{(x_k, y_k, z_k) \in Sp} \int_{I_k} |xyz - x_k y_k z_k| \, d\mu_2(Sp) \\
&\leq \sum_{(x_k, y_k, z_k) \in Sp} \int_{I_k} \varepsilon(n) \, d\mu_2(Sp) \\
&= \varepsilon(n) \xrightarrow[n \to \infty]{} 0.
\end{aligned}$$

Thus, $\lim_{n\to\infty} \frac{1}{n^4} F_D(n)$ exists if and only if there exists $\lim_{n\to\infty} \min_{|Sp|=n} S_2(Sp)$, and in the case of existence both limits coincide.

A.3. Convergence. In the previous subsection, we realized that it is sufficient to consider the problem of finding a partition Sp that minimizes $S_2(Sp)$. In this subsection we prove that $\lim_{n\to\infty} \min_{|Sp|=n} S_2(Sp)$ exists. Later we will see that $\lim_{n\to\infty} \min_{|Sp|=n} S_2(Sp) = C_P$, where C_P is the optimal value of the functionals in the primal and dual problems.

From the definition of C_P the following statement immediately follows.

PROPOSITION A.5. *For every partition Sp an inequality $S_2(Sp) \geq C_P$ holds.*

Indeed, $S_2(Sp)$ is the integral of xyz by some $(3,1)$-stochastic measure, and C_P is the minimum for all $(3,1)$-stochastic measures.

PROPOSITION A.6. *The sequence $s_k = \min_{|Sp|=k} S_2(Sp)$ admits a limit.*

Proof. The sequence s_k is bounded below by $C = C_P$. First, we check that $s_{n+k} \leq \left(\frac{n}{n+k}\right)^4 s_n + \frac{k}{n+k}$. Indeed, let Sp_n be a partition with $S_2(Sp_n) = s_n$. We construct a partition $Sp_{n+k} = Sp_n \cup \{(i,i,i) \mid n+1 \leq i \leq n+k\}$ and verify inequality $S_2(Sp_{n+k}) \leq \left(\frac{n}{n+k}\right)^4 s_n + \frac{k}{n+k}$:

$$\begin{aligned}
S_2(Sp_{n+k}) &= \sum_{(x_i,y_i,z_i)\in Sp_n} \int_{\frac{z_i-1}{n+k}}^{\frac{z_i}{n+k}} \int_{\frac{y_i-1}{n+k}}^{\frac{y_i}{n+k}} \int_{\frac{x_i-1}{n+k}}^{\frac{x_i}{n+k}} (n+k)^2 xyz \; dxdydz \\
&\quad + \sum_{i=n+1}^{n+k} \int_{\frac{i-1}{n+k}}^{\frac{i}{n+k}} \int_{\frac{i-1}{n+k}}^{\frac{i}{n+k}} \int_{\frac{i-1}{n+k}}^{\frac{i}{n+k}} (n+k)^2 xyz \; dxdydz \\
&\leq \sum_{(x_i,y_i,z_i)\in Sp_n} \int_{\frac{z_i-1}{n}}^{\frac{z_i}{n}} \int_{\frac{y_i-1}{n}}^{\frac{y_i}{n}} \int_{\frac{x_i-1}{n}}^{\frac{x_i}{n}} (n+k)^2 \left(\frac{n}{n+k}\right)^6 uvw \; dudvdw + \frac{k}{n+k} \\
&= \left(\frac{n}{n+k}\right)^4 \sum_{(x_i,y_i,z_i)\in Sp_n} \int_{\frac{z_i-1}{n}}^{\frac{z_i}{n}} \int_{\frac{y_i-1}{n}}^{\frac{y_i}{n}} \int_{\frac{x_i-1}{n}}^{\frac{x_i}{n}} n^2 uvw \; dudvdw + \frac{k}{n+k} \\
&= \left(\frac{n}{n+k}\right)^4 s_n + \frac{k}{n+k},
\end{aligned}$$

where $u := \frac{n+k}{n}x$, $v := \frac{n+k}{n}y$, $w := \frac{n+k}{n}z$.

We also verify that $s_{nk} \leq s_n + \varepsilon(n)$. As in the proof of the previous statement, assume that Sp_n is a partition with $S_2(Sp_n) = s_n$. We construct another partition $Sp_{nk} = \{(u_{k(i-1)+j}, v_{k(i-1)+j}, w_{k(i-1)+j})\} = \{(k(x_i-1)+j, k(y_i-1)+j, k(z_i-1)+j)\}$, where $1 \leq i \leq n$, $1 \leq j \leq k$. It is easy to check that Sp_{nk} is a partition.

We estimate $S_2(Sp_{nk})$. For indices i and j,

$$\begin{aligned}
\int_{I_{k(i-1)+j}} n^2k^2 xyz \; dxdydz &\leq \int_{\frac{z_i-1}{n}}^{\frac{z_i}{n}} \int_{\frac{y_i-1}{n}}^{\frac{y_i}{n}} \int_{\frac{x_i-1}{n}}^{\frac{x_i}{n}} \frac{n^2}{k}(xyz + \varepsilon(n)) \; dxdydz \\
&\leq \frac{n^2}{k} \int_{\frac{z_i-1}{n}}^{\frac{z_i}{n}} \int_{\frac{y_i-1}{n}}^{\frac{y_i}{n}} \int_{\frac{x_i-1}{n}}^{\frac{x_i}{n}} xyz \; dxdydz + \frac{1}{nk}\varepsilon(n).
\end{aligned}$$

From this we get

$$\begin{aligned}
S_2(Sp_{nk}) &= \sum_{i=1}^{n}\sum_{j=1}^{k} \int_{I_{k(i-1)+j}} n^2k^2 xyz \; dxdydz \\
&\leq \sum_{i=1}^{n}\sum_{j=1}^{k} \left(\frac{n^2}{k} \int_{\frac{z_i-1}{n}}^{\frac{z_i}{n}} \int_{\frac{y_i-1}{n}}^{\frac{y_i}{n}} \int_{\frac{x_i-1}{n}}^{\frac{x_i}{n}} xyz \; dxdydz + \frac{1}{nk}\varepsilon(n) \right) \\
&= \varepsilon(n) + \sum_{i=1}^{n} \int_{\frac{z_i-1}{n}}^{\frac{z_i}{n}} \int_{\frac{y_i-1}{n}}^{\frac{y_i}{n}} \int_{\frac{x_i-1}{n}}^{\frac{x_i}{n}} n^2 xyz \; dxdydz \\
&= s_n + \varepsilon(n).
\end{aligned}$$

From these inequalities we find that for $1 \le i \le n$,

$$s_{kn+i} \le \left(\frac{kn}{kn+i}\right)^4 s_{kn} + \frac{i}{kn+i} \le s_{kn} + \frac{1}{k+1} \le s_n + \varepsilon(n) + \frac{1}{k+1}.$$

As $\frac{1}{k+1} \to 0$, we get $s_m \le s_n + 2\varepsilon(n)$ for all sufficiently large m.

Set $C_1 = \liminf s_n$. We prove that $\lim_{n\to\infty} s_n = C_1$. Indeed, for any $\varepsilon > 0$ there exists such N such that $s_N < C_1 + \frac{\varepsilon}{2}$ and $2\varepsilon(N) < \frac{\varepsilon}{2}$. Then for all sufficiently large m the inequality $s_m \le s_N + 2\varepsilon(N) < C_1 + \varepsilon$ holds. In addition, for all sufficiently large m inequality $s_m > C_1 - \varepsilon$ holds; otherwise, there exists a convergent subsequence, with a limit not greater than $C_1 - \varepsilon$. Thus, $\lim_{n\to\infty} s_n = C_1$; in particular, this sequence is convergent. ∎

From this statement it follows that it suffices to find partitions Sp_t of an arbitrary size for which $\lim_{t\to\infty} S_2(Sp_t) = C_P$.

A.4. Discrete measure approximation. Let $\widetilde{\mu}$ be a measure solving the primal problem. For a given n we define another measure $\widetilde{\mu}_n$. We require that $\widetilde{\mu}_n$ is uniform on every

$$I_{ijk} = \left[\frac{i-1}{n}, \frac{i}{n}\right] \times \left[\frac{j-1}{n}, \frac{j}{n}\right] \times \left[\frac{k-1}{n}, \frac{k}{n}\right],$$

for $1 \le i, j, k \le n$, and satisfies $\int_{I_{ijk}} 1\, d\widetilde{\mu} = \int_{I_{ijk}} 1\, d\widetilde{\mu}_n$. The latter quantity will be denoted by ρ_{ijk}. The resulting measure will be $(3,1)$-stochastic.

Set $c_{ijk} = \min(xyz \mid (x,y,z) \in I_{ijk})$. Then for all $(x,y,z) \in I_{ijk}$ there holds $|c_{ijk} - xyz| < \varepsilon(n)$. Hence, it is possible to estimate $|\int_{[0,1]^3} xyz\, d\widetilde{\mu} - \int_{[0,1]^3} xyz\, d\widetilde{\mu}_n|$:

$$\begin{aligned}
&\left|\int_{[0,1]^3} xyz\, d\widetilde{\mu} - \int_{[0,1]^3} xyz\, d\widetilde{\mu}_n\right| \le \sum_{1\le i,j,k,\le n} \left|\int_{I_{ijk}} xyz\, d\widetilde{\mu} - \int_{I_{ijk}} xyz\, d\widetilde{\mu}_n\right| \\
&\le \sum_{1\le i,j,k,\le n} \left|\int_{I_{ijk}} c_{ijk}\, d\widetilde{\mu} - \int_{I_{ijk}} c_{ijk}\, d\widetilde{\mu}_n\right| + \varepsilon(n)\left(\int_{I_{ijk}} 1\, d\widetilde{\mu} + \int_{I_{ijk}} 1\, d\widetilde{\mu}_n\right) \\
&= \varepsilon(n)\left(\int_{[0,1]^3} 1\, d\widetilde{\mu} + \int_{[0,1]^3} 1\, d\widetilde{\mu}_n\right) = 2\varepsilon(n).
\end{aligned}$$

For the following discussion we need the following theorem.

THEOREM A.7 (Dirichlet's theorem on the Diophantine approximation). *Assume we are given a set of real numbers $(a_1, a_2, \dots, a_d)$. Then for every $\varepsilon > 0$ there exist a natural number m and integers $b_1, b_2, \dots, b_d$ such that $|a_i m - b_i| < \varepsilon$ for all $1 \le i \le d$.*

Applying this theorem for the set $n\rho_{ijk}$, we find that for any ε_1 there exists a natural m, such that $\rho_{ijk} = \frac{t_{ijk}+\varepsilon_{ijk}}{nm}$, where $|\varepsilon_{ijk}| < \varepsilon_1$ and all t_{ijk} are integers. We construct the measure $\nu_{n,m}$ as follows: on each cube I_{ijk} we define a uniform measure in such a way that the measure of the whole cube I_{ijk} is equal to $\frac{t_{ijk}}{nm}$.

We verify that this measure is $(3,1)$-stochastic provided $\varepsilon_1 < \frac{1}{n^2}$. For this, it suffices to verify that the sum of all $\frac{t_{ijk}}{nm}$ with one argument fixed is equal to $\frac{1}{n}$. Without loss of generality, we fix i. Then $\frac{1}{n} = \sum_{1\le j,k\le n} \rho_{ijk} = \sum_{1\le j,k\le n} \frac{t_{ijk}}{nm} + \sum_{1\le j,k\le n} \frac{\varepsilon_{ijk}}{nm}$ or $m = \sum_{1\le j,k\le n} t_{ijk} + \sum_{1\le j,k\le n} \varepsilon_{ijk}$. All t_{ijk} are natural numbers, and $|\sum_{1\le j,k\le n} \varepsilon_{ijk}| \le n^2\varepsilon_1 < 1$; thus $\sum_{1\le j,k\le n} t_{ijk} = m$, as required.

Estimate the difference $|\int_{[0,1]^3} xyz\ d\widetilde{\mu}_n - \int_{[0,1]^3} xyz\ d\nu_{n,m}|$:

$$\left|\int_{[0,1]^3} xyz\ d\widetilde{\mu}_n - \int_{[0,1]^3} xyz\ d\nu_{n,m}\right| \leq \sum_{1\leq i,j,k,\leq n} \left|\int_{I_{ijk}} xyz\ d\widetilde{\mu}_n - \int_{I_{ijk}} xyz\ d\nu_{n,m}\right|$$
$$\leq \sum_{1\leq i,j,k,\leq n} \left|\int_{I_{ijk}} c_{ijk}\ d\widetilde{\mu}_n - \int_{I_{ijk}} c_{ijk}\ d\nu_{n,m}\right| + \varepsilon(n)\left(\int_{I_{ijk}} 1\ d\widetilde{\mu}_n + \int_{I_{ijk}} 1\ d\nu_{n,m}\right)$$
$$= \sum_{1\leq i,j,k\leq n} c_{ijk}\left|\rho_{ijk} - \frac{t_{ijk}}{nm}\right| + \varepsilon(n)\left(\int_{[0,1]^3} 1\ d\widetilde{\mu}_n + \int_{[0,1]^3} 1\ d\nu_{n,m}\right)$$
$$\leq \frac{n^3\varepsilon_1}{nm} + 2\varepsilon(n) \leq n^2\varepsilon_1 + 2\varepsilon(n).$$

Assume we have found a partition Sp_{nm} and the corresponding $\mu_2(Sp_{nm})$ such that every I_{ijk} contains exactly t_{ijk} small cubes with sides $\frac{1}{nm}$. Then one can control the difference $|\int_I xyz\ d\nu_{n,m} - \int_I xyz\ d\mu_2(Sp_{nm})|$ in the same way as above. One can easily check that the upper bound is $2\varepsilon(n)$; hence $|\int_I xyz\ d\mu_2(Sp_{nm}) - C_P| \leq 6\varepsilon(n) + n^2\varepsilon_1$. This number can be less than any preassigned ε: first, we choose n, such that $6\varepsilon(n) < \varepsilon/2$, then choose ε_1, such that $n^2\varepsilon_1 < \varepsilon/2$.

Thus, to complete the main proof of this appendix, it is sufficient to show that for given numbers $t_{ijk}, 1 \leq i,j,k \leq n$, it is always possible to construct a partition with the required property. Namely, using the fact that for a fixed i the sum $\sum_{1\leq j,k\leq n} t_{ijk}$ is equal to m, we build a partition $Sp_{nm} = \{(x_i, y_i, z_i) \mid 1 \leq i \leq nm\}$ with

$$\{x_1, \dots, x_{nm}\} = \{y_1, \dots, y_{nm}\} = \{z_1, \dots, z_{nm}\} = \{1, 2, \dots, nm\}$$

such that for fixed $i, j, k \in \{1, \dots, n\}$ the number of indices t satisfying

$$m(i-1) < x_t \leq mi, \quad m(j-1) < y_t \leq mj, \quad m(k-1) < z_t \leq mk$$

equals t_{ijk}.

In order to do this, we construct a correspondence between the numbers $1, \dots, nm$ and the triples (i,j,k), $1 \leq i,j,k \leq n$, in such a way that to every index we assign exactly one triple, and every triple (i,j,k) corresponds to exactly t_{ijk} indices lying in the half-open interval $(m(i-1), mi]$. The construction is accomplished step by step. The interval $(m(i-1), mi]$ containing the first t_{i11} numbers corresponds to the triple $(i,1,1)$, the following t_{i12} numbers correspond to the triple $(i,1,2)$, and so on. The last t_{inn} numbers are associated with (i,n,n). This procedure is possible because $\sum_{1\leq j,k\leq n} t_{ijk} = m$.

Similarly, we construct the correspondences in the second and third coordinates. As a result, every triple (i,j,k) corresponds to a set of numbers $a_{(i,j,k),1}, \dots, a_{(i,j,k),t_{ijk}}$ from $(m(i-1), mi]$, numbers $b_{(i,j,k),1}, \dots, b_{(i,j,k),t_{ijk}}$ from $(m(j-1), mj]$, and numbers $c_{(i,j,k),1}, \dots, c_{(i,j,k),t_{ijk}}$ from $(m(k-1), mk]$. Then we set

$$Sp_{nm} = \{a_{(i,j,k),t}, b_{(i,j,k),t}, c_{(i,j,k),t} \mid 1 \leq i,j,k \leq n, 1 \leq t \leq t_{ijk}\}.$$

Clearly, this will be a partition of size nm, since the values of the numbers $a_{(i,j,k),t}$, $b_{(i,j,k),t}$, and $c_{(i,j,k),t}$ are exactly the set $\{1, \dots, nm\}$.

This completes the proof of Theorem A.2. ∎

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