On Harris–Kleitman type inequalities

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Abstract

We prove an inequality for the expected values of functions on the hypercube, generalizing both the Harris–Kleitman inequality and a previous result by the authors.

1 Introduction

The Harris–Kleitman inequality is a result in probability theory independently discovered by Harris [H60] and Kleitman [K66], each from a different perspective. Harris used it to show that the critical probability for a bond percolation on $\mathbb{Z}^2$ is at least $\frac{1}{2}$; Kleitman proved it as a generalization of a conjecture by Erdös on subset families with no disjoint subsets.

We consider our probability space to be the hypercube $H_n = 2^{[n]}$ and μ to be a probability product measure on it. For $x, y \in H_n$ we say $x \preceq y$ if x is coordinatewise less or equal to y. The subset of H_n is called closed upwards if with each x it contains all y such that $x \preceq y$. In particular, the indicator function of each closed upward subset is nondecreasing.

The Harris–Kleitman inequality says that any two closed upwards events have a non-negative correlation.

Theorem 1.1. *Let A and B be two subsets of H_n closed upwards. Then, for a probability $\mathbf{P}$ given by a product measure μ, one has*

$$\mathbf{P}(A \cap B) \geq \mathbf{P}(A)\mathbf{P}(B).$$

One can think of Theorem 1.1 the following way. Let M_2 be the diamond poset given by Figure 1.

Theorem 1.2. *Let H_n be split into parts S_a, S_{b_1}, S_{b_2} and S_c such that if $x \preceq y$ then x belongs to a part indexed by an element of M_2 not less than the part of y. Then*

$$\mu(S_a)\mu(S_c) \geq \mu(S_{b_1})\mu(S_{b_2}).$$

As we will see in Example 2.4, this is the only condition on $\mu(S_a)$, $\mu(S_{b_1})$, $\mu(S_{b_2})$ and $\mu(S_c)$.

In the applications arising from Bernoulli percolation, H_n is mapped to various posets. So in Section 2, we study the inequalities for the sizes of the preimages of poset elements under these mappings. In Section 3 we give the applications of these inequalities to connection probabilities in Bernoulli bond percolation.

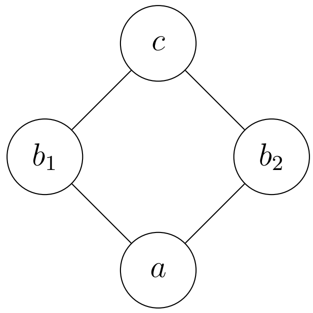

Figure 1: Poset M_2

Definition 1.3. We say that for the given poset $\mathcal{P}$ a vector of probabilities $\{m_p\}_{p\in\mathcal{P}}$ is *realizable* if there exists n such that H_n can be split into subsets S_p indexed by $p \in \mathcal{P}$ such that if $x \in S_a$ and $y \in S_b$ are such that $x \preceq y$, then $a \leq_{\mathcal{P}} b$ and there is a product measure μ on H_n such that $\mu(S_p) = m_p$.

Example 2.4 gives a characterization of realizeable vectors for M_2 poset. We only have necessary conditions for a vector of probabilities to be realizeable, but we pose Conjecture **??** saying that this condition is also sufficient.

In Section 4, we provide a computationally tractable way to check whether a vector of probabilities for a poset $\mathcal{P}$ satisfies all our inequalities. Finally, in Section **??** we provide conjectures on realizeability.

2 Main results

Theorem 2.1. *Let μ be a probability product measure on $2^{[n]}$. Let $g(x, y)$ be a function on $2^{[n]} \times 2^{[n]}$ such that for any $x \preceq y$, $z \preceq t$ one has*

$$g(x, z) + g(y, t) \leq g(x, t) + g(y, z). \tag{1}$$

Then

$$\mathbf{E}_{\mu\times\mu} g(x, y) \geq \mathbf{E}_{\mu} g(x, x). \tag{2}$$

Moreover, if the sign in (1) is reversed, the sign in (2) reverses as well.

Proof. The proof proceeds by the induction on n. For $n = 0$, equation (2) turns into equality. Consider the last coordinate and let μ' be the projection of μ onto the rest of the coordinates. It will also be a product measure. Moreover, the projection of μ to the last coordinate will assign probability p to 0 and $1 - p$ to 1. Let x and y be generated independently according to μ' and x^-, x^+, y^- and y^+ be defined as x and y supplied with the last coordinate equal to 0 and 1 respectively. From the induction hypothesis, we know that $\mathbf{E}_{\mu'\times\mu'} g(x^-, y^-) \geq \mathbf{E}_{\mu'\times\mu'} g(x^-, x^-)$ and $\mathbf{E}_{\mu'\times\mu'} g(x^+, y^+) \geq \mathbf{E}_{\mu'\times\mu'} g(x^+, x^+)$.

Combining this with the condition (1) applied to x^-, x^+, y^- and y^+ we get

$$
\begin{aligned}
&\mathbf{E}_{\mu\times\mu} g(x,y) \\
&= \mathbf{E}_{\mu'\times\mu'}\big(p^2 g(x^-,y^-) + p(1-p)g(x^-,y^+) + p(1-p)g(x^+,y^-) + (1-p)^2 g(x^+,y^+)\big) \\
&\qquad = p(1-p)\mathbf{E}\big(g(x^-,y^+) + g(x^+,y^-) - g(x^-,y^-) - g(x^+,y^+)\big) \\
&\qquad\qquad + p\mathbf{E}_{\mu'\times\mu'} g(x^-,y^-) + (1-p)\mathbf{E}_{\mu'\times\mu'} g(x^+,y^+) \\
&\qquad\qquad \geq 0 + p\mathbf{E}_{\mu'} g(x^-,x^-) + (1-p)\mathbf{E}_{\mu'} g(x^+,y^+) \\
&\qquad\qquad\qquad = \mathbf{E}_{\mu} g(x,x).
\end{aligned}
$$

If the sign in (1) is reversed, we switch the direction of the only inequality in the chain, completing the proof. □

In particular, Theorem 2.1 generalizes the original Harris–Kleitman theorem. We now present a functional generalization of the Harris–Kleitman inequality, which can be particularly useful in certain applications. The original version is recovered when f_1 and f_2 are characteristic functions of A and B respectively.

Theorem 2.2. *Let f_1 and f_2 be nondecreasing functions on H_n, then f_1 and f_2 correlate nonnegatively with respect to μ. In other words,*

$$\mathbf{E}_\mu\left(f_1(x)f_2(x)\right) \geq \mathbf{E}_{\mu\times\mu} f_1(x)f_2(y) = \mathbf{E}_\mu f_1(x)\mathbf{E}_\mu f_2(y)$$

Proof. Consider $g(x,y) = f_1(x)f_2(y)$. Then for $x \preceq y$, $z \preceq t$ one has

$$
\begin{aligned}
g(x,z) + g(y,t) - g(x,t) - g(y,z) &= f_1(x)f_2(z) + f_1(y)f_2(t) - f_1(x)f_2(t) - f_1(y)f_2(z) \\
&= (f_1(y) - f_1(x))(f_2(t) - f_2(z)) \geq 0.
\end{aligned}
$$

So, the condition of Theorem 2.1 holds and the conclusion is

$$\mathbf{E}_\mu\left(f_1(x)f_2(x)\right) \geq \mathbf{E}_{\mu\times\mu} f_1(x)f_2(y) = \mathbf{E}_\mu f_1(x)\mathbf{E}_\mu f_2(y)$$

which shows that f_1 and f_2 have nonnegative correlation with respect to μ. □

We notice that it is convenient to consider g to be constant on subsets of H_n. This motivates the following partial case of Theorem 2.1. It can be seen as the set of restrictions on realizable vectors of probabilities for a given poset.

Theorem 2.3. *Let $\mathcal{P}$ be a poset of size m and H_n be split into subsets S_p indexed by $p \in \mathcal{P}$ such that if $x \in S_a$ and $y \in S_b$ are such that if $x \preceq y$ then $a \leq_{\mathcal{P}} b$. Let A be an $m \times m$ matrix satisfying the condition*

$$A_{a,c} + A_{b,d} \leq A_{a,d} + A_{b,c}, \tag{3}$$

whenever $a <_{\mathcal{P}} b$ and $c <_{\mathcal{P}} d$. Then for any probability product measure μ we have

$$\sum_{a,b\in\mathcal{P}} A_{a,b}\mu(S_a)\mu(S_b) \geq \sum_{a\in\mathcal{P}} A_{a,a}\mu(S_a) \tag{4}$$

Proof. We use Theorem 2.1 for the function g that is constant within the subsets S_p. For $x \in S_a$ and $y \in S_b$ we put $g(x,y) = A_{a,b}$. It is easy to check that condition (3) implies (1) for g and so

$$\sum_{a,b\in\mathcal{P}} A_{a,b}\mu(S_a)\mu(S_b) = \mathbf{E}_{\mu\times\mu} g(x,y) \geq \mathbf{E}_\mu g(x,x) = \sum_{a\in\mathcal{P}} A_{a,a}\mu(S_a).$$

□

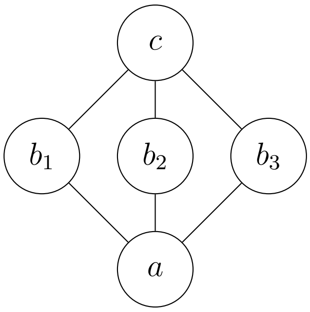

Figure 2: Poset M_3

An example poset to keep in mind here is the poset M_3 that is the poset of the smallest nondistributive lattice (see Figure 2). This poset was used in [G24a]. It corresponds to a partition lattice of a set $\{1, 2, 3\}$. So for bond percolation one can think of inequalities on sizes of S_p as inequalities on Boolean combinations of the events "vertex i is connected to vertex j" (see Theorem 3.2).

Example 2.4. For a diamond poset (see Figure 1), the vector of probabilities

$$\{m_a, m_{b_1}, m_{b_2}, m_c\}$$

is realizable if and only if $m_a m_c \geq m_{b_1} m_{b_2}$.

Proof. Indeed, if the vector is realizable, then we apply Theorem 2.3 with

$$A = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 0 & 0 & 1 \\ b & 0 & 0 & -1 & 0 \\ c & 0 & -1 & 0 & 0 \\ d & 1 & 0 & 0 & 0 \end{array},$$

which gives the needed inequality. Conversely, if $m_a m_c \geq m_{b_1} m_{b_2}$, denote by p the excess of m_c over its minimal attainable value: $p = m_c - \frac{m_{b_1} m_{b_2}}{m_a}$ and $1 - p = \frac{(m_a + m_{b_1})(m_a + m_{b_2})}{m_a}$.

Then consider the following product measure on $\{0, 1\}^3$: the first coordinate is 0 with probability $1 - p = \frac{(m_a + m_{b_1})(m_a + m_{b_2})}{m_a}$ and 1 with probability p, the second coordinate is 0 with probability $\frac{m_a}{m_a + m_{b_1}}$ and 1 with probability $\frac{m_{b_1}}{m_a + m_{b_1}}$ and the third coordinate is 0 with probability $\frac{m_a}{m_a + m_{b_2}}$ and 1 with probability $\frac{m_{b_2}}{m_a + m_{b_2}}$. Then consider the sets

$$\begin{aligned} S_a &= \{(0, 0, 0)\}; \\ S_{b_1} &= \{(0, 1, 0)\}; \\ S_{b_2} &= \{(0, 0, 1)\}; \\ S_c &= \{(0, 1, 1), (1, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1)\}. \end{aligned}$$

One can see that the probabilities of these sets are equal to the corresponding m_p. □

Remark 2.5. Poset M_2 is a partial case of a hypercube poset. For hypercube posets the measures on them given by realizable vectors of probabilites are called FUI in [K22]. Example 2.4 is a partial case of a more general statement [G24a, Proposition 3.1]. If a vector $\{m_p\}_{p \in H_k}$ on a hypercube H_k satisfies an FKG property

$$m_x m_y \leq m_{x \vee y} m_{x \wedge y},$$

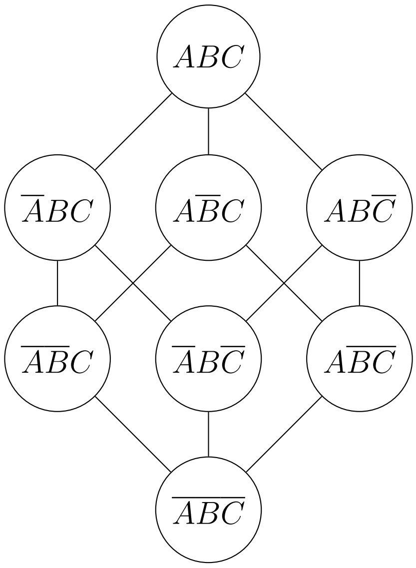

Figure 3: Cube poset

for any $x, y \in H_k$, then it is realizeable. The proof gives a bound that it is realizeable by a measure on H_n for $n = 2^k - 1$.

One can bootstrap the inequality from H_n to H_k as in [G24a] and show that Theorem 2.1 and 2.3 hold for any FUI measure μ, in particular proving the FKG-inequality [FKG71].

For general posets, it is unclear what are the conditions on the vector $\{m_p\}$ to be realizable. One can merge some of the nodes of $\mathcal{P}$ to form a diamond poset and apply the Harris–Kleitman inequality to the merged parts. This operation gives restrictions that are a partial case of the restrictions given by Theorem 2.3.

In 2001, Richards [R04] stated a third-degree inequality for a cube poset $\mathcal{P}_{cube}$ (Figure 3). The proof was later found to contain significant problems and the statement remains a conjecture.

Conjecture 2.6. For any realizable vector $\{m_p\}$ corresponding to the poset $\mathcal{P}_{cube}$, define m_A, m_B and m_C as the sum of elements where the corresponding letter doesn't have an overline and m_{AB}, m_{AC} and m_{BC} as the sum of vectors where both letters don't have an overline. Then

$$2m_{ABC} + m_A m_B m_C \geq m_A m_{BC} + m_B m_{AC} + m_C m_{AB}.$$

Sahi and Lieb were able to prove partial cases of this conjecture [S08, LS22].

Definition 2.7. We say that for the given poset $\mathcal{P}$ the vector of probabilities $\{m_p^-\}_{p\in\mathcal{P}}$ can be *glued* with the vector $\{m_p^+\}_{p\in\mathcal{P}}$ if there exists some n, a probability product measure μ on H_n and two subdivisions of H_n into parts S_p^- and S_p^+ such that

- S_a^- is contained in $\bigcup_{b<_{\mathcal{P}}a} S_b^+$;
- $\mu(S_p^-) = m_p^-$;
- $\mu(S_p^+) = m_p^+$.

Theorem 2.8. *If Let A be an $m \times m$ matrix satisfying the condition*

$$A_{a,c} + A_{b,d} \leq A_{a,d} + A_{b,c}, \tag{5}$$

whenever $a <_{\mathcal{P}} b$ and $c <_{\mathcal{P}} d$. Suppose in addition that whenever $a \leq_{\mathcal{P}} b$, $A_{a,b} \geq 0$. Then for any poset $\mathcal{P}$ and any vectors $\{m_p^-\}_{p\in\mathcal{P}}$ and $\{m_p^+\}_{p\in\mathcal{P}}$ that can be glued together, one has

$$\sum_{a,b\in\mathcal{P}} A_{a,b} m_a^- m_b^+ \geq 0$$

3 Applications to Bernoulli percolation

3.1 Inequalities for connectivity events

Initially, Harris–Kleitman inequality was used in Bernoulli bond percolation to show that the critical probability p_c for the square lattice is at least $\frac{1}{2}$. Our inequalities also tell something about connectivity probabilities in percolation. We use the notation from [G24b].

Definition 3.1. Consider a Bernoulli bond percolation on a finite graph $G = (V, E)$ where each edge $e \in E$ has a probability w_e of being open. Let E_o be the random set of open edges. We call the connected components of $G_o = (V, E_o)$ *clusters*. We denote by "$v_{11}v_{12}\dots v_{1i_1}|v_{21}\dots v_{2i_2}|\dots|v_{n1}\dots v_{ni_n}$" the event that the vertices $v_{11}, \dots, v_{1i_1} \in V$ belong to the same cluster, vertices $v_{21}, \dots, v_{2i_2}$ belong to the same cluster, ..., vertices $v_{n1}, \dots, v_{ni_n}$ belong to the same cluster, and, moreover, these clusters are all different. By $\mathbf{P}(v_{11}v_{12}\dots v_{1i_1}|v_{21}\dots v_{2i_2}|\dots|v_{n1}\dots v_{ni_n})$ we denote the probability of this event in the underlying bond percolation. In particular, $\mathbf{P}(abc)$ denotes the probability that vertices $a, b, c \in V$ lie in the same cluster, and $\mathbf{P}(a|b|c)$ is the probability that a, b and c belong to 3 different clusters.

The following theorem was proven in [G24a] (see also an alternative proof in [GZ24]). We put the proof in the context of our results.

Theorem 3.2. *Let $G = (V, E)$ be a finite graph and a, b, c are vertices in V. Let $\mathbf{P}$ be taken over Bernoulli percolation on G. Then*

$$\mathbf{P}(abc)\mathbf{P}(a|b|c) \geq \mathbf{P}(ab|c)\mathbf{P}(ac|b) + \mathbf{P}(ab|c)\mathbf{P}(a|bc) + \mathbf{P}(ac|b)\mathbf{P}(a|bc).$$

Proof. Since events ab, ac and bc are all increasing and if two of them happen, the third is forced, the events abc, $ab|c$, $ac|b$, $a|bc$ and $a|b|c$ form the poset on Figure 4.

Consider the following matrix A labeled by the elements of the poset

$$A = \begin{array}{c|ccccc} & abc & ab|c & ac|b & a|bc & a|b|c \\ \hline abc & 0 & 0 & 0 & 0 & 1 \\ ab|c & 0 & 0 & -1 & -1 & 0 \\ ac|b & 0 & -1 & 0 & -1 & 0 \\ a|bc & 0 & -1 & -1 & 0 & 0 \\ a|b|c & 1 & 0 & 0 & 0 & 0 \end{array} \tag{6}$$

It is easy to check the condition (3) for A, so, by Theorem 2.3, we get

$$2\big(\mathbf{P}(abc)\mathbf{P}(a|b|c) - \mathbf{P}(ab|c)\mathbf{P}(ac|b) - \mathbf{P}(ab|c)\mathbf{P}(a|bc) - \mathbf{P}(ac|b)\mathbf{P}(a|bc)\big) \geq 0.$$

□

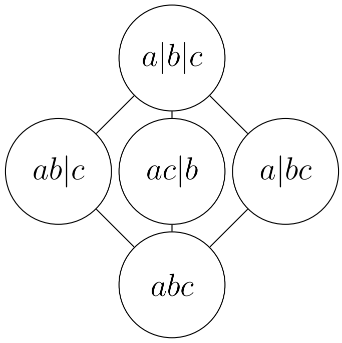

Figure 4: Partition lattice of $\{a, b, c\}$

We can consider the larger partition lattices. The partition lattice on 4 element set $\{a, b, c, d\}$ is shown on Figure 5.

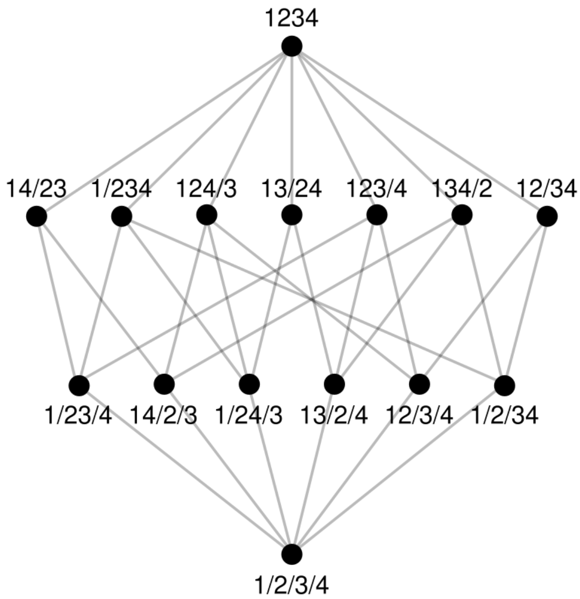

Figure 5: Partition lattice of $\{a, b, c, d\}$

Our method allows one to find new inequalities for the probabilities of the connectivity events.

Theorem 3.3. *Let $G = (V, E)$ be a finite graph and a, b, c, d are vertices in V. Let $\mathbf{P}$ be taken over Bernoulli percolation on G. Then*

$$\mathbf{P}(ab \wedge cd) - \mathbf{P}(ab)\mathbf{P}(cd) \geq \mathbf{P}(ab \vee cd)(\mathbf{P}(ac|bd) + \mathbf{P}(ad|bc)) + \mathbf{P}(ac|bd)\mathbf{P}(ad|bc). \quad (7)$$

Proof. Consider the following matrix A:

$$A = \begin{array}{c|ccccccccccccccc}
 & a|b|c|d & a|b|cd & a|bc|d & a|bd|c & a|bcd & ab|c|d & ab|cd & ac|b|d & ac|bd & ad|b|c & ad|bc & abc|d & abd|c & acd|b & abcd \\
\hline
a|b|c|d & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
a|b|cd & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & -1 & -1 & 0 & 0 \\
a|bc|d & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
a|bd|c & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
a|bcd & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & -1 & -1 & 0 & 0 \\
ab|c|d & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & -1 & 0 \\
ab|cd & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
ac|b|d & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
ac|bd & 0 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 \\
ad|b|c & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
ad|bc & 0 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & -1 & 0 \\
abc|d & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & -1 & 0 \\
abd|c & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & -1 & 0 \\
acd|b & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & -1 & -1 & 0 & 0 \\
abcd & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{array}$$

It acts as a certificate for Theorem 2.3 to show the inequality (7). □

3.2 Inequalities for conditional measures

The work [BHK06] studies correlations for different connectivity events in the percolation measure μ conditioned on the event $a|b$. It shows that any two events with positive dependence on the component of a correlate nonnegatively and all such events correlate nonpositively with the events with positive dependence on the component of b.

In the heart of the proof [H07] there is a sequence of product measures on hypercubes of increasing dimension that approximate the Bernoulli percolation conditioned on $a|b$. Additionally, it shows that the events positively dependent on the component of a are closed upwards in these hypercubes while events positively dependent on the component of b are closed downwards. In particular, if we only care about 4 vertices, it shows that the events in Figure 6 give rise to a realizable vector $\{m_p\}_{p\in\mathcal{P}}$. We interpret elements of poset $\mathcal{P}$ as the events in Bernoulli percolation. Then the union $\bigcup_{p\in\mathcal{P}} p = a|b$ and so $m_p = \frac{\mathbf{P}(p)}{\mathbf{P}(a|b)}$.

Papers [BHK06, BK01] found the inequalities that follow from this poset by applying the Harris–Kleitman inequality to the pair of the increasing events ac and ad and the pair of increasing events ac and $b|d$. However, there are more restrictions on the probability vectors realizable by the poset. In particular, one can notice a hidden M_3 in this poset.

Corollary 3.4. *Let $G = (V, E)$ be a finite graph and a, b, c, d are vertices in V. Let $\mathbf{P}$ be taken over Bernoulli percolation on G. Then*

$$(\mathbf{P}(acd|b) + \mathbf{P}(ad|b|c) + \mathbf{P}(ac|b|d))(\mathbf{P}(a|bc|d) + \mathbf{P}(a|bd|c) + \mathbf{P}(a|bcd)) \geq \mathbf{P}(ad|bc)\mathbf{P}(ac|bd) + (\mathbf{P}(ad|bc) + \mathbf{P}(ac|bd))(\mathbf{P}(a|b|c|d) + \mathbf{P}(a|b|cd)). \quad (8)$$

Proof. Consider the following A:

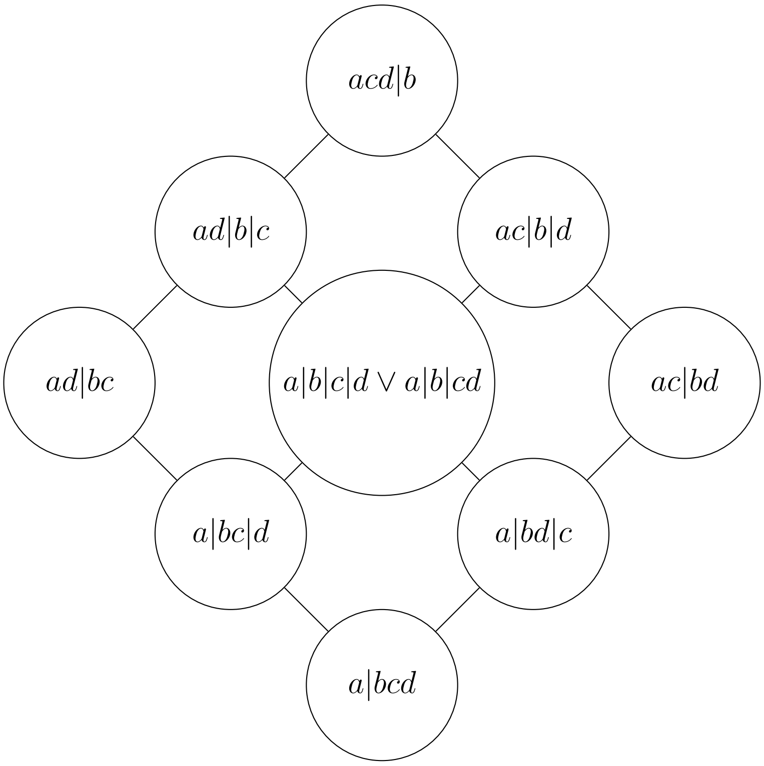

Figure 6: Poset of events conditioned on $a|b$

$A =$

	$a\|bcd$	$a\|bc\|d$	$ad\|bc$	$a\|bd\|c$	$a\|b\|c\|d \vee a\|b\|cd$	$ad\|b\|c$	$ac\|bd$	$ac\|b\|d$	$acd\|b$
$a\|bcd$	0	0	0	0	0	1	0	1	1
$a\|bc\|d$	0	0	0	0	0	1	0	1	1
$ad\|bc$	0	0	0	0	-1	0	-1	0	0
$a\|bd\|c$	0	0	0	0	0	1	0	1	1
$a\|b\|c\|d \vee a\|b\|cd$	0	0	-1	0	0	0	-1	0	0
$ad\|b\|c$	1	1	0	1	0	0	0	0	0
$ac\|bd$	0	0	-1	0	-1	0	0	0	0
$ac\|b\|d$	1	1	0	1	0	0	0	0	0
$acd\|b$	1	1	0	1	0	0	0	0	0

One can see that it satisfies the condition (3). Indeed, it just comes from the matrix (6) for the poset M_3 one gets by merging together 3 top vertices of the poset as well as 3 bottom vertices. Writing the corresponding conclusion of Theorem 2.3 and multiplying it by $\mathbf{P}(a|b)^2$ to get rid of conditional probabilities we get the needed (8). $\square$

The paper [BHK06] also defines a way to use two sets of vertices S, T instead of two vertices a, b. If we apply their method to $S = \{a, b, c\}$ and $T = \{d\}$, we will get the poset M_3 in Figure 7. Here $\bigcup_{p \in \mathcal{P}} p = S|T := a|d \wedge b|d \wedge c|d$, so

$$m_p = \frac{\mathbf{P}(p)}{\mathbf{P}(a|d \wedge b|d \wedge c|d)}.$$

Writing the inequality corresponding to matrix (6) and multiplying it by the common denominator of $\mathbf{P}(a|d \wedge b|d \wedge c|d)^2$, we get the following inequality.

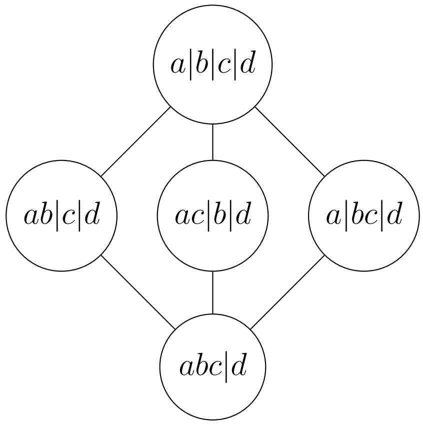

Figure 7: Poset of connection events between a, b, c, d conditioned on $a|d \wedge b|d \wedge c|d$

Corollary 3.5. *Let $G = (V, E)$ be a finite graph and a, b, c, d are vertices in V. Let $\mathbf{P}$ be taken over Bernoulli percolation on G. Then*

$$\mathbf{P}(abc|d)\mathbf{P}(a|b|c|d) \geq \mathbf{P}(ab|c|d)\mathbf{P}(ac|b|d) + \mathbf{P}(ab|c|d)\mathbf{P}(a|bc|d) + \mathbf{P}(ac|b|d)\mathbf{P}(a|bc|d).$$

3.3 Bunkbed conjecture for 2 posts

The bunkbed conjecture was first posed by Kasteleyn [BK01].

A bunkbed graph G_b consists of two isomorphic graphs G, called the upper and lower bunks, and some additional edges, called posts; each post connects a vertex in the upper bunk with the corresponding isomorphic vertex in the lower bunk. We assign a probability to each edge, with each edge in the upper bunk assigned the same probability as the corresponding isomorphic edge in the lower bunk. The probabilities on the posts are arbitrary. We then run the Bernoulli bond percolation on the bunkbed graph with the prescribed edge probabilities. The Bunkbed Conjecture states that in the resulting random subgraph the probability that a vertex x in the upper bunk is connected to some vertex y in the upper bunk is greater than or equal to the probability that x is connected to y', the isomorphic copy of y in the lower bunk.

There are many ways to reduce the conjecture to its partial cases. In particular, one can assume that all posts have a probability of 0 or 1. In this case one can denote the set of vertices of G adjacent to posts with probability 1 by W and contract each vertex of W with its copy. In fact, the conjecture in this form appears in [BK01]. The vertices of W are called transversal in [L09].

The partial case where $|W| = 1$ follows from the Harris–Kleitman inequality [L09, Lemma 2.4].

4 Testing measure realizability on a poset

For a poset $\mathcal{P}$ consider a matrix $F_{\mathcal{P}}$ defined by the following rules:

Definition 4.1. We say that a *covers* b (written as $a \lessdot b$) if $a <_{\mathcal{P}} b$ and there are no elements between a and b.

Assume poset $\mathcal{P}$ has m elements and p pairs of elements (a, b) in a covering relation. Then $F_{\mathcal{P}}$ is an $m \times p$ matrix where each row corresponds to an element of $\mathcal{P}$ and each column corresponds to a cover $a \lessdot b$ and

$$F_{e,ab} = \begin{cases} -1, \text{ if } e = a, \\ 1, \text{ if } e = b, \\ 0 \text{ otherwise.} \end{cases}$$

Definition 4.2. A matrix A is called *completely positive* (see [BS03]) if it can be written as a product $A = BB^T$ for some elementwise nonnegative matrix B. In particular, any completely positive matrix is elementwise nonnegative and positive semidefinite.

Theorem 4.3. *If vector $x = \{m_p\}_{p \in \mathcal{P}}$ is realizable, then there exists a completely positive matrix $p \times p$ matrix M such that*

$$FMF^T = \text{diag}(x) - xx^T. \tag{9}$$

Proof. Assume x is realizable by a product measure μ on a hypercube H_n. For $n = 0$ one can take $M = 0$ and the statement would be true. For $n \geq 1$ we use induction. Split the cube into the upper and lower halves. Let p be the probability of the upper half and μ' be the probability measure on H_{n-1} obtained as the projection of μ to $n-1$ first coordinates. Then μ' coincides with μ conditioned on the upper or lower part of H_n. Let M^+ and M^- be the matrices and x^+ and x^- be the vectors corresponding to μ^+ and μ^-.

To use the fact that the upper and lower parts can be glued together, consider the subdivisions S^- and S^+ indexed by elements of $\mathcal{P}$. Let y be the vector indexed by the pairs $a \lessdot b$ from poset, such that $y_{ab} = \mu'(\{S_a^- \wedge S_b^+\})$. By definition of F, $x^+ - x^- = Fy$. Note that all entries of y are nonnegative.

Now

$$\begin{aligned} &\text{diag}(x) - xx^T \\ &= p\text{diag}(x^+) + (1-p)\text{diag}(x^-) - p^2 x^- x^{+T} - p(1-p)(x^+ x^{-T} + x^+ x^{+T}) - (1-p)^2 x^- x^{-T} \\ &= p(FM^+F^T) + (1-p)(FM^-F^T) + p(1-p)(x^+ - x^-)(x^+ - x^-)^T \\ &= F(pM^+ + (1-p)M^- + p(1-p)yy^T)F^T, \end{aligned}$$

and it is easy to see that $M = pM^+ + (1-p)M^- + p(1-p)yy^T$ is completely positive. □

Testing if a particular vector x for a particular poset $\mathcal{P}$ satisfies condition (9) for some positive semidefinite matrix M with nonnegative entries is a convex optimization problem – the restrictions of M being nonnegative and positive semidefinite are convex and can be solved using modern semi-definite programming optimizers. So, instead of checking all inequalities coming from various matrices A satisfying condition (3), one can run one instance of a semi-definite program.

Computationally, it is much more efficient. Even for a relatively small poset P_4 from Figure 5, possible matrices A form a 100-dimensional cone with more than 10 000 generators. So having a simple test helps.

Moreover, this test is as strong as checking all the generators. Let $\langle A, B\rangle = \text{Tr}(A^T B)$ be the Frobenius product of two matrices – the component-wise inner product of two matrices as though they are vectors. Then (3) is equivalent to

$$\langle A, Fe_i e_j^T F^T \rangle \geq 0 \tag{10}$$

for all $0 \leq i, j \leq p$ and (4) is equivalent to

$$\langle A, \mathrm{diag}(x) - xx^t \rangle \tag{11}$$

for $x = \{m_p\}_{p \in \mathcal{P}}$.

Proposition 4.4. *If a vector x satisfies condition (9) for some nonnegative matrix M, then for every matrix A satisfying (10), it satisfies (11) as well.*

Proof. Note that any nonnegative matrix M is a sum of terms of the form $m_{ij} e_i e_j^T$. Then

$$\begin{aligned}\langle A, \mathrm{diag}(x) - xx^T \rangle &= \langle A, FMF^T \rangle \\ &= \langle A, F \left(\sum m_{ij} e_i e_j^T \right) F^T \rangle = \sum m_{ij} \langle A, F e_i e_j^T F^T \rangle \geq 0\end{aligned}$$

□

This shows that the realizability test of Theorem 4.3 is stronger than this of Theorem 2.3 since it also adds the restriction that M is nonnegative definite.

Definition 4.5. We say one vector $\{m_p^+\}_{p \in \mathcal{P}}$ *dominates* another vector $\{m_p^+\}_{p \in \mathcal{P}}$

5 Inequalities of degree at least 3

Our inductive method allows to prove inequalities of higher degree. The main lemma is

Theorem 5.1. *Let μ be a probability product measure on $2^{[n]}$. Let $g(x_1, x_2, \ldots, x_k)$ be a function $(2^{[n]})^k \to \mathbb{R}$ such that for any $x_1 \preceq x_1'$, …, $x_k \preceq x_k'$ and for any i one has*

$$g(x_1, x_2, \ldots, x_k) + g(x_1, x_2, \ldots, x_k) \leq g(x, t) + g(y, z). \tag{12}$$

Then

$$\mathbf{E}_{\mu^k} g(x_1, x_2, \ldots, x_k) \geq \mathbf{E}_\mu g(x, x, \ldots, x). \tag{13}$$

Proof. □

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