Day 3: Primer on Noetherian Stuff

Tom Gannon

August 17, 2017

1 Introduction

Today we'll deal with everything regarding the finitely generated and Noetherian properties. A lot of times finitely generated things turn out to be the right thing to work with in algebra, and we'll show why this is with a few strange pathologies. We'll also prove a property we'll need to use tomorrow that any submodule of R^n is free and of rank $\leq n$ if R is a PID.

1.1 Motivation from Principal Ideal Domains

Recall that a *Principal Ideal Domain* R is an integral domain R such that any ideal $I \subset R$ can be written $I = Ra$ for some element $a \in R$, and specifically recall a part of the proof that principal ideals are unique factorization domains:

Exercise 1.1. *(PIDs Are Factorization Domains): Show that if R is a PID, then any nonzero element $r \in R$ has some factorization $r = s_1...s_n$ where s_i are not necessarily distinct irreducible elements of R. (Hint: If $r_i = r_{i+1}t_{i+1}$ was reducible, then $(r_i) \subsetneq (r_{i+1})$. Let $I = \cup_i (r_i)$.)*

1.2 The Initial Definition

The definition of a Noetherian ring generalizes the above proof technique we used:

Definition 1.2. *We say a ring A is* ***Noetherian*** *if any chain of ideals $I_1 \subset I_2 \subset ... \subset A$ stabilizes*[1] *at some point–that is, there exists an $n \in \mathbb{N}$ such that $m \geq n$ implies that $I_n = I_m$.*

Another (equivalent) way to think of Noetherian rings is that the chain $I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq ...$ can't happen for ideals $I_k \subset A$. This is called the **ascending chain condition** on ideals. Also, note that ideals of A are just A modules that *just so happen* to be a submodule of the A module A. In particular, it is possible to define the concept of a **Noetherian Module**, and you should do so. Feel free to do so before I tell you to in this next exercise. That way, you can feel like you're ahead of the game.

Exercise 1.3. *Define what it means for an A module M to be* ***Noetherian*** *using the ascending chain condition on submodules. Show that equivalently, an A module M is Noetherian if and only if each submodule of M is finitely generated as an A module. Determine what this condition specifically means in terms of rings and ideals.*

Exercise 1.4. *(A Non-Noetherian Ring/Module) Show that the ring $\mathbb{R}[x_i]_{i \in \mathbb{N}^{>0}} = \mathbb{R}[x_1, x_2, ...]$ is not a Noetherian ring.*

If you ever come across something with "Noetherian" in the hypothesis, it's a good idea to throw the above ring in the theorem and see if it still holds.

2 Primer on Stock Results on Noetherian Modules that I Like To Keep Around

Exercise 2.1. *(Submodules/Quotients of Noetherian modules are Noetherian) Show that if $N \subset M$ is a submodule of an A module M, then N and M/N are Noetherian.*

[1] A word on notation. "$\subset$" and "$\subseteq$" will mean the same thing–"is a subset of" and $\subsetneq$ means that the two sets aren't equal.

Exercise 2.2. *(Finitely Generated Module with Non Finitely Generated Submodule) Let* $R := \mathbb{R}[x_i]_{i \in \mathbb{N}^{>0}} = \mathbb{R}[x_1, x_2, ...]$. *Show that* R *is a finitely generated* R *module, but* R *has a submodule which is not finitely generated as an* R *module.*

This shows that the hypothesis of a ring/module being Noetherian is stronger than requiring it to be finitely generated, because any Noetherian ring/module is finitely generated.

But on the other hand:

Exercise 2.3. *Show that any finitely generated module of a Noetherian ring* R *is a Noetherian module. (Easy mode: Assume that* R^n *is Noetherian. Hard Mode: Prove it.)*

We also have a very helpful:

Theorem 2.4. *(Hilbert's Basis Theorem) If* A *is a Noetherian ring, so is* $A[x]$.

Exercise 2.5. *Prove this theorem. (Hint: Don't actually try to prove this theorem, just go look it up in a book somewhere. Like Dummit and Foote. To check if you really understand the proof, modify the idea of the proof slightly to show that if* A *is a Noetherian ring, then the ring of formal power series of* A, $A[[x]]$, *is a Noetherian ring.)*

3 Primer on the Module R^n

Note that in the world of being "well behaved", modules are just the absolute worst. We have above that a finitely generated module can have a not finitely generated submodule. Also even if the submodules are finitely generated (i.e. the module is Noetherian), we don't have any good rank bounds:

Exercise 3.1. *Show that the ring* $R := \mathbb{Q}[x, y]$ *has* ***rank*** *1–that is, there is a subset of* R *consisting of one linearly independent element over* R *but no set of two linearly independent elements over* R. *However, construct a submodule (ideal) of* R *with rank 2.*

Definition 3.2. *An* $A-$*module* M *is free on generators* $m_1, ..., m_k \in M$ *if* $r_1 m_1 + ... + r_k m_k = 0$ *implies* $r_i = 0$ *for all* i.

Exercise 3.3. *(Rank one submodule that isn't a free submodule) Let* $R = \mathbb{Z}[x]$ *and let* I *denote the submodule/ideal* $(2, x)$. *Show that any two elements are linearly dependent and no single element spans.*

This sort of tomfoolery[2] doesn't happen when you restrict to PIDs. In particular,

Theorem 3.4. *If* R *is a PID, then any submodule of* R^n *is a free submodule*[3] *of rank* $\leq n$.

Exercise 3.5. *Prove the above theorem. (Hint: Let* $M_i := (Re_1 + ... + Re_i) \cap M$, *where* M *is the submodule and* $\{e_i\}$ *is the standard basis of* R^n. *For the inductive hypothesis at step* k, *let* I *denote the set of all elements* r_k *that have the form* $r_1 e_1 + ... + r_k e_k$ *and are in* M. *Pick some* v *whose last coordinate is a generator of* I, *and show that* $M_k \cong M_{k-1} \oplus Rv$*).*

We'll use this theorem tomorrow to help us prove that there's only one quality that a module M over a PID R can have, called *torsion*, that stops M from looking like R^n for some $n \in \mathbb{N}$.

[2]Side note. If you can think of a good adjective to describe a person that has the word "Tom" in it, I'd love to hear it. Currently there are only bad ones (e.g. "Peeping Tom")

[3]It could also be the zero submodule, by the way. You can think of R^0 as the zero module. But note that for a general ring, R^0 isn't the *only* module of rank zero!