What I Learned Today

Tom Gannon

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(8/1/2022) Today I learned that the notion of a graded ring in the context of homotopy theory is not given either by a particular direct sum or a direct product of rings, in general, but rather a sequence of abelian groups and associated multiplication maps between them. This can also be viewed as a commutative algebra object in the category of functors from $\mathbb{Z}$ (viewed as a discrete category) to the category of abelian groups.

(8/2/2022) Today I learned the explicit computation of the height of a formal group over some $\mathbb{F}_p$-algebra. Specifically, one can take the *p-series* of a formal group law f for the formal group, defined recursively via $[0](f) = 0$ and $[n](t) = f([n-1](t), t)$. From this, one can compute that the p-series of any formal group law over an $\mathbb{F}_p$-algebra will have first term vanishing. In fact, it will look like $\lambda x^{p^{?-1}}$ for some value of ?, and this formal group law is said to have height $\geq$?.

(8/3/2022) Today I learned that for the complex oriented cohomology theory $\mathbb{Q}[\beta\pm 1]$, one can consider the associated parity sheaves of spectra, and that these precisely identify with the 'induced' parity sheaves given by $IC_w \otimes_{\mathbb{Q}} \mathbb{Q}[\beta^{\pm 1}]$.

(8/4/2022) Today I learned a bit of the construction on Lusztig's character formulas for a *fixed* dominant integral weight λ. Specifically, one can write $\lambda = \lambda_0 + \lambda_1 p + ...$ where each λ_i is a restricted weight (and these eventually vanish), and define the n^{th} stage character formula in $\mathbb{Z}[X^\bullet(T)]$ recursively by E^1_λ given by the character for the characteristic p quantum group and the formula

$$E^n_\lambda = E^1_{\lambda_0}(E^1_{\lambda_1})^{[1]}...(E^{n-1}_{\lambda_{n-1}})^{[n-1]}(E^0_{\lambda_n p^n + ...})^{[n]}$$

where the E^0 refers to the Weyl character formula and the $[i]$ denotes an i-fold Frobenius twist.

(8/6/2022) Today I learned that the rank of the $K(i)$-theory of the flag variety for GL_n is $|W|$ for any $n \geq 2$ when $p > n$. This follows from the Atiyah-Hirzebruch spectral sequence collapsing when the dimension of the given manifold is bounded by $2(p^i - 1)$, noting that

$$\dim_{\mathbb{R}}(\mathrm{GL}_n/B) = n(n-1) \leq p^2 - p \leq p^2 - 2 < 2(p^2 - 1).$$

(8/7/2022) Today I learned that to any 2-sided cell of some endoscopic Weyl group with associated endoscopic parameter $[\lambda]$, one can find a unique 2-sided cell as a subset of the product $W \times W[\lambda]$ associated to it for which Lusztig-Yun show there is a bijection of unipotent character sheaves for the endoscopic group and the character sheaves for the original group with that unique 2-sided cell (and associated semisimple parameter the W-orbit of $[\lambda]$).

(8/8/2022) Today I learned the Tate valued Frobenius is a map of $\mathbb{E}_\infty$ ring spectra. In particular, one has a map of $\mathbb{E}_\infty$ ring spectra given by the Tate valued Frobenius for KU, and this map factors through the inversion of multiplication by p. This Tate-valued Frobenius also turns out to be give the p-stable Adams operation composed with the inclusion into the completion of K-theory.

(8/10/2022) Today I learned the definition of the Frobenius twist of a scheme defined over a maybe at least perfect field k of characteristic p, although I guess the definition makes sense in general. Specifically, the Frobenius gives a ring map $k \to k$, and we can use it to declare that $k^{(-1)}$ is the k-algebra whose underlying ring is k but equipped with this new k-algebra structure. We can then base change any k-scheme X over the map $\mathrm{Spec}(k^{(-1)}) \to \mathrm{Spec}(k)$ to obtain the *Frobenius twist* $X^{(1)}$, and this admits a map known as the *relative Frobenius* $X \to X^{(1)}$.

(8/11/2022) Today I learned that one can define the notion of 2-sided cells as a subset of the product $W \times \mathfrak{o}$, where $\mathfrak{o}$ is an orbit of semisimple parameters, and moreover (at least when $\mathcal{L} \in \mathfrak{o}$ has $W_{\mathcal{L}}^{\circ} = W_{\mathcal{L}}$) these cells are in bijective correspondence with the 2-sided cells of $W_{\mathcal{L}}^{\circ}$.

(8/12/2022) Today I learned (or at least computed some strong evidence for the claim that) if C is the cohomlogy ring of the flag variety SL_3/B, then the dimension of $\mathrm{Tor}_i(\mathbb{Q}, \mathbb{Q})$ is $i+1$.

(8/13/2022) Today I learned that there is no nonzero 'wrong way' ring map $\mathbb{F}_p^{tC_p} \to \mathbb{F}_p$. This follows from the observation that $\mathbb{F}_p^{tC_p}$ is 1-periodic and therefore the fact that the first (say) homotopy group of $\mathbb{F}_p^{tC_p}$ vanishes implies that the zeroth homotopy group goes to zero.

(8/14/2022) Today I learned more details about the Steinberg-Whittaker localizaiton. Specifically, I learned that, given two characters of the prounipotent radical of the Iwahori, one can identify the category of bi-Whittaker sheaves (for those two characters) with a certain block of the category of modules for the associated affine Kac-Moody group at integral, noncritical level, which satisfy an analogous Whittaker equivariance condition.

(8/15/2022) Today I learned Donkin's theorem on tensor product of tilting objects, which says that if one has some λ which has the form $(p-1)\rho+\xi$ for ξ some restrited weight and μ is any weight, then one has an isomorphism $T_{\lambda+p\mu} \cong T_{\lambda} \otimes T_{\mu}^{(1)}$, where the superscript denotes the Frobenius twist.

(8/16/2022) Today I learned that it's not the case that the bi-Iwahori-Whittaker invariants agree with the Whittaker invariants when applied to the loop group. Furthermore, I learned that, under the local geometric Langlands conjecture, the Whittaker invariants correspond to the part living over $\mathrm{QCoh}(\mathrm{LocSys}_{G^\vee}(\mathring{D}))$, whereas the Iwahori-Whittaker invariants corresponds to viewing the category over $\mathrm{QCoh}(\mathrm{LocSys}_{G^\vee}^{\geq -1}(\mathring{D}))$, i.e. the local systems with regular singularities.

(8/17/2022) Today I learned a path to prove that, for λ any dominant weight and p any fixed prime, the endomorphisms of the tilting object $T_{p\lambda}$ admit a representation of the Weyl group. Specifically, using the Finkelberg-Mirkovic conjecture for quantum groups (a theorem of ABG) and Koszul duality, one can identify these tilting objects as those objects in the essential image of a long Whittaker averaging functor from the Iwahori-Whittaker category for loops on G, and then use the paradigm of the Gelfand-Graev action to get the desired representation. Actually maybe this is the injective objects...

(8/18/2022) Today I learned that, if one knows the maps between all Frobenius twists of mixed lifts of an object, then one knows the endomorphisms of that object, at least as a vector space. In particular, the endomorphisms of a tilting object (which are concentrated in degree zero) can be identified with the ext algebra of that given IC sheaf.

(8/19/2022) Today I learned an explicit identification of the Finkelberg-Mirkovic conejcture, which matches the simple $L_{w\cdot -2\rho}$ with the IC sheaf indexed by w, for any w which takes -2ρ to a dominant weight under the $W^{\mathrm{aff}}, \cdot$ action.

(8/22/2022) Today I learned the claim that the quotient category of I-monodromic sheaves on the affine Grassmannian by the kernel of the Whittaker averaging functor is given by $\mathrm{Rep}(G\vee) \otimes C$-mod, where C denotes the coinvariant algebra for W.

(8/24/2022) Today I learned you can explicitly describe the group scheme of regular centralizers in the following way. Specifically, there is a map $\mathcal{J} \to T$ which factors as a map $\mathcal{J} \to (T \times \mathfrak{t}^*) /\!\!/ W$ as a map of schemes over $\mathfrak{t}^* /\!\!/ W$, and one can view $\mathcal{J}$ as an affine blow up of this scheme via this map.

(8/25/2022) Today I learned a description of generic G-categories. Specifically, given the fact that generic G-categories can be identified as the smallest 2-subcategory of G-module categories containing $\mathcal{D}(G/_{[\lambda]}B\text{-mon})$, for generic λ we can identify the category as endomorphisms of this object, which, via the Mellin transform, identifies with module categories for $\mathrm{IndCoh}((\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}})^{\wedge}_{[\lambda]})$.

(8/26/2022) Today I learned a bit more of the construction of Ho-Li on graded sheaves. Specifically, given some sheaf defined over a finite field, one can consider the mixed complexes on them. All of these mixed complexes admit an action of mixed sheaves on a point. The definition of mixed implies that these objects are filtered by objects on which Frobenius acts semisimply by certain eigenvalues in certain circles in $\mathbb{C}$ whose radius depends on the finite field. These define a symmetric monoidal functor to the category of graded $\mathbb{Q}_\ell$-sheaves, and tensoring with this we obtain a category of graded sheaves.

(8/27/2022) Today I learned that one can explicitly identify the space $T /\!\!/ W$ for $G = \mathrm{GL}_n$ with the set of divisors of degree n inside $\mathbb{A}^1$, given as the set of solutions to the characteristic polynomial counting multiplicity.

(8/28/2022) Today I learned that there is a four dimensional topological field theory which gives rise (via evaluation at the circle) to a certain 2-category known as the *2-representations* of $\mathfrak{sl}_2$. This four dimensional topological field-theory, and in particular it admits a functor to the monoidal unit of 2-categories. However, this forgetful functor is not even monoidal!

(8/29/2022) Today I learned that, given the value of a fully extended topological field theory on some n-sphere, one can immediately obtain it for all $m > n$-spheres as well. This follows from writing the m-sphere as a coproduct of two disks and, using monoidality, gives that the value on the m-sphere is the trace of the value on the $m-1$ sphere.

(8/30/2022) Today I learned that if one has an equivalence between a sheared filtered category and some other category over $\mathbb{A}^1_\hbar$, where $\hbar$ lives in cohomological degree 2, then one can compute the associated graded of the category by changing the grading on the associated graded of the filtered category.

(8/31/2022) Today I learned a theorem of Venkatesh which says that, for K some non-Archimedian local field, one can define the derived Hecke algebra as the exterior algebra of the derived endomorphism ring of the compact induction of the trivial $G(\mathcal{O})$ representation. One can show that this admits a W-equivariant map to the derived Hecke algebra for some split maximal torus, and that this natural map factors through W-invariants, and this factored map yields an isomorphism.

July 2022

(7/1/2022) Today I learned that the functor $\pi^! : \mathrm{IndCoh}(\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}) \to \mathrm{IndCoh}(\mathfrak{t}^*)$ satisfies the conditions of Barr-Beck. In particular, the fact that this functor and its adjoint are also t-exact implies that we can identify $\mathrm{IndCoh}(\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}})$ with modules over some classical ring N, which I expect to be the so called nil-Hecke ring. This implies a bunch of formal properties about the category $\mathrm{IndCoh}(\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}})$, such as the existence of enough projectives and injectives and the fact that it is the derived category of its heart.

(7/2/2022) Today I learned a theorem known as *Nagata's compactification theorem*, which says that a separated and finite type morphism to a Noetherian scheme can be factored as the composite of an open embedding and then a proper morphism (where $\mathbb{A}^1 \hookrightarrow \mathbb{P}1 \to *$ is a convenient way to remember the order of the factorization). This in particular gives a hands on description to the definition of the !-pullback for IndCoh–specifically, one can take a factorization as above and define the !-pullback for open embeddings as the usual $*$-pullback, and this reduces the definition of the IndCoh !-pullback to defining it for proper morphisms and exhibiting that the definition is independent of the factorization, i.e. the category of such factorizations is contractible. In fact, the hypothesis that the scheme is Noetherian can be weakened to it just being qcqs.

(7/3/2022) Today I learned that if $X = \text{Spec}(A)$ is the spectrum of some classical integral domain A and that $M \in A\text{-mod}^{\heartsuit}$ has the property that the fiber for any closed subscheme $Z \overset{i}{\hookrightarrow} X$, the (derived) fiber $i^*(M)$ vanishes. Then either $M = 0$ or there is an injection of A-modules $K \hookrightarrow M$, where K denotes the fraction field of A. The proof is: If M is nonzero, there exists some nonzero map $\phi : A \to M$ sending $1 \in A$ to some $m \in M$. First, note that if $f \in A$, we have a cofiber sequence of A-modules

$$A/f \to A \xrightarrow{f\cdot -} A$$

and if we let $i : \text{Spec}(A/f) \hookrightarrow A$ denote the closed subscheme given by f, we obtain that

$$i^*(M) \to M \xrightarrow{f\cdot -} M$$

is also a cofiber sequence. Now if f is not a unit, i is a proper closed subscheme and so we see that the map which multiplies by f is an isomorphism, since the (derived) fiber vanishes. This analysis shows that ϕ is an injection, and moreover that ϕ induces a map $\tilde{\phi} : K \to M$ via the universal property of localization. Finally, $\tilde{\phi}$ is an injection because if $\frac{a}{b}m = 0$ for some $a, b \in A$ we would have that $am = 0$ as well.

(7/4/2022) Today I learned that if Y is a classical integral finite dimensional finite type scheme and $M \in \text{QCoh}(Y)$ such that for all closed proper subschemes $i : Z \hookrightarrow Y$, $i^!(\Upsilon_Y(M)) \simeq 0$, then $i^*(M) \simeq 0$ for all closed proper subschemes $i : Z \hookrightarrow Y$. The argument essentially follows by induction on the dimension–either the module on Z is supported on some generic point or it is not, and in the former case, the generic point is in the smooth locus and so Υ is an equivalence, and in the latter case, we can use the inductive hypothesis.

(7/5/2022) Today I learned an explicit computation which states that the coinvariant algebra for the additive action of W on $\text{Sym}(\mathfrak{t})$ is equivalent to the coinvariant algebra for the multiplicative action of W on $\mathcal{O}(T)$.

(7/6/2022) Today I learned that the equivariant K-theory of a space upon rationalization is *not* isomorphic to the periodification of rational equivariant cohomology. For example, in the case of $\mathbb{P}^1$ with its usual B-action, the equivariant rational cohomology as an ungraded ring gives via Spec the union of two copies of $\mathbb{A}^1$ whereas the equivariant K-theory gives two copies of $\mathbb{A}^1 \setminus 0$.

(7/7/2022) Today I learned a standard argument which says that the dimension of the endomorphisms of the big projective is bounded above by $|W|$. First, you can argue that every Verma module has a unique simple submodule, and then you can argue that this submodule is itself a Verma module, and thus the Verma = simple labelled by 2ρ. Then BGG reciprocity says that there is at most one dimension's worth of maps between the projective $P_{-2\rho}$ and a given Verma. Then, because each Verma appears in a standard filtration of the big projective exactly once, you can use the exactness of $\text{Hom}(P_{-2\rho}, -)$ to get the desired inequality.

(7/8/2022) Today I learned a quick way to get the antidominant weight labelled by a certain weight, explicitly. Specifically, given a simple labelled by some weight, the Verma which surjects onto it has a unique simple sub, which also turns out to be a Verma module. The Vermas which are also simple are precisely those labelled by antidominant weights, which gets you the associated antidominant weight in the block!

(7/9/2022) Today I learned that one can create a formal group law $x + y + pxy$ over $\mathbb{Z}_p$, and that this formal group law is likely not Landweber exact, since modulo p the formal group law has associated p-series given by 0, since the associated group law is additive modulo p.

(7/12/2022) Today I learned that a result of Bezrukavnikov and Tolmachov which says that one can identify the averaging of the Whittaker sheaf $\text{Av}_*^{\Delta_G}(\psi)$ with the center averaging of the constant sheaf on the regular unipotent locus of the group.

(7/13/2022) Today I learned the definition of a schtuka with one leg, and the bounded version. Specifically, for a fixed point x in a smooth projective geometrically connected curve over $\mathbb{F}_q$, one has the notion of a Hecke stack at that point, which admits two maps to Bun_G given by the source and target. A schtuka with one leg is defined to be the pullback of this map by the map $id, \text{Frob} : \text{Bun}_G \to \text{Bun}_G \times \text{Bun}_G$ where Frob denotes the pullback by the Frobenius on the test schemes.

(7/14/2022) Today I learned that one can define the stack of Langlands parameters and, for a local field or a global function field, this stack, before modding out by conjugacy, is a disjoint union of affine schemes. I also learned that r-iterated schtukas have a *partial Frobenius*, i.e. a particular endomorphism whose r-th power gives the usual Frobenius. This comes from shifting the identifications all over by one and twisting the last one by the Frobenius.]

(7/15/2022) Today I learned the proof of the fact that the formal completion of $\mathbb{A}^n$ at some field valued point $\text{Spec}(K) \to \mathbb{A}^n$ can be identified with the formal completion of $\mathbb{A}^n_K$ at the associated closed point. This essentially follows from the fact that any k-algebra A with a map from $k[x_1, ..., x_n]$ whose reduced locus factors through the fraction field necessarily is an algebra for the fraction field itself. In turn, this follows because if $\phi(f) \in A$ is invertible up to nilpotence, then you can use the power series trick to explicitly compute the inverse in A.

(7/16/2022) Today I learned that, if one has an action of a Weyl group on an abelian category, then to give an equivariant object is equivalent to giving an equivariant object for the associated braid group action. This is because an equivariant object for a discrete group acting is given by isomorphisms $F_w(A) \xrightarrow{\sim} A$ satisfying natural compatibility conditions, none of which use the associated identifications.

(7/17/2022) Today I learned a theorem of Lusztig which gives a bijection between the nilpotent orbits of the Langlands dual group of some group and the two sided *cells* of the affine Weyl group, which is a certain set of equivalence classes for some equivalence relation on the affine Weyl group.

(7/18/2022) Today I learned some motivation for why the Hitchin system with the map to the base actually gives a globalization of the characteristic polynomial story. Specifically, one would like to apply $\text{Hom}(C, -)$ to the characteristic polynomial map when C is a smooth projective curve. However, the GIT quotient $\mathfrak{g} /\!\!/ W$ has no nontrivial maps from a projective curve, so this motivates further quotienting by $\mathbb{G}_{\gg}$ and working with some nontrivial line bundle.

(7/19/2022) Today I learned that the Mellin transform takes the restricted exponential $\mathcal{D}$-module on $\mathbb{G}_m$ to a $k[v]$-module which is not finitely generated! This is given by direct computation: the fiber at each field-valued point of $k[v]$ is given by k since it's the cohomology of the associated twisted sheaf, and moreover if it were finitely generated this would imply it would have to be free of rank one by the classification. However it is not since one can check that the v-action can't drop powers of t, identifying global differnetial operators on $\mathbb{G}_m$ with $k[t^{\pm 1}, v := t\partial_t]$/relation.

(7/20/2022) Today I learned that the equivariance on $\mathcal{D}(T)^{N_G(T)}$ is only braided monoidal, not symmetric monoidal!

(7/21/2022) Today I learned that the irreducible perverse sheaves which give rise to the Gelfand-Graev action on $\mathcal{D}(G/N)$ have compactly supported cohomology isomorphic to the usual cohomology of the associated Deligne-Lusztig variety, which is a theorem of Arnaud Eteve.

(7/22/2022) Today I learned some consequences of the definition of cells, following section 4.1 and 4.2 of Lusztig's 'Cells in Affine Weyl Groups.' Specifically, I learned that it can never be the case that 1 is larger than any element (in the left or right order) because, in the notation of Lusztig, $\mathcal{L}(1) = \mathcal{R}(1)$ is the set of simple reflections and this is the only element for which this is the case. This implies that 1 is in its own equivalence class, and a dual argument gives w_0 is as well. Furthermore, I learned that with respect to the left ordering $s \not\leq st$ because $\mathcal{L}(1) \subseteq \mathcal{L}(st)$, but $\mathcal{L}(1) \subseteq \mathcal{L}(ts)$ so $s \leq st$ in the right ordering and thus the two sided ordering, assuming the other condition about the Hecke algebra which I didn't check. This implies that there are at least 4 left cells as $1, s, st$, and w_0 are all not identified under this equivalence relation, whereas there are exactly three two sided cells.

(7/24/2022) Today I learned the broad outline of the conjecture of Braverman and Kazhdan, which constructs a function on the irreducible representations of $G(\mathbb{F}_q)$ and conjectures that this function is given by the trace of Frobenius by a certain equivariant sheaf known as a ρ-Bessel sheaf.

(7/25/2022) Today I learned that, for Harish-Chandra bimodlues of a given central character of some dominant weight, the center from the other action on a given element acts by some generalized central character given by some integral translate. If this generalized central character also comes from a regular weight, this can be identified as a subcategory of the BGG category $\mathcal{O}_0$.

(7/27/2022) Today I learned the proof of the construction of parity sheaves on a flag variety. Specifically, after formally arguing the uniqueness of these sheaves, one can show that pushforward and pullback of constructible sheaves preserve parity and that you can use these to create a Bott-Samuelson so that, in particular, you get a new parity sheaf by support reasons.

(7/28/2022) Today I learned that one can take the Lazard ring and kill off certain generators and localize at the ideal given by p, and that this is flat as a map to the moduli stack of formal groups. In particular it gives rise to a cohomology theory known as *Brown-Petersen theory.*

(7/29/2022) Today I learned that the characters of Soergel bimodules in characteristic p agree with their characteristic zero counterparts if and only if a certain intersection form has the same rank over the characteristic p field as it does over the rationals. A key step in this proof is to notice that the product of $b_w b_{w'}$ in the Hecke algebra can be written as the sum of $b_{w''}$ with integral coefficients (since we can write it first with coefficients in $\mathbb{Z}[v^{\pm 1}]$ and then use the fact that $b_w b_{w'}$ is self dual with respect to the Kazhdan-Lusztig involution on the Hecke algebra. This then gives the fact that these coefficients are positive since the associated $b_{w''}$ are simple.

(7/30/2022) Today I learned that the exponential map identifies the multiplicaitve and the additive coinvariant algebra if one inverts all primes less than or equal to $\ell(w_0)$. This in particular gives that the additive and multiplicative Soergel modules agree after inverting these primes.

June 2022

(6/1/2022) Today I learned that the cohomological amplitude of the functor $I^! : \mathcal{D}(\mathrm{SL}_2)^{N\times N} \to \mathcal{D}(\mathrm{SL}_2)^{N\times N}_{\mathrm{deg}}$ has cohomological amplitude in [0,1] by an explicit calculuation. This passes a basic sanity check because $I^!$ is left t-exact, and maps object in the heart of the degenerate category to the heart by fully faithfulness. I also learned the probable fact that the result of Ben-Zvi and Nadler on the center being the full subcategory generated by the essential image of pulling then pushing along the correspondence $B\backslash G/B \leftarrow G/_{\mathrm{ad}}B \to G/_{\mathrm{ad}}G$ also works when the Hecke category is replaced with $\mathcal{D}(N\backslash G/N)^{T\times T,w}$, although this is less exciting because all this would translate to is that $\mathcal{D}(G/G)$ is generated by the essential image of ch, which is true because ch is conservative.

(6/2/2022) Today I learned that the closed subscheme of $(G/N \times G/N)/T$ cut out by the diagonal G-orbit of the closed subscheme $(B/N \times B/N)/T$ admits a W-action which commutes with the G-action. This is because this diagonal G-orbit is isomorphic to $G \overset{B}{\times} (B/N \times B/N)/T$, and W acts on the right of this expression by conjugation. In other words, the action of W by conjugation on $G \times (B/N \times B/N)/T$ descends to $G \overset{B}{\times} (B/N \times B/N)/T$!

(6/3/2022) Today I learned that the horocycle functor applied to a very central $\mathcal{D}$-module acquires a W-equivariant structure which always descends to the coarse quotient! Specifically, the fact that an equivariant $\mathcal{D}$-module $\mathcal{F}$ on G is very central implies that we can identify $\mathrm{hc}(\mathcal{F}) \simeq$

$\mathcal{F} \star \mathrm{hc}(\delta_1) \in \mathcal{D}(G \overset{B}{\times} (B/N \times B/N)/T)^{G,\heartsuit}$, and because the space $G \overset{B}{\times} (B/N \times B/N)/T$ has a W-action which commutes with the G-action and we can explicitly compute that, modulo G, $\mathrm{hc}(\delta_1)$ maps to the monoidal unit of $\mathcal{D}(T)$, we can equip it with W-equivariance.

(6/4/2022) Today I learned that the global sections of the (multiplicative) Grothendieck-Springer resolution $\tilde{G}$ can be identified with the global sections of the (affine) variety $G \times_{T/\!/W} T$. This follows from the fact that the canonical map $\tilde{G} \to G \times_{T/\!/W} T$ is an isomorphism over the regular locus of G, along with the fact that regular functions which are defined away from a subset of codimension 2 on a Noetherian normal scheme extend uniquely.

(6/5/2022) Today I learned that a $W = \mathbb{Z}/2\mathbb{Z}$-equivariant sheaf on $\mathbb{A}^1$ which descends to the coarse quotient may have many other equivariances. For example, the sheaf $k[x]^{\oplus 2}$ has at least three–two coming from the coarse quotient (one of these is obtained by tensoring by the sign representation) and the third is a W-equivariance which swaps the two factors.

(6/6/2022) Today I learned that the variety $\mathfrak{g} \times_{\mathfrak{t}^*/\!/W} \mathfrak{t}^*$ is not smooth, even when $\mathfrak{g} = \mathfrak{sl}_2$, but has rational singularities in general, as exhibited by the Grothendieck-Springer resolution. I also learned a reference for many of these facts for the first time, all in Gaitsgory's notes on geometric representation theory. Specifically, the fact that this map actually exhibits the 'last' condition on being a resolution of singularities follows from Gaitsgory's Theorem 7.13, although I'm not sure if he would claim any originality for this.

(6/8/2022) Today I learned that one can identify the quotient of $G \times G$ modulo the action of the diagonal Borel subgroup acting as the moduli space of two Borels, together with an x which conjugates one Borel to the other.

(6/10/2022) Today I learned that if you identify the space $(G \times G)/B$ with the moduli space of two Borels and an isomorphism given by conjugation of some element of G between them, then the associated projection map to the diagonal quotient $(G \times G)/G$ can be identified with the map which maps to the coset whose first coordinate is the identification and the second coordinate is 1.

(6/11/2022) Today I learned that the pullback of D-modules given by $T \to T /\!/ W$ does not likely induce a fully faithful functor $\mathcal{D}(T /\!/ W) \to \mathcal{D}(T)^W$. This is because we can check both on the associated groupoid given by the maps from the torus to the stack quotient and GIT quotient, and this would imply that the pullback from $\mathrm{Spec}(k[x,y]/(xy))$ to the union of two lines is fully faithful at the level of D-modules.

(6/12/2022) Today I learned a heuristic for the Gauss-Manin connection, explained to me by Kendric Schefers. Specifically, I learned that, given a map of varieties $X \to Y$, one can equip a connection on a specific bundle on Y such that at each point it returns the fiber of the map at that point. I also learned about the Ehrehsmen fibration theorem, which says that for a smooth map of complex varieties, the fibers are all diffeomorphic as complex varieties.

(6/13/2022) Today I learned a result of Brion and Fu which says that any conical symplectic variety, i.e. a normal variety whose smooth locus admits a homogeneous symplectic form, which is of weight one (i.e. the weight of the sympelctic form of the canonical bundle) with a symplectic resolution is necessarily the resolution given by a cotangent bundle of a partial flag variety. I also learned a result of Namikawa which says that in fact *all* conical symplectic varieties of *maximal* weight one (the weight of all generators of the variety is 1 $\Longrightarrow$ conical symplectic variety is weight zero or weight one) are given by either affine space or this resolution.

(6/15/2022) Today I learned that there is a bijective correspondence between the equivariant covers of nilpotent orbits of some semisimple group G and pairs of a Levi subgroup and a nilpotent cover such that the affinization of the nilpotent cover is $\mathbb{Q}$-*factorial*, ie the pullback map to the regular locus has cokernel only torsion, and *terminal*, ie, for schemes with symplectic singularities, just means the regular locus has codimension ≥ 4.

(6/16/2022) Today I learned a theorem of Losev, Mason-Brown, and Matvieievskyi, which gives a canonical bijection between the quantizations of the functions on equivariant covers of nilpotent orbits and the equivariant covers of all coadjoint orbits of $\mathfrak{g}^*$. I learned furthermore that, under this bijection, the coadjoint orbit covering a *nilpotent* orbit gives a canonical quantization of its ring of functions.

(6/17/2022) Today I learned a specific example of a $\mathbb{A}^1$-bundle which cannot be promoted to a structure of a vector bundle. Specifically, if $G = \mathrm{SL}_2$, we can identify G/T with the open subscheme of $\mathbb{P}^1 \times \mathbb{P}^1$ away from the diagonal. This is a bundle over $\mathbb{P}^1 = G/B$ via the projection map, however, one can show that there is no global section of this bundle, so in particular there is no *zero* section.

(6/18/2022) Today I learned that there is an explicit description of parabolic restriction obtained recently due to Ginzburg. Specifically, one can identify parabolic restriction or its lifted variant, at least at the level of abelian categories, as the functor which takes some adjoint G-equivariant sheaf on G, takes its global sections, takes the G invariants of this representation, and then tensors via $\Gamma(\mathcal{D}_T) \otimes_{\Gamma(\mathcal{D}_T)^W} -$.

(6/19/2022) Today I learned that $\mathcal{D}$-affine varieties have the underlying quasicoherent sheaf of any $\mathcal{D}$-module generated by global sections, essentially by definition. This is because (admittedly roundabout reasoning but:) there is a t-exact equivalence of $\mathcal{D}$-modules with the category of modules for the sheaf of differential operators which sends the sheaf to the global sections, and any module has the global sections surjecting onto it.

(6/20/2022) Today I learned that, given a filtered object in some abelian category with a subobject or quotient, one can define an induced filtration on the subobject or quotient in a pretty natural way. Specifically, the subobject obtains a filtration via pulling back the filtration, and the quotient obtains a filtration via pushforward. Given a short exact sequence obtained by a subobject and quotient, I also learned the associated graded functor is exact.

(6/22/2022) Today I learned the equivalence of categories which identifies the Kirillov model of a category $\mathcal{C}$ with an action of $\mathbb{G}_a \rtimes \mathbb{G}_m$, i.e. the kernel of $\mathrm{Av}_* : \mathcal{C}^{\mathbb{G}_m} \to \mathcal{C}^{\mathbb{G}_a \rtimes \mathbb{G}_m}$, with the standard Whittaker model of a category, is t-exact up to cohomological shift if the action of $\mathbb{G}_a \rtimes \mathbb{G}_m$ is compatible with the t-structure. This is because the functor giving the equivalence of the Kirillov model and the Whittaker model is given by $\mathrm{Av}_*^{\psi}[1]\mathrm{oblv}^{\mathbb{G}_m}$, which has cohomological amplitude in [-1, 0], which is equivalently $\mathrm{Av}_!^{\psi}[-1]\mathrm{oblv}^{\mathbb{G}_m}$ which has cohomological amplitude in [0, 1]. In particular it's a t-exact equivalence of categories, so its inverse is also a left and right adjoint so is t-exact since it's right and left t-exact.

(6/23/2022) Today I learned that the base change of a parity sheaf over $\mathbb{Z}_p$ preserves parity sheaves, but it need not preserve indecomposable parity sheaves.

(6/24/2022) Today I learned that the natural $\mathbb{G}_m$-action given on the flag variety as a closed subscheme of the affine Grassmannian via the Steinberg embedding is trivial, which essentially follows from the fact that the $\mathbb{G}_m$ action on t^α is trivial for any coroot α. I also learned that if one has a complete local ring R, the indecomposible Soergel module associated to a give realization is indecomposable after base change to the residue field, which is Lemma 4.1 in Jensen-Williamson.

(6/25/2022) Today I learned the notion of an *ample* vector bundle on some variety X, which is defined as some vector bundle $\mathcal{V}$ such that for every coherent sheaf $\mathcal{F}$ on the variety, the higher cohomology $H^i(X, \mathcal{F} \otimes \mathcal{V}^{\otimes n})$ vanishes for $n \gg 0$. I also learned that Mori proved *Harthsorne's conjecture*, which says that the only projective varieties with ample tangent bundles are projective spaces.

(6/27/2022) Today I learned that the only connected symplectic manifold whose associated symplectic form is exact is a point. This is because the top exterior power of a symplectic manifold gives a volume form and, by Stokes' theorem, anything which is exact on a symplectic manifold without boundary will vanish.

(6/28/2022) Today I learned the difference between a category being *Krull-Schmidt* vs. just having objects split into a direct sum of indecomposable subs uniquely. Specifically, Krull-Schmidt categories are those for which objects splits into a direct sum of indecomposable subs *which have local endomorphism rings.*

(6/29/2022) Today I learned a theorem of Kumar, Lauritzen and Thomsen, which states that the higher cohomology of all symmetric powers of the tangent bundle of a flag variety in characteristic zero vanishes, whereas the *tensor* powers do not vanish by a result of Belmans-Smirnov for partial flag varieties and for full flag varieties by a computation of Knutson.

(6/30/2022) Today I learned that, on a quasicompact scheme X, a locally finitely generated coherent sheaf which is globally generated is globally generated by finitely many global sections. This is because you can cover your scheme by affine opens and pick finitely many of the given surjection which 'covers' all the local generators, and quasicompactness says we can do this for finitely many affine opens.

May 2022

(5/1/2022) Today I learned an interesting fact due to Arinkin, which says that one can check whether an object of the derived category (unbounded on both sides, even!) of quasicoherent sheaves on a locally Noetherian scheme X is zero by checking the (derived) fiber at each point! Of course, since locally Noetherian schemes are covered by $\text{Spec}(A)$ for Noetherian A, it suffices to prove this when $X = \text{Spec}(A)$. Now, consider some A-module M and consider all closed subschemes $i : Z \hookrightarrow X$ such that $i^*(M)$ is nonzero. Since X is Noetherian, there is a minimal closed subscheme with this property. Taking the fiber of M at this closed subscheme, we may replace this minimal closed subscheme by X itself and assume that no fiber of M vanishes. In particular, for any $f \in A$ which is nonzero, since we have a fiber sequence $A/f \to A \xrightarrow{f \cdot -} A$, we have that multiplication by any nonzero $f \in A$ yields an equivalence $f \cdot - : M \xrightarrow{\sim} M$.

If A is not an integral domain, then there exists some nonzero $f, g \in A$ whose product is zero. We therefore see that the zero map is an isomorphism since the composite $M \xrightarrow{f \cdot -} M \xrightarrow{g \cdot -} M$ is an isomorphism. Now assume A is an integral domain, and fix some $i \in \mathbb{Z}$. The above analysis gives any nonzero $f \in A$ acts invertibly on $H^i(M)$ for all i. Therefore, we have that, as classical A-modules, $H^i(M) \cong K \otimes_A H^i(M)$, where K denotes the field of fractions of A. However, since localization is an exact functor of abelian categories, the functor $K \otimes_A -$ is a t-exact functor of derived categories. Thus, we have

$$0 = H^i(K \otimes_A M) \cong K \otimes_A H^i(M) \cong H^i(M)$$

where the first step uses the assumption that the fiber of M at every point vanishes. Since $H^i(M) = 0$ for all i, we have that M itself vanishes in A-mod by the left and right completeness of the t-structure on A-mod.

(5/2/2022) Today I learned a direct consequence of the algebraic Hartog's lemma, which says that the restriction functor j^* for some open embedding of codimension 2 or higher is fully faithful on the abelian category of vector bundles, at least on $\mathbb{A}^n$. This is because the only vector bundle on $\mathbb{A}^n$ is trivial and therefore it suffices to show for the structure sheaf itself. For the structure sheaf, fully faithfulness at the abelian categorical level is equivalent to the restriction map on global sections yielding an isomorphism–the surjective part of this is given by Hartog and the injectivity is given because $\mathbb{A}^n$ is irreducible.

(5/3/2022) Today I learned that if X denote any discrete set of points acting on some ind-scheme Γ, one can define a t-structure on $\text{IndCoh}(\Gamma/X)$ via setting $\text{IndCoh}(\Gamma/X)^{\leq 0}$ to be the full subcategory generated under colimits by objects which are pushed forward via $\text{IndCoh}(\Gamma)^{\leq 0}$. I also learned that the functor $\text{Av}_*^{T,w}\text{oblv} : \mathcal{C}^{T,w} \to \mathcal{C}^{T,w}$ corresponds to direct summing your object $X^\bullet(T)$ many times, which can be shown in the universal case by the Mellin transform in the $\mathcal{D}(T)$-module case.

(5/4/2022) Today I learned that the IndCoh pushforward map from $\mathfrak{t}^*/X^\bullet(T) \times_{\mathfrak{t}^* /\!/ \tilde{W}^{\text{aff}}} \mathfrak{t}^*/X^\bullet(T) \to \mathfrak{t}^*/X^\bullet(T)$ is conservative on eventually coconnective objects. One way to see this is that, given any eventually coconnective object in the category, there exists some field-valued point where the !-restriction to $\text{Spec}(K) \times_{\mathfrak{t}^* /\!/ \tilde{W}^{\text{aff}}} \mathfrak{t}^*/X^\bullet(T)$ will not vanish. Moreover, this restriction will be left t-exact and thus preserve eventually coconnective objects. This is convenient for us because the IndCoh pushforward of $\text{Spec}(K) \times_{\mathfrak{t}^* /\!/ \tilde{W}^{\text{aff}}} \mathfrak{t}^*/X^\bullet(T) \to \text{Spec}(K)$ is equivalently the pushforward from a disjoint union of copies of the coinvariant algebra, which is conservative on eventually coconnective objects.

(5/5/2022) Today I learned a potential way to get an entirely algebraic proof of the existence of a characteristic polynomial map $\mathfrak{g} \to \mathfrak{t} /\!/ W$. Specifically, we have that the Grothendieck-Springer resolution map $\mu : \tilde{\mathfrak{g}} \to \mathfrak{g}$ is a proper map. Now recall the regular locus $\mathfrak{g}^{\text{reg}}$ is, by definition, the restriction to the locus of elements whose centralizer in the Lie algebra is of minimal dimension. Any element in the regular locus has at most $|W|$ many Borel subalgebras containing it, and so this implies that $\mu|_{\mu^{-1}(\mathfrak{g}^{\text{reg}})}$ quasifinite. Since it's also proper (and of Noetherian schemes since both schemes are quasiprojective), we obtain that the map $\mu|_{\mu^{-1}(\mathfrak{g}^{\text{reg}})}$ is itself finite.

Now, something I'm seeing written down a few places but not finding a proof of is that $\mathfrak{g}^{\widetilde{\text{reg}}} \simeq \mathfrak{g}^{\text{reg}} \times_{\mathfrak{t}/\!/W} \mathfrak{t}$ (of course, this is using the Chevalley restriction map!), which in particular implies that somehow one can extend the W-action that one can get on $\mathfrak{g}^{\widetilde{\text{reg}}, \text{ss}}$ to the entire locus of regular elements. Using this and the fact that $\mu|_{\mu^{-1}(\mathfrak{g}^{\text{reg}})}$ is finite (and so in particular, the underlying map of topological spaces is closed), I think one can use this to show that $\mu|_{\mu^{-1}(\mathfrak{g}^{\text{reg}})}$ is a geometric quotient with respect to this W-action. In particular, this would imply it's a categorical quotient by a result of Mumford and so, assuming one could also show that the map $\mathfrak{g}^{\widetilde{\text{reg}}} \to \mathfrak{t}^*$ is W-equivariant, or at least that the composite to $\mathfrak{t}^* /\!/ W$ kills the W-action, one can get the universal property of the categorical quotient giving you a map $\mathfrak{g}^{\text{reg}} \to \mathfrak{t}^* /\!/ W$. However, $\mathfrak{t}^* /\!/ W$ is a polynomial algebra by the Chevalley-Shephard-Todd theorem, so a map to $\mathfrak{t}^* /\!/ W$ is given by specifying some global functions on $\mathfrak{g}^{\text{reg}}$. However, $\mathfrak{g}^{\text{reg}}$ has codimension ≥ 2 and so the global functions agree with the usual global functions on $\mathfrak{g}$ itself, via the algebraic Hartog lemma.

(5/6/2022) Today I learned an explicit computation of the moduli of Borel subgroups containing a given element, and, in particular, I learned that the moduli of Borel subgroups containing the element e with a 1 in the upper right corner and 0's elsewhere is given by $\text{Spec}(A)$ with $A := k[\epsilon]/\epsilon^2$. This can be given explicitly as follows. We are interested in the moduli space of Borel subalgebras of $\mathfrak{sl}_{2,A}$ containing e. A Borel subalgebra, mimicking SGA 3.XIV.4, is defined as a smooth(!) Lie-A-subscheme, where by smoothness I mean the map to $\text{Spec}(A)$ is smooth. The constant standard Borel subscheme $\mathfrak{b}_A$ base changed gives an example because base change preserves smoothness and $\mathfrak{b}$ is smooth as a k-scheme. In particular, for any $x \in k$, we can take the matrix $m := \begin{pmatrix} 1 & 0 \\ x\epsilon & 1 \end{pmatrix} \in \mathfrak{sl}_{2,A}(A)$ and conjugate this Borel to obtain another Borel subalgebra! Direct computation yields that

$$m \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix} m^{-1} = \begin{pmatrix} a - bx\epsilon & b \\ 2ax\epsilon - b(x\epsilon)^2 & bx\epsilon - a \end{pmatrix}$$

and if x is nonzero, this Borel subalgebra does not contain the element $h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, but it does still contain the element e! This is exciting because it agrees with the statement implicitly made by the isomorphism $\widetilde{\mathfrak{g}_{\text{reg}}} \simeq \mathfrak{g}_{\text{reg}} \times_{\mathfrak{t}/\!/W} \mathfrak{t}$, which says that the moduli of Borels containing e is the dual numbers, whereas the moduli of the Borels containing h is two separate points. (The ϵ^2 term is explicitly written out to show that $\epsilon^2 = 0$ is required for e to still be in the Borel so you can't do this trick indefinitely!)

(5/7/2022) Today I learned that if you take the $\mathbb{Z}/2\mathbb{Z}$-action on $\mathbb{A}^1 \setminus 0$ given by $x \mapsto x^{-1}$, the GIT quotient of this action is actually $\mathbb{A}^1$! One can directly compute that the fixed points of the ring, which we will denote $R := k[x^{\pm 1}]^{\mathbb{Z}/2\mathbb{Z}}$, under this action is given by those polynomials $p(x)$ such that $p(x) = p(x^{-1})$. However, this turns out to be a polynomial algebra! Specifically, let $t = x + x^{-1}$. Then $k[t]$ is a subring of $k[x^{\pm 1}]$ because $\{1, t, t^2, ...\}$ are linearly independent for (x-)degree reasons. Moreover, any polynomial of R is in the image of this inclusion map, which you can show by induction on the x-degree!

Moreover, the inclusion map of rings $\phi : R \to k[x^{\pm 1}]$ corresponding to the GIT quotient is finite flat. To see this, note we have just argued that R is a polynomial algebra in one generator, and so it's in particular a PID. All finitely generated non-torsion elements over a PID are flat by the fundamental theorem of modules over a PID, so this map is not flat.

(5/8/2022) Today I learned the explicit computation of the homotopy groups of the Tate construction of E_1. Specifically, one has

$$\pi_{-*}E_1^{tC_p} \simeq E_1^*(*)((x))/([p](x)) = \mathbb{Z}_p[\beta^{\pm 1}]((x))/((1+x)^p - 1)$$

where the first result can be checked as, for example, Lemma 2.1 in Ando-Morava-Sadofsky, and the second follows from the homotopy groups of $E_1^*(\mathbb{C}P^\infty)$ and the p-series of the multiplicative group, which can be checked in Lecture 12 of Lurie's chromatic homotopy theory class notes. The ideal of the above is given by $x^p = p?$ for some correct value of ?, and so p is a unit, and this ring can be identified with

$$\mathbb{Q}_p[\beta^{\pm 1}]((x))/((1+x)^p - 1) = \mathbb{Q}_p[\beta^{\pm 1}]((t-1))/((t-1)\Phi_p(t)) = \mathbb{Q}_p[\beta^{\pm 1}]((t-1))/(\Phi_p(t)) = \mathbb{Q}_p(\zeta_p)[\beta^{\pm 1}]$$

setting $t := x + 1$, where $\Phi_p(t)$ is the minimal polynomial for a p^{th} root of unity, and the second equivalence follows since $t - 1$ is invertible.

(5/9/2022) Today I learned the full definition of an *algebraic space*. Specifically, an algebraic space is equivalent to the data of a scheme X equipped with a closed subscheme of $X \times X$ which forms an equivalence relation and such that the projection to the factors is étale.

(5/10/2022) Today I learned a fact which says that one can identify the exterior algebra of the trivial representation for some characteristic $p > h$ field with the ring of functions on the nilpotent cone. In particular, the extensions of *any* module by the trivial one forms a quasicoherent sheaf on this space!

(5/11/2022) Today I learned about the interesting scaling invariant section $t^{-1}dt$, a differential form on $\mathbb{G}_m$. Specifically, I learned that the k-representation given by the k-span of this differential form is given by the sign representation of the Weyl group.

(5/13/2022) Today I learned the sketch of the proof which says that a holonomic, or more generally any complex of ℓ-adic sheaves on a torus vanishes if and only if each twisted compactly supported cohomology vanishes for each character of the field, in the case $T = \mathbb{G}_m$. This is because the fact that compactly supported cohomology is left t-exact tells you that any irreducible subobject of some object in the heart whose compactly supported cohomology vanishes has to have 0th cohomology 0. There is a result which says that the Euler characteristic of a perverse ℓ-adic sheaf must be nonnegative, and therefore the Euler characteristic is zero. We then see (apparently) that using something called the Grothendieck-Ogg-Shafarevich formula that this implies that, as a constructible sheaf on $\mathbb{P}^1$, the sheaf has to have moderate growth at zero and infinity and therefore has to be a line bundle, which contradicts the cohomology zero at each field-valued point computation.

(5/15/2022) Today I learned that if you have some W-equivariant sheaf supported at the origin (say), you can check whether it descends to the coarse quotient by checking whether it descends to just the 0th and the -1st cohomology groups. This is because you can explicitly show that in this case, using the fact that it reduces to simple reflections, that the induced map from the fixed points is a surjection, and that the kernel also descends so it also admits a surjection from something pulled back from the coarse quotient. Since the pullback functor is t-exact and fully faithful, we see that any such sheaf is a quotient of two such objects and thus itself lies in the essential image.

(5/16/2022) Today I learned a slick way to argue that the globalized Springer sheaf and the Springer sheaf admit an action of W. The main point comes from the fact that one can identify the pushforward $(\tilde{G} \to G)_*(\underline{k})$ with the intermediate restriction of the analogous pushforward on the associated regular singular loci $(\tilde{G}^{\text{rs}} \to G^{\text{rs}})_*(\underline{k})$, which obtains an action of W since this restricted projection map is a W-torsor, so the constant sheaf is canonically W-equivariant.

(5/17/2022) Today I learned that, using the fact that the endomorphisms of the Springer sheaf are given by $k[W]$ for k a field of characteristic zero, one can argue that there is a full subcategory of perverse sheaves on the nilpotent cone cut out given by the Springer sheaf and all its direct summands. Because all simple objects in the equivariant category of perverse sheaves on the nilpotent cone are given by an orbit and a representation of the component of the centralizer, this gives an injection between the representations of W and an associated orbit and representation.

(5/18/2022) Today I learned a proof that, if one equips the functor of parabolic induction from a given torus with the standard W-representation structure, the W-representation on parabolic *restriction* of the delta sheaf δ_{1G} is not given by the trivial representation, and therefore by one dimensionality is given by the sign representation for a simple Lie group. This is because

$$\pi_0\text{Hom}_{\mathcal{D}(G)^G}(\text{Ind}(\delta_{1T})^W, \delta_{1G}) \simeq \pi_0\text{Hom}_{\mathcal{D}(G)^G}(\mathcal{S}^W, \delta_{1G}) \simeq 0$$

where $\mathcal{S}$ denotes the Springer sheaf, where the last equality follows since the W-invariant subrepresentation of $\mathcal{S}$ is given by the constant sheaf shifted to be constant, which has no maps to δ_{1G} a distinct simple perverse sheaf for $G \neq 1$. Therefore by adjointiness we see that

$$\pi_0\text{Hom}_{\mathcal{D}(T)^W}(\delta_{1T}, \text{Res}(\delta_{1G}))$$

must also be zero, but since the underlying sheaf of $\text{Res}(\delta_{1G})$ is δ_{1T}, this says that the representation given is the *sign* representation.

(5/19/2022) Today I learned an explicit category which comes up in nature which is not the derived category of its heart! Specifically, we can take the category $\mathcal{D}(B\mathbb{G}_m) \simeq k[\xi]$-mod where ξ lives in cohomological degree 1. The standard t-structure on this gives the heart of Vect, whose generator is the constant sheaf. However, the constant sheaf corresponds to the $k[\xi]$-module k, and we have that $\underline{\text{End}}(k)$ is nonzero in multiple cohomological degrees since $k[\xi]$ is a nontrivial extension of some shift of k.

(5/20/2022) Today I learned the fact that $\text{IndCoh}(\mathfrak{t}^* /\!/ \tilde{W}^{\text{aff}})$ embeds fully faithfully into $\mathcal{D}(G/G)$ via the functor $\text{Ind}(- \otimes k_{\text{sign}})^W$. This is true because it suffices to show this on the pushforward-ed sheaves given by all field-valued points, and by base change it suffices to prove this for all closed points, where it reduces to a result of Tsao-Hsien where he shows that the unit map of the associated adjunction is an equivalence on such delta sheaves. '

(5/22/2022) Today I learned a statement of Ben-Zvi and Nadler, which says that the center of the Hecke category can be identified with the *character* $\mathcal{D}$-modules, i.e. the full subcategory of $\mathcal{D}(G)^G$ generated by the essential image of the horocycle functor given by pulling then pushing along the correspondence $B\backslash G/B \leftarrow G/_{\text{ad}}B \to G/_{\text{ad}}G$.

(5/23/2022) Today I learned that the affine plane is not *maximally affine*. In other words, there exists some affine variety X such that $\mathbb{A}^2_{\mathbb{C}}$ can be written as a locally closed subscheme but not a closed subscheme of X.

(5/24/2022) Today I learned a quick restriction on the class of $\mathcal{D}$-affine varieties. Specifically, since $\mathcal{D}$-affine varieties have the global sections functor exact, the higher cohomology of the structure sheaf should vanish. In particular, since the genus of a smooth projective curve is defined as the dimension of the first cohomology of the structure sheaf, all $\mathcal{D}$-affine smooth projective curves are $\mathbb{P}^1$.

(5/25/2022) Today I learned that for the group $G := \text{GL}_n$, the parabolic restriction functor is conservative, and induces an equivalence of abelian categories $\mathcal{D}(G)^{G,\heartsuit} \simeq \mathcal{D}(T)^{W,\heartsuit}$. Using this equivalence, one can argue that the two summands of the global Springer sheaf have no maps in the category of perverse sheaves between each other for $G := \text{GL}_2$.

(5/26/2022) Today I learned (from Joakim Færgeman!) that if G is a connected group and $q : * \to BG$ denotes the quotient map, then $q_{*,dR}(k)$ is a compact generator of the category $\mathcal{D}(BG)$! This is because one can apply Verdier duality to the QCA stack BG, and then it suffices to check that $q_{!,dR}(k)$ is a compact generator. The generator part follows from the conservativity of $q^!$, and the compactness part follows since $q^!$ is continuous.

(5/27/2022) Today I learned that the de Rham pushforward functor induced by $q : * \to B\mathbb{G}_m$ acquires a lift $\text{Vect} \to \mathcal{D}(B\mathbb{G}_m)^W$ where W denotes the group $\mathbb{Z}/2\mathbb{Z}$ which is fully faithful! This is because one can check that the endomorphisms of this object are given by the de Rham homology, whose associated W-representation is different in the two nonzero degrees.

(5/28/2022) Today I learned that there is an I think bijective correspondence between the *character* $\mathcal{D}$-modules for some algebraic group H, i.e. objects of $\mathcal{D}(H)^H$, and categorical representations of H where the underlying category is Vect. Specifically, given such a categorical representation, we equivalently obtain a monoidal functor $\mathcal{D}(H) \to \text{End}_{\text{DGCat}}(Vect) \simeq \text{Vect}$ which one can dualize and take the diagonal H equivariance of the map and get a functor $\text{Vect} \to \mathcal{D}(H)^H$.

(5/29/2022) Today I learned a fun abelian categorical lemma. Assume that M, N, and X are all objects in some abelian category and $f : M \to N$ is some map of abelian categories for which the pullback map $\text{Hom}(N, X) \xrightarrow{-\circ f} \text{Hom}(M, X)$ is an isomorphism. Then the $\text{End}(M)$-module structure on $\text{Hom}(M, X)$ factors through the quotient of the ideal given by the kernel of the pullback map $\text{End}(M) \xrightarrow{f\circ -} \text{Hom}(M, N)$. This is because if $\phi : M \to X$ and $e : M \to M$ then there exists some $\xi : N \to X$ with $\xi f = \phi$ then $\phi \circ e = \xi \circ f \circ e = 0$ if $f \circ e = 0$.

(5/30/2022) Today I learned that there are surjections of filtered objects which need not induce surjections on associated graded. A trivial example is the map $V_1 \to V_0$, where V_i is the one dimensional vector space whose filtration starts being nonzero in degree i. This is a surjective map but is not surjective on associated graded.

(5/31/2022) Today I learned that the following diagram is almost certainly Cartesian:

$$\begin{array}{ccc} G \overset{B}{\times} (B \times B)/B & \xrightarrow{\text{Av}_*^{N\times N}} & G \overset{B}{\times} (B/N \times B/N)/T \\ \downarrow & & \downarrow \\ (G \times G)/B & \xrightarrow{\text{Av}_*^{N\times N}} & \text{Hor} \end{array}$$

where the downward arrows are the associated closed embeddings and the horocycle space Hor :=

$(G/N \times G/N)/T$. At least a sanity check on this is that if you mod out by the left action of the diagonal copy of G, the top arrow becomes the arrow to identify this with the pushforward of $B/B \to T/T$ as expected to match this with parabolic restriction.

April 2022

(4/1/2022) Today I learned the following paradigm for determining whether the decomposition theorem holds for a given closure of a Schubert variety of a flag variety. Specifically, pick some point w in the Schubert variety, and consider its normal slice, which is an affine space. We can take the fiber of the and consider its embedding into this normal slice, which will give us an intersection pairing. Then it's a theorem of Junteau-Mautner-Williamson that the decomposition theorem holds over a field of characteristic p for the Schubert variety if and only if for all points of the flag variety, the rank of this pairing over $\mathbb{Q}$ agrees with the rank of the pairing over the field of characteristic p.

(4/2/2022) Today I learned some of the basics of modular forms. Specifically, because of the definition of modular forms, if we are given a modular form f we have that $f(z) = f(z+1)$, so that if we set $q = e^{2\pi i z}$, then the fact that $f(q)$ is a holomorphic function on $\{q : 0 < |q| < 1\}$ implies it extends to the closure of this disk and so it admits a power series representation. The first term in this power series representation is called the *constant term*, and the rest are known as the Whittaker coefficients.

(4/3/2022) Today I learned that the pushforward of the Springer resolution in characteristic 2 is parity, since it is an even resolution. I also learned that the pushforward of this sheaf does not split into a direct sum of simples in characteristic 2 because, for example, one can compute the associated IC sheaf because the IC sheaf at the regular orbit is not parity, and that there are only two simple objects so one can directly argue that this pushforward is not a sum of these two or multiplicities of these two.

(4/4/2022) Today I learned a theorem of Nadler, which states that Betti geometric Langlands is true over an elliptic curve. I also learned the definition of a Kac-Moody Lie algebra implies it is a *two*-dimensional central extension of the loop algebra, and that the *commutator* of the Lie algebra with itself can be identified, before completion, with the affine Lie algebra, a *one*-dimensional central extension of the loop algebra.

(4/5/2022) Today I learned that the tensor product of tilting modules is tilting, and moreover there is a theorem which allows one to compute the tensor product of the indecomposable tilting indexed by some weight of the form $\lambda + p\mu$ for λ p-restricted, and it is literally given by the tensor product of the indecomposable tilting at λ and the Frobenius pullback of the tilting at μ. In fact, this tensor product is always a tilting object but it is only for $p \geq 2h-2$ for which this tilting object is proved to be indecomposable.

(4/6/2022) Today I learned that there are two interesting t-structures one can put on $\mathbb{Z}$-equivariant constructible sheaves on a point! An equivalent fact is that the t-structure is not invariant under the duality functor $\mathrm{Hom}(-, \mathbb{Z})$ because it moves the cohomological degree of torsion. This gives an intuitive pictures for the t-structures–one needs to pick which degrees torsion is allowed to live.

(4/7/2022) Today I learned an explicit computation for the group scheme of regular centralizers for PGL_2. Specifically, we can view this group scheme as a one dimensional family over the affine line, and over zero, this centralizer returns $\mathbb{G}_a$, while at all other points, it returns $\mathbb{G}_m$!

(4/8/2022) Today I learned that one can identify the cotangent stack to Bun_G with the space of Higgs fields, i.e. a G-bundle equipped with a certain linear map from the bundle to the bundle tensored with the sheaf of one forms. Moreover, once one is given a trivialization of the sheaf of one forms, one can define what it means for the map to be nilpotent, and this is the global analogue of the Kostant section.

(4/9/2022) If $\mathcal{Y}$ is a 0-coconnective 0-truncated prestack and admits a free group action from some group ind-classical-scheme $\mathcal{G}$, then the prestack quotient $\mathcal{Q} := \mathcal{Y}/\mathcal{G}$ is a 0-coconnective 0-truncated prestack. The proof is given by showing that the based loops on any component of $* \times_{\mathcal{Q}(S)} *$ is a point, which follows from the freeness.

(4/13/2022) Today I learned a way to construct Ringel duality at the level of sheaves. Specifically, one can define the *long intertwining functor*, which is given by restricting to the correspondence of the $1, w_0$-cell of $G/B \times G/B$ and applying a pull push. One can check, for example, that one version of this sends the delta sheaf to the costandard at the origin, as expected via Ringel duality.

(4/14/2022) Today I learned an extension of the endomorphismensatz to the big monodromic tilting object. Specifically, this isomorphism says that one can identify the endomorphisms with the tensor product of the completion of the representation ring of the torus with itself over its W-fixed points. One can take the tensor product and kill one of the sides to derive the classical endomorphismensatz.

(4/15/2022) Today I learned that, in defining the formal completion of a closed embedding of a smooth variety into another smooth scheme separately from using the de Rham prestack, one can define the de Rham prestack itself by the quotient of the groupoid obtained by the formal completion of the diagonal inside the product of the variety itself, which forms an equivalence relation in sets that we may quotient by. This extends to the prestack setting because, essentially by definition, the de Rham prestack of a 0-coconenctive 0-truncated prestack is is 0-coconnective and 0-truncated as a prestack.

(4/16/2022) Today I learned that there exists a dual concept to the notion of a Kan extension–namely, a Kan lift. This is dually defined as the left/right adjoint, respectively, to the map of functor categories given by post-composition.

(4/17/2022) Today I learned the notion/purpose of Day convolution. Specifically, given two symmetric monoidal categories $\mathcal{C}$ and $\mathcal{D}$, one can define a convolution product on the category of functors between them, formally given by a certain left Kan extension, as I've defined below. This convolution structure is defined so that the commutative algebra objects of the functor category with Day convolution structure give lax symmetric monoidal functors.

(4/19/2022) Today I learned one restatement of Tannaka duality for qcqs schemes, and more generally algebraic spaces X in the sense of Lurie's Elliptic 1. Specifically, this theorem says that one can take such an X and view it as a functor on all DG categories via the 'Spec' of its quasicoherent sheaves, and Tannaka duality says that this X is determined by its restriction to all $\text{QCoh}(A)$ for all connective rings A.

(4/21/2022) Today I learned the *Atiyah-Singer index theorem*, which says that the index of an elliptic differential operator on a closed symplectic manifold equals the integral over X of the $\hat{A}$-genus of X multiplied by a certain character of the manifold X.

(4/22/2022) Today I learned the computation of the 1-shifted Cartier dual of $B\mathbb{G}_{m,dR}$, or at least the details (Justin Campbell explained it to me a few weeks ago...). Here's the proof. You have a cofiber sequence $B\hat{\mathbb{G}_m} \to B\mathbb{G}_m \to B\mathbb{G}_{m,dR}$, where the hat refers to the formal completion, given by the fact the sequence without the B is a cofiber sequence and the fact that B commutes with colimits. Taking 1-shifted Cartier duals, we obtain a *fiber* sequence in the 'opposite direction' given by $\mathbb{A}^1 \leftarrow \mathbb{Z} \leftarrow ?$ where the question mark refers to the 1-shifted Cartier dual in question. In particular, this implies that $? \simeq \mathbb{Z} \times_{\mathbb{A}^1} *$, and because $\mathbb{Z} \setminus 0 \times_{\mathbb{A}^1} * \simeq \emptyset$, we obtain that $? \simeq * \times_{\mathbb{A}^1} *$ as desired!

(4/23/2022) Today I learned the connected components of the $\mathbb{G}_m$-fixed points of the affine

Grassmannian are in bijective correspondence with the dominant coweights, and the proof outline is as follows: given a $G(\mathcal{O})$-orbit, one can first examine the associated G-orbit, which forms a partial flag variety, and then the remaining orbit of the congruence group $\ker(G(\mathcal{O}) \to G)$ forms an 'infinite rank vector bundle' on this partial flag variety, on which the associated $\mathbb{G}_m$-action contracts fibers!

(4/24/2022) Today I learned that you can use the ideas in Riche-Williamson's paper on Smith-Treumann theory to prove the linkage principle for characteristic zero, although this is much less exciting. Specifically, one can use the full '$p = \infty$' version of this theorem and argue that the $p = \infty$ Smith-Treumann category of a point is given by the quotient of $\mathrm{IndCoh}(\Omega\mathbb{A}^1)/\,\mathrm{QCoh}(\Omega\mathbb{A}^1)$, where the endomorphisms of the constant sheaf at the level of abelian categories is k!

(4/25/2022) Today I learned that for an affine smooth scheme X of dimension d, the dual to the 'indcoh global sections functor' t_*^{IndCoh} is given by $t^![-d]$, since global sections is by definition given by mapping out of the constant sheaf. This in particular implies that the Cartier dual to the forgetful functor $(T \to T_{dR})^!$ is given by a *shifted* global sections functor, which is the working out of a technical point about the Mellin transform, and in particular, 'explains' why it is only t-exact up to cohomological shift for ind-coherent sheaves. I also learned an interpretation of the *Hochster-Roberts theorem*, which says that if H is any reductive group over a field of characateristic zero which acts on some regular ring R, then the ring of invariants R^H has a graded polynomial subring $P \subseteq R^H$ such that R^H is finite as a P-module! I also learned a really, really cool converse of the Chevalley-Shephard-Todd theorem, which says that the ring of invariants of a finite matrix group over an algebraically closed field acting on the associated polynomial ring *only if* the finite matrix group is a *pseudo-reflection group*, i.e. it is generated by *pseudo-reflections*, i.e. elements which pointwise fix some hyperplane but aren't the identity. (Note that by the Jordan decomposition theorem, any finite order matrix is diagonalizable over a field of characteristic zero, so for finite order matrices we don't have to require them to be diagonalizable upfront.)

(4/26/2022) Today I learned that a quasicoherent sheaf on the coarse quotient on any *pseudo-reflection* group H acting on affine space descends if and only if it descends pointwise, and, using this, one can show that an H-equivariant sheaf defined on affine space descends to the coarse quotient if and only if it descends for each pseudo-reflection!

(4/27/2022) Today I learned that the projection formula also gives the fact that the subquotients S factor through C/C_0^+ gives that $S \cong i_*(S')$ for $i : */H \hookrightarrow \mathrm{Spec}(C)/H$ the closed embedding, where $S' \in \mathrm{QCoh}(*/H) \simeq \mathrm{Rep}(H)$ has no nontrivial subrepresentations. Then one can equivalently view the above computation as

$$t_*(i_*(S') \otimes_C M) \simeq t_* i_*(S' \otimes_k i_*(M))$$

where t is the terminal map, by the projection formula. This makes the H-equivariance in the above computation manifest, because the tensor product is now being taken in $S' \in \mathrm{QCoh}(*/H) \simeq \mathrm{Rep}(H)$.

(4/28/2022) Today I learned some interesting answers to the question of whether the *tautological ring*, the subring of the Chow ring of the moduli space of genus g curves generated by the powers of Chern classes of the tautological bundle of the universal bundle, generates the entire ring. I learned that there are different proofs for each $2, 3, 4$ and $5, 6,$ and 7 through 9 and that for $g = 12$ the claim is false, and a similar argument shows that a slightly weaker claim regarding the compactification of the moduli space of genus g curves has a non-tautological Chow ring if $g \geq 12$.

(4/29/2022) Today I learned that the relation given by quotienting two points on $\mathbb{A}^1$ (say) if they agree up to a square zero extension is not an equivalence relation! Specifically, one can test this on the ring $A := k[\epsilon_1, \epsilon_2]/(\epsilon_i^2)$. Moreover, I learned that quotienting by all higher order differential operators, and one can show that this relation also shows that in characteristic zero these are the only relations and shows the associated connection one can extract from this data is flat.

(4/30/2022) Today I learned a potential way to derive a quantized Bezrukavnikov's equivalence from a quantized ABG. Specifically, assuming that a quantized ABG holds and that one has similar $\hbar$-Morita equivalences for G and the associated Harish-Chandra category, one obtains $\mathcal{D}_\hbar(N\backslash G/N)^{T\times T,w} \simeq \underline{\text{End}}_{\mathcal{HC}_\hbar}(\mathcal{D}_\hbar(G/N)^{G\times T,w})$. In particular, applying a quantized ABG which we assume is $\hbar$ monoidal, one might hope a similar dualizablility holds on the A-side.

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(3/1/2022) Today I learned the definition of a cusp form and a cuspidal representation. Specifically, given some global field k and a $G(k)$-invariant function on $G(\mathbb{A})$, one can define the *cusp forms* with a given continuous, unitary central character $\omega : Z(\mathbb{A}) \to \mathbb{C}^\times$ as the functions on $G(\mathbb{A})$ with central character ω for which the associated function lies in $L^2(Z(\mathbb{A})G(k)\backslash G(\mathbb{A}))$ and has for every g and every proper parabolic P the function 'integrate over $[P,P](k)\backslash[P,P](\mathbb{A})$ against g' vanishes. The space of cusps forms breaks up as a Hilbert space as a direct sum of irreducible representations known as *cuspidal representations*.

(3/2/2022) Today I learned the t-structure on IndCoh(X) is right-complete for any scheme X. This is because we have a t-exact equivalence $\Psi_X : \text{IndCoh}(X)^{\geq 0} \xrightarrow{\sim} \text{QCoh}(X)^{\geq 0}$. Because Ψ_X is exact, we see that $\Psi_X(\mathcal{K}) \simeq \text{fib}(\Psi_X(\phi^{\mathcal{F}}))$. Because Ψ is continuous and t-exact, we obtain a canonical identification $\Psi_X(\phi^{\mathcal{F}}) \simeq \phi^{\Psi_X(\mathcal{F})}$. However, $\phi^{\Psi_X(\mathcal{F})}$ is an equivalence since the t-structure on QCoh(X) is right-complete. Thus $\Psi_X(\mathcal{K}) \simeq 0$, and since Ψ_X is in particular conservative on IndCoh$(X)^{\geq 0}$, we see that $\mathcal{K} \simeq 0$ in this case.

(3/3/2022) Today I learned a theorem which says that, viewing 2QCoh(X) as a fully dualizable 2-category, one can attach a TFT whose value at a point returns this 2-category, and that the value this TFT assigns to a surface is the global functions of the mapping space of the surface to X.

(3/4/2022) Today I learned a more general statement of the theorem of Ben-Zvi—Gunningham—Orem. Specifically, I learned about the notion of a *proper dualizable* bimodule, and that it in particular gives you a fully faithful embedding of modules over the associated "Hecke"-like category to the category of G-modules. It is precisely when tensoring with this bimodule is conservative is when you get this equivalence.

(3/5/2022) Today I learned an equivalence of categories $\mathcal{D}(B(N_G(T))) \simeq \mathcal{D}(B(T \rtimes W)$, which follows from identifying the rings for which the modules are categories over, i.e. $H^*(BN_G(T)) \simeq H^*(BT)\#k[W]$. Here, the # refers to the smash product, which uses the Hopf algebra structure on $k[W]$.

(3/6/2022) Today I learned a bit more about the notion and motivation for the definition behind the p-canonical basis of some realization of some reflection group. Specifically, over the *real* numbers (or, likely, any field of characteristic zero) one can show that one has an alternate description for the category of Soergel bimodules obtained as follows. Specifically, it is a result of Elias-Williamson that the category of *Bott-Samuelson bimodules*, i.e. objects of the form $C \otimes_{C^{s^1}} ... \otimes_{C^{s^m}} C$ is equivalent to the *diagrammatic Hecke category* via a canonical map, and in particular one can define Soergel bimdoules as the Karobi completion of either category. However, it turns out that the correct category to work with (i.e. the category for which the *Soergel categorification theorem*, the theorem which says that each Bott-Samuelson bimodule has a unique new indecomposable summand and that the Grothendieck group of the category of Soergel bimodules can be identified with the Hecke algebra) in the characteristic p case, when the diagrammatic Hecke category functor to the category of Soergel bimodules is not an equivalence, is the diagrammatic Hecke category. One therefore can define the *p-canonical basis* of an element as the associated class in the Hecke algebra = Grothendieck group of category of 'Soergel bimodules'.

(3/7/2022) Today I learned that (almost certainly) one can identify the horocycle functor *hc* applied to the $G \times G$-category $\mathcal{C} := \mathcal{D}(N^-, \psi\backslash G) \otimes \mathcal{D}(G/_\psi N^-, \psi)$ with the usual pullback functor by $\mathfrak{t}^*/X^\bullet(T) \to \mathfrak{t}^*//\tilde{W}^{\text{aff}}$. Both functors are definitely monoidal right adjoints with unexpected left adjoints and the source and targets agree.

(3/8/2022) Today I learned that one can define the notion of a cosimplicial object $S^\bullet$ giving an *effective limit*, meaning that the canonical maps from $S^\bullet$ to the partial totalizations form an effective limit, i.e. the cokernels are weakly nilpotent. In particular, a nice fact about these is that eaxct functors preserve effective limits, which follows from realizing the totalization $S^\bullet$ as a limit of its tower of partial totalizations (which can be done since the limit is effective) and the Dold-Kan correspondence.

(3/10/2022) Today I learned an example of a vertex operator algebra is given by taking the Verma like construction on the universal enveloping algebra of the affine Lie algebra associated to some semisimple Lie algebra. Furthermore, I learned that for a representation of the affine Lie algebra (possibly which is integrable?) one can also define the global Lie algebra as those objects of the form $\mathfrak{g} \otimes \mathcal{O}(C\backslash *)$ for C a projective curve. These form a Lie subalgebra because the residue vanishes, and the quotient of the simple module for the affine Lie algebra by this global Lie action is finite dimensional and gives a vector bundle of finite rank on the moduli of genus g curves with n marked points.

(3/11/2022) Today I learned some useful facts about the Springer sheaf. Specifically, I learned that one way you can define the Springer sheaf which makes the W-equivariance manifest is by first taking the W-torsor on the locus of regular elements on G and using the regular W-representation to define a vector bundle with connection on the regular locus. One can $!*$-extend this vector bundle to the whole space, and it retains a W-representation by the functoriality of $!*$. This gives a kind of universal Springer sheaf (also attainable by parabolic induction of the constant sheaf on T) and then the restriction to the unipotent locus gives the classical Springer sheaf, which is also attainable by parabolically inducing the delta sheaf at the origin of T. Furthermore, there is an alternate construction of the Springer sheaf by defining it as the pushforward $(N \to \mathcal{U}/G)_*(\omega_N)$ up to cohomological shift, where N is the unipotent radical of some Borel.

(3/12/2022) Today I learned the proof of *BGG reciprocity*, which says that $(P_\lambda : \Delta_\mu)$ i.e. the multiplicity of Δ_μ in a filtration by Vermas of P_λ, is equal to the index $[\Delta_\mu, L_\lambda]$. The strategy is to show that both sides are also equal to the dimension of the space of maps in the abelian category $\mathcal{O}$ $\text{Hom}_{\mathcal{O}}(P_\lambda, \nabla_\mu)$. The fact that the space of maps $(P_\lambda : \Delta_\mu)$ is equal to the dimension of this hom space uses the more general fact that for any P with a standard filtration, this dimensional equality holds. This is done by induction on the length of the standard filtration (where the base case, both sides are $\delta_{\lambda\mu}$), and in particular uses the fact that there are no Ext's between Vermas. The fact that $[\Delta_\mu, L_\lambda]$ is given by $\dim(\text{Hom}_{\mathcal{O}}(P_\lambda, \nabla_\mu))$ is a consequence of two equalities: one is $[\nabla_\mu : L_\lambda] = \dim(\text{Hom}_{\mathcal{O}}(P_\lambda, \nabla_\mu))$ (which holds more generally if ∇_μ is replaced with any object M of $\mathcal{O}$ and is obtained by induction on the length of M, where the simples in the $\lambda = \mu$ case follows by the fact that each P_λ is a projective *cover* of L_μ and for the $\lambda \neq \mu$ case follows by the fact that no two projective covers are the same) and the fact that $[\nabla_\mu : L_\lambda] = [\Delta_\mu : L_\lambda]$ via duality.

(3/13/2022) Today I learned the fine print of the notion of a $\mathbb{Z}/2\mathbb{Z}$-action on a (1,1)-category. Specifically, such an action can be identified with an endofunctor of some given (1,1)-category $F : \mathcal{C} \to \mathcal{C}$ and some natural isomorphism $\eta : F^2 \xrightarrow{\sim} \text{id}$ such that $F\eta \simeq \eta F : F^3 \to F$. Given any natural transformation $\zeta : F_1 \to F^2$, one can check (by naturality) that the two possible definitions of the map $F_1^2 \to F_2^2$ give the same functor (where again by (1,1)-category-ness this is a property and not additional structure). This allows us to define the notion of two $\mathbb{Z}/2\mathbb{Z}$-categories being equivalent as a natural transformation $\zeta : F_1 \to F_2$ such that $(F_1^2 \to F_2^2) \circ \eta_1 = \eta_2$ as maps $\text{id}_{\mathcal{C}} \to F_2^2$. Using this and the fact that the natural isomorphisms from the abelian category of vector spaces to itself are given by scaling by a nonzero scalar, one can show that the $\mathbb{Z}/2\mathbb{Z}$-actions on $\text{Vect}^\heartsuit$ up to equivalence is given by $k^\times/k^{\times,2}$.

(3/14/2022) Today I learned that you can check whether a G-linear functor $F : \mathcal{C} \to \mathcal{D}$ commutes with limits (i.e. is a right adjoint) by checking that the associated functor on N-invariants commutes with limits. This follows since the functor hc is a conservative right adjoint and the fact the following diagram commutes:

$$\begin{array}{ccc} \mathcal{D}(G) \otimes_G \mathcal{C} & \xrightarrow{F} & \mathcal{D}(G) \otimes_G \mathcal{D} \\ \downarrow{\scriptstyle \text{hc}} & & \downarrow{\scriptstyle \text{hc}} \\ \mathcal{D}(G/N) \otimes_T \mathcal{C}^N & \xrightarrow{\text{id}\otimes F^N} & \mathcal{D}(G/N) \otimes_T \mathcal{D}^N \end{array}$$

(3/15/2022) Today I learned that yesterday's argument above actually gives the fact that the N-invariants functor is conservative! This is again because the horocycle functor is conservative. I also learned that the shift of the Mellin transform is monoidal one (for IndCoh!) since the Mellin transform needs to match the monoidal units.

(3/16/2022) Today I learned that one can use the Adams spectral sequence to argue that the rank of the Morava K-theories converge to the rank of the $\mathbb{F}_p$-cohomology of a space. Specifically, one can argue that there is a spectral sequence whose E_2 page is given by the Ext group in the category of modules (maybe comodules?) for the polynomial algebra adjoint a Steenrod operation. Since this operation has degree which depends on n (specifically the degree is $2p^n - 2$), if the CW complex is finite the differentials vanish for n large enough.

(3/18/2022) Today I learned that the nilpotent orbits of the Lie algebra $\mathfrak{sp}_{2n}$ are classified by partitions of n where the odd integers occur an even number of times. I also learned the notion of a *distinguished* nilpotent orbit, which is those nilpotent orbits such that any element in the centralizer is nilpotent itself. A nilpotent orbit for $\mathfrak{sp}_{2n}$ is distinguished if and only if the integers in the partition are all distinct.

(3/19/2022) Today I learned that the parity sheaves on the flag variety associated to a Kac-Moody group categorifies the diagrammatic Hecke category. In particular, one can define the p-canonical basis via the usual multiplicity of stalk formalism using the multiplicity of standard sheaves in parity sheaves. I also learned that the theory of Soergel bimodules only holds for *reflection faithful* representations and so that in particular there are infinite reflection groups without reflection faithful representations, so that the bimodule theory isn't expected to categorify the Hecke algebra.

(3/20/2022) Today I learned that for any (compact?) Lie group G, $\pi_2(G)$ vanishes. The idea is to construct a Morse function on the space $\Omega(G)$ and argue that the index of each critical point of this Morse function has even index. Since this implies that the cells in forming the cell structure of based loops are all also even, this implies that $\Omega(G)$ can be built with a CW structure without one cells, and therefore has no fundamental group!

(3/21/2022) Today I learned that one can tensor a DG category/stable category/∞-category $\mathcal{C}$ with any topological space by using the space to obtain a CW complex and taking the constant diagram and taking the colimit over it. I also learned how to define the notion of a *Calabi-Yau* monoidal category, which is a monoidal category $\mathcal{A}$ for which there is a canonical map $\mathcal{A}\otimes_{\mathcal{A}\otimes\mathcal{A}^{\text{op}}}\mathcal{A} \to \text{Vect}$ for which the pullback by $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes_{\mathcal{A}\otimes\mathcal{A}^{\text{op}}} \mathcal{A}$ is the counit of a duality.

(3/22/2022) Today I learned one way to argue that the natural map $U\mathfrak{g} \otimes U\mathfrak{t} \to \Gamma(\mathcal{D}_{G/N})^T$ factors through the invariants of the center. Specifically, one can use the fact that the center of the universal enveloping algebra is a polynomial algebra in Casimir elements and if c is such an element, the fact that $c \in Z\mathfrak{g} \simeq \Gamma(\mathcal{D}_G)^{G\times G}$ then one can commute c with any differential operator, so $c \mapsto h_\alpha^2 + h_\alpha$, just as the Harish-Chandra map does.

(3/23/2022) Today I learned that the intersection form of an irreducible d-dimensional compact smanifold in a $2d$-dimensional manifold is given by the Euler class of the normal bundle of the inclusion of the d-dimensional manifold into the full manifold. One reason for this is that the Euler class of a vector bundle is given by taking a section which is transverse to the zero section, taking its class in homology, and taking its Poincaré dual. Since a manifold included is always transverse to its normal bundle N, we see that for the inclusion of the fiber F (which by assumption is a manifold!) we obtain $[F]\cap[F] = [F]^*([F]) = e(N)([F]) = p_!(e(F))$, where the last fact follows since if you have any vector bundle V on some M, the pairing $\langle e(V), [M]\rangle$ is the Euler characteristic of M.

(3/24/2022) Today I learned a semi-conceptual explanation behind shearing and filtered lifts of categories. Specifically, a *filtered lift* of a DG category $\mathcal{C}$ is an object of

$$\text{QCoh}(\mathbb{A}^1/\mathbb{G}_m)\text{-modcat} \times_{\text{QCoh}(\mathbb{G}_m/\mathbb{G}_m)\text{-modcat}} \{\mathcal{C}\}$$

where the $\mathbb{G}_m$-action on $\mathbb{A}^1 = \text{Spec}(k[t])$ places t in weight 2. Given a filtered lift of a category, one can remember only the $\text{QCoh}(\mathbb{A}^1/\mathbb{G}_m)$-modcat part. Then one can shear and use the fact that shearing gives a symmetric monoidal equivalence $\text{QCoh}(\mathbb{A}^1/\mathbb{G}_m) \simeq \text{QCoh}(\mathbb{A}^1_\hbar/\mathbb{G}_m)$ where $\mathbb{A}^1_\hbar = \text{Spec}(k[\hbar])$ where $\hbar$ is in weight 2 *and* cohomological degree 2. Now, given any $\text{QCoh}(\mathbb{A}^1_\hbar/\mathbb{G}_m)$-module category, we can forget the grading via pulling back by the map $\mathbb{A}^1_\hbar \to \mathbb{A}^1_\hbar/\mathbb{G}_m$, which (likely) gives a *different* category than forgetting via the usual map $\mathbb{A}^1 \to \mathbb{A}^1/\mathbb{G}_m$ obtained before shearing. This gives the associated 'sheared degrading' functor.

(3/25/2022) Today I learned a way to relate shifted symplectic geometry to the usual notion of Hamiltonian reduction and to quasi-Hamiltonian reduction. Specifically, given a space M with a Hamiltonian G-action, one obtains the moment map $M \xrightarrow{\mu} \mathfrak{g}^*$, which is G-equivariant. Therefore, we obtain a map $M/G \to \mathfrak{g}^*/G$, and we may identify $T^*[1](BG) \simeq \mathfrak{g}^*/G$. Since BG itself (embedded as the zero section) is also a Lagrangian in this shifted symplectic space, we can intersect it with the Lagrangian given by M/G and obtain a space which turns out to be the usual Hamiltonian reduction! Similarly, G/G itself is 1-shifted symplectic and contains BG as a Lagrangian.

(3/26/2022) Today I learned the proof of the claim that, assuming the character map on Soergel *bi*modules maps each indecomposable Soergel bimodules B_y to the Kazhdan-Lusztig basis element b_y, then the Kazhdan-Lusztig conjecture holds in category $\mathcal{O}$. To see this, note that using BGG reciprocity it suffices to show P_y projects to the Grothendieck group of $\mathcal{O}_0$ to $b_y|_{v=1}$, since the coefficient expansion of b_y in the Hecke algebra is given by the Kazhdan-Lusztig polynomials. We show this by induction, noting that $[P_1] = [\Delta_1] = 1$ in the Grothendieck group, which agrees with $b_1|_{v=1} = 1|_{v=1} = 1$. Now, inductively fix a y for which $sy < y$ for some simple reflection s. We have that $\theta_s(P_{sy})$ is projective and acts as $[\Delta_1]+[\Delta_s]$ at the level of the Grothendieck group. In particular, $\theta_s(P_{sy}) \cong P_{sy} \oplus \bigoplus_{y'<y} P_y^{m_{y'}}$. We see that $C \otimes_{C^s} \mathbb{V}(P_{sy}) \simeq \mathbb{V}(P_y) \oplus \bigoplus_{y'<y} \mathbb{V}(P_y))^{m_{y'}}$, and since the indecomposable Soergel module $\overline{B}_x \cong B_x \otimes_C k$ we see that $C \otimes_{C^s} \overline{B_{sy}} \simeq \overline{B}_y \oplus \bigoplus_{y'<y} \overline{B}_{y'}^{m_{y'}}$ and so since the indecomposable Soergel modules are tensors of the indecomposable Soergel bimodules we see $C \otimes_{C^s} B_{sy} \simeq B_y \oplus \bigoplus_{y'<y} B_{y'}^{m_{y'}(v)}$, where the v indicates possible grading shifts. Applying the character map and using our inductive hypothesis (and using the monoidality of the character map) we see that $b_s b_{sy} = b_y + \sum_{y'} m_{y'} b_{y'}$ and that in particular the equality holds when $v = 1$ in the Hecke algebra. Thus since the multiplicity of $\theta_s(P_{sy})$ is given by $b_s b_{sy}|_{v=1}$ (since multiplication by $b_s|_{v=1} = 1+s$ is how the wall crossing functor acts on the Grothendieck group) and each multiplicity m_y is given by $m_{y'} b_{y'}|_{v=1}$, we see that the equality $b_y|_{v=1} = [P_y]$ holds as well, as desired.

(3/28/2022) Today I learned a result Ginzburg-Riche, which states that for any weight λ and any perverse sheaf on the affine Grassmannian, one has an isomorphism $H^*_{T\rtimes\mathbb{G}_m}(i^!_\lambda(\mathcal{F})) \simeq (S(\mathcal{F}) \otimes^{(\lambda)} \Gamma(\mathcal{D}^\hbar_{G/N}))^{G\times T_\lambda}$ where the superscript $^{(\lambda)}$ denotes a twisting by λ of the action of $\mathrm{Sym}\mathfrak{t}[2]$ on the ring of differential operators by a perscribed algebra isomorphism, and S is the functor given by geometric Satake for some perverse sheaf on the affine Grassmannian.

(3/29/2022) Today I learned that for any scheme with a T-action X, one can localize the Euler class of some vector bundle to a T-fixed point of the action, yielding a class in $H_T(*) = \mathrm{Sym}(\mathfrak{t})[-2]$, and that the localization of the Euler class of this equivariant vector bundle is given by the product of T-weights on the induced representation on the stalk at that point.

(3/29/2022) Today I learned that one can define the *attracting locus* of a variety X with a $\mathbb{G}_m$ action as the mapping space $\mathrm{Maps}^{\mathbb{G}_m}(\mathbb{A}_1, X)$, and for any algebraic space X (i.e. an object which étale locally looks like an affine scheme) and that this is representable as a scheme. Moreover, I learned that if the variety X is smooth, then the restricton of this attracting locus to any one connected component is a locally closed embedding, and that this restriction is necessary as the example of $\mathbb{P}^1$ with its standard $\mathbb{G}_m$-action shows. Moreover, I learned one can identify the Schubert variety at some point as the attracting locus for some $\mathbb{G}_m$-action given by restricting the torus action by a regular dominant coweight.

(3/30/2022) Today I learned the notion of a *cell*. Specifically, the Hecke algebra comes labelled with a canonical basis (the Kazhdan-Lusztig basis), and one can place an ordering on the Weyl group via the ideals generated by each of these elements (with containment). With respect to this ordering, the element 1 is largest. Furthermore, I learned that there is a bijection between cells and *special nilpotent orbits* of the associated group, and I learned that these special nilpotent orbits are in canoncical bijection with the special nilpotent orbits of the Langlands dual group.

February 2022

(2/1/2022) Today I learned that the Steenrod operations for $\mathbb{F}_p$-cohomology correpsond to the Tate-valued Frobenius in the following sense. Specifically, given some X, we can take the Tate valued Frobenius, which lands in $H^*(X^{t\mathbb{Z}/p\mathbb{Z}}; \mathbb{F}_p)[\hbar, \epsilon]/\epsilon^2$ where $\hbar$ lies in degree 2. One can then take the map $y \mapsto y - y^p\hbar^{-(p-1)}$, which exhibits the fact that the first Steenrod operation on y is given by $y \mapsto y^p$.

(2/2/2022) Today I learned an equivalence, given a compactly generated DG category $\mathcal{C}$, of the dual of $\mathcal{C}$ (defined as $\mathrm{Ind}(C^{c,\mathrm{op}})$ with the category of exact functors $\mathrm{Fun}_{\mathrm{ex}}(\mathcal{C}^c, \mathrm{Vect}) \simeq \mathrm{Fun}_{\text{exact, cont}}(\mathcal{C}, \mathrm{Vect})$. Informally, this can be summarized by saying that taking the dual as maps to spectra is equivalence (for a DG category) to taking maps to Vect.

(2/3/2022) Today I learned a few useful facts about Lie algebroids, which is the natural generalization of Lie algebras which are not necessarily over a point. Specifically, given a Lie algebroid over some base scheme X, a representation of the tangent Lie algebroid $TX \to X$ gives rise to a vector bundle with connection on X. In particular, if X admits the structure of the Lie group, then the Lie algebroid is parallelizable (i.e. $TG \simeq G \times \mathfrak{g}$) and so any line bundle on G can be equipped with a connection.

(2/4/2022) Today I learned a fact that there exists some abelian group object of prestacks (i.e. a functor from derived rings to commutative spaces) for which the Cartier is non-classical even though the usual prestack is classical. The particular example of this is $B\mathbb{G}_{m,dR}$. Specifically, using the fact that we have an exact sequence of groups $1 \to B\hat{\mathbb{G}}_m \to B\mathbb{G}_m \to B\mathbb{G}_{m,dR} \to 1$, we can take the Cartier dual and see that the Cartier dual of $B\mathbb{G}_{m,dR}$ is precisely the kernel of $\mathbb{Z} \to \mathbb{G}_a$, which is equivalently given by $\{0\} \times_{\mathbb{G}_a} \mathbb{Z} \simeq \Omega\mathbb{A}^1$. I learned of this from Justin Campbell (although any errors in recording this are my own!).

(2/5/2022) Today I learned that, given two symmetric monoidal categories $\mathcal{C}$ and $\mathcal{D}$, one can define a symmetric monoidal structure on the category of functors between them known as *Day convolution*. This convolution is given informally by the formula

$$(F_1 \star F_2)(C) \simeq \text{colim}_{C_1 \otimes C_2 \to C} F_1(C_1) \otimes F_2(C_2)$$

and, furthermore, a commutative algebra object in this functor category is the same thing as a symmetric monoidal functor $\mathcal{C} \to \mathcal{D}$.

(2/6/2022) Today I learned a very fun lemma known as Zariski's lemma, which says that any finitely generated k-algebra K which is a field is actually a finite extension of k! One can use this to show that any nonempty quasiprojective k-scheme X over an algebraically closed field k contains a k-point, and if X furthermore has positive dimension it has infinitely many k-points.

(2/7/2022) Today I learned that any open subscheme of a $\mathcal{D}$-affine variety (in the sense that the derived global sections functor is conservative) is $\mathcal{D}$-affine. This follows from the fact that if $j : U \to X$ denotes the open embedding, one can use functorial properties of $j_{*,dR}$ to show that $j_{*,dR}(\mathcal{D}_U) \simeq \mathcal{D}_X$ and then use the fully faithfulness of $j_{*,dR}$ to conclude that $j^!(\mathcal{D}_X) \simeq \mathcal{D}_U$. In particular, we obtain a functor with a conservative right adjoint and therefore preserves compact generators, and so $\mathcal{D}_U$ generates $\mathcal{D}(U)$.

(2/8/2022) Today I learned that a certain fact about vector bundles in algebraic varieties is actually remarkable. Namely, defining a vector bundle over a scheme as a map from the total space to a scheme, it is actually remarkable that the transition functions actually are elements of GL_n, i.e. that the transition functions themselves are linear! Using this, one can show that any vector bundle V admits a $\mathbb{G}_m$-action by scaling.

(2/9/2022) Today I learned a theorem of Orlov which states that you can recover a smooth proper variety with ample canonical bundle or anti-ample canonical bundle from its (derived) category of coherent sheaves. I also learned an extension of this theorem due to Arinkin, who used this to show you can recover an abelian variety from its derived category of D-modules.

(2/10/2022) Today I learned some interesting computations on the normal cone. For example, I learned that the normal cone for the closed embedding of the singular point into $\text{Spec}(A)$ with $A := k[x, y]/(xy)$ is given by two lines, since its associated ring of functions is

$$A \oplus (x, y)/(x^2, y^2) \oplus (x^2, y^2)/(x^3, y^3) \oplus ... \simeq A[x] \oplus A[y]$$

which is the union of two lines. In particular, the exceptional fiber of the blow up of $\text{Spec}(A)$ at this point corresponds to the projectivization of the normal cone, which has exactly two directions i.e. two points.

(2/11/2022) Today I learned the existence of some category of mixed constructable sheaves for a scheme over $\mathbb{F}_q$ which fits in between the category of constructable mixed ℓ-adic sheaves on the scheme itself and the constructable sheaves on the base change to the algebraic closure, a result of Ho and Li. In fact, this holds for any Artin stack of finite type over the algebraic closure $\overline{\mathbb{F}_q}$ and recovers the category of Soergel bimodules.

(2/12/2022) Today I learned that the fact that the symmetric monoidal category of correspondences has all objects self dual, along with the symmetric monoidality of the functor IndCoh, gives a manifestation of Serre duality. Specifically, this symmetric monoidality along with the fact all objects are dualizable gives that the category $\text{IndCoh}(X)$ for any scheme X is self dual and the dual to pullback is pushforward, and one can show that this duality functor identifies on coherent sheaves on a classical scheme X as $\underline{\text{Hom}}(-, \omega_X)$.

(2/14/2022) Today I learned a potential strategy to show that all $\mathcal{D}$-affine sheaves are flag varieties. Specifically, it suffices to show that the group of automorphisms acts transitively on the variety, and to do this, I'm pretty sure it suffices to show that the tangent space of automorphisms at the identity (which identifies as a vector space with global vector fields!) has a large dimension by an orbit-stabilizer argument.

(2/15/2022) Today I learned a weak version of the Arnold conjecture, which is now a theorem. Specifically, the Arnold conjecture states that the number of fixed points of a nondegenerate Hamiltonian automorphism is $\geq$ than the sum of the Betti numbers. I also learned two interpretations/examples of this. For example, the rotation of the sphere is Hamiltonian, and this has two fixed points ($\geq$ the sum of the Betti numbers). Alternatively, one can view a conceptual version of this by replacing these automorphisms with intersections of Lagrangians in $T^*(S^1)$. A property of being a Hamiltonian is that the area is not changed, and so one can see that if one draws a S^1 which does not change the area in the tangent bundle, there are at least two intersection points. Riccardo Pedrotti showed me all of these examples, and he is great.

(2/16/2022) Today I learned that, given a irreducible representation of some locally compact locally nonarchimedian group, this representation is admissible, and moreover, given any admissible representation π of our group into some vector space V and some given locally constant compactly supported function on G, one can define an operator on V which takes a function f on V to a new function Φ on V given by $v \mapsto \int_g f(g)\pi(g)(v)$. I learned that for these integrals and representations, the notion of the trace of this map makes sense, therefore defining a distribution sending f to the trace of $\Phi(f)$. I learned that it's a theorem of Harish-Chandra that this distribution is actually representable by (i.e. is integration against) a locally constant function on the regular singular locus.

(2/17/2022) Today I learned the formal definition on what it means for a local Noetherian ring to be Gorenstien. Specifically, a Noetherian local ring R is *Gorenstein of dimension zero* if and only if the abelian group of R-module maps from $R/\mathfrak{m}$ to R is one dimensional, and a Noetherian local ring in general is said to be *Gorenstein* if there is a regular sequence for which the quotient is Gorenstein of dimension zero. Furthermore, a dimension zero finite k-dimensional graded ring is Gorenstein if and only if it satisfies Poincaré duality in the sense that the top graded piece is one dimensional and the pairing map to the top graded piece is a perfect pairing.

(2/18/2022) Today I learned that the functions on BG are trivial, even derivedly, for G a reductive group. This is because one can identify these functions with the algebra given by pull-push by the terminal map. The pull-push of the one dimensional vector space can be identified in turn with the fixed points of the trivial representation, which is trivial and has no derived parts precisely because G is reductive over a field of characteristic zero, so invariants is exact.

(2/19/2022) Today I learned a neat proof of the following fact. Let C denote the coinvariant algebra of the Weyl group acting on $\mathfrak{t}^*$. Then if $M \in C\text{-mod}^W$ has the property that $k \otimes_C M$ is the trivial representation, then $M^W \xrightarrow{\sim} (k \otimes_C M)^W$. To see this, it suffices to show that $(C^+ \otimes_C M)^W$ vanishes. In turn, to see this, note that C^+ is graded by degree and that W preserves the degree of the polynomials. In particular, C^+ is filtered by objects whose C-module structure factors through the augmentation map $C \to C/C^+ = k$. Therefore, for each of these subquotients S, we see that

$$(S \otimes_C M)^W \simeq (S \otimes_k k \otimes_C M)^W$$

is the tensor product of some W-rep over k with an entirely trivial representation. Since S itself is entirely nontrivial, this entire representation vanishes.

(2/20/2022) Today I learned that for any finite, non-modular (i.e. the associated reflection representation is over a field whose characteristic does not divide the order of the group) pseudo-reflection group H, the coinvariant algebra is the regular representation for H. I also learned about the concept of *pseudo-reflections* themselves, which is a linear map of some finite dimensional vector space over some field of arbitrary characteristic which leaves some hyperplane fixed (but is not required to be diagonalizable), whereas a (generalized) reflection is a diagonalizable such map. Finally, I learned that in the non-modular case, a finite order pseudo-reflection is a generalized reflection.

(2/21/2022) Today I learned the full proof of the fact that a sheaf $\mathcal{F} \in \text{IndCoh}(\mathfrak{t}^*/\tilde{W}^{\text{aff}})$ satisfies Coxeter descent if and only if for every $\overline{x} \in \mathfrak{t}^*//\tilde{W}^{\text{aff}}(K)$ and every field K that the canonical W_x-representation on the pullback $\overline{x}^!(\mathcal{F})$ is trivial.

(2/22/2022) Today I learned a theorem of Chen, which says that for any $*$-central holonomic object $\mathcal{F}$ of $\mathcal{D}(T)^W$ (or, equivalently, a holonomic object of $\mathcal{D}(T)^W$ satisfying Coxeter descent) that the enhanced parabolic induction functor $\text{ind}(\mathcal{F})^W$ has the property that its N-averaging is entirely supported on the $N\backslash B/N$-orbit. I also learned that this implies that the trace of a certain perverse equivariant sheaf gives an interesting functions on representations of $G(\mathbb{F}_q)$.

(2/23/2022) Today I learned the notion of a *conical symplectic variety*, which is a subvariety of affine n-space for some n closed under the standard $\mathbb{G}_m$-action which is smooth in codimension one and whose regular locus has some symplectic form satisfying a scale invariance property and an extension property for every resolution of singularities. Given a resolution of one of these things whose associated 2-form extends to a symplectic form, the resolution of the conical symplectic variety is called a *symplectic resolution* of a conical symplectic variety. I learned a result of Brion and Fu, which says that all such symplectic resolutions of normal conical symplectic varieties are given by the cotangent bundle of some partial flag variety to the orbit closure of some nilpotent orbit in the associated group.

(2/24/2022) Today I learned about the Stolz-Teichner program, which conjectures a map from supersymmetric quantum field theories indexed by a base manifold X into topological modular forms also indexed over X which induces an equivalence of these SQFT's modulo deformations. (2/25/2022) Today I learned that for any stable ∞-categories that the shift functor (which may be defined as a limit) also commutes with all colimits! This just follows from Yoneda's lemma and the fact that the shift functor gives an equivalence of categories, but can be used to show the essential image of the inclusion of the degenerate category $\mathcal{D}(G/N)_{\text{deg}}$ is closed under the trucnation functors.

(2/25/2022) Today I learned the definition of the cotensor product of some left $\mathcal{A}$-module M with some right $\mathcal{A}$-module category N. Specifically, one can view $M \otimes N$ as an $\mathcal{A} \otimes \mathcal{A}^{\text{op}}$-bimodule and define the cotensor product as the internal mapping Hom from $\mathcal{A}$ to $M \otimes N$ as a bimodule. One can also show that if $\mathcal{A}$ is rigid monoidal, then $\mathcal{A}$ is its own dual which in turn entails that the cotensor product and the tensor product are isomorphic up to reordering the term and possibly modifying the multiplication on N in a twisted way (when $\mathcal{A}$ is not symmetric).

(2/26/2022) Today I learned the definition of an automorphic representation of the group $G(\mathbb{A}_K)$ where K is the function field of some smooth projective geometrically connected curve over a finite field. Specifically, it is any irreducible admissible (admissible $V \Leftrightarrow$ *smooth* i.e. $V = \cup_{K \subseteq G(\mathbb{A}_K)} V^K$ and has the property that V^K is finite dimensional, both for K the compact opens) isomorphic to a subquotient of the space of *automorphic functions*, i.e. a subquotient of the space of automorphic forms–functions on $G(K)\backslash G(\mathbb{A}_K)$ whose G-span itself is an admissible representation.

(2/27/2022) Today I learned that the union of graphs for the action of W on the torus of SL_3 over a field of characteristic three is not flat over the base torus by explicitly computing that the fiber at the singular point has dimension ten over the base field.

(2/28/2022) Today I learned that there is an isomorphism of stacks $\tilde{N}/G \simeq \mathfrak{n}/B$. This follows from the isomorphism of stacks $\tilde{\mathfrak{g}}/G \simeq \mathfrak{b}/B$, which is an isomorphism over $\mathfrak{g}/G$. In particular, taking the fiber at the zero point, we obtain our desired isomorphism.

January 2022

(1/1/2022) Today I learned an interesting way to motivate the definition of the stack X_{crys}, analogous to the construction of the de Rham prestack. Namely, the associated space $X_{crys}(A)$ for a ring A should be defined in terms of $X(?(A))$ where $?(A)$ is some currently unknown parameter. In particular, this implies that the construction of $X(?(A))$ should commute with both limits and colimits. In particular, it suffices to compute this when X is an affine (derived) scheme, and, moreover, since the category of simplicial commutative rings is the non-abelian derived category of the ordinary category of polynomial rings, it thus suffices to see what $?(A)$ is for the ring $X = \mathbb{A}^1 \cong \mathbb{G}_a$. Now, given any ring A, we can realize the de Rham prestack as the quotient of $\mathbb{G}_a$ by the formal completion at the identity, and this is an equivalence of functors from rings to rings. Therefore, using this, one can similarly define $\mathbb{G}_{a,crys}$ as the quotient of $\mathbb{G}_a$ by the *pd-completion* $\mathbb{G}_a^{\#}$, and this gives the definition in general.

(1/2/2022) Today I learned a kind of interesting point about the equivalence $\mathcal{D}(N\backslash G/N)^{T\times T,w} \simeq \mathrm{IndCoh}(\Gamma_{\tilde{W}^{\mathrm{aff}}})$. Namely, the fine print of the right hand category implies that, since $\Gamma_{\tilde{W}^{\mathrm{aff}}}$ has $\pi_1(G^\vee)$ many connected components, that this should reflect on the left hand side of the equivalence somehow. The way this appears for, say, semisimple groups, is as follows. Note that if Z is the center of the group, then the center of the group corresponds under Langlands duality to the fundamental group of the Langlands dual group, and the center is a subgroup of the torus. Therefore, the canonical trivial action of the diagonal copy of Z induces an equivalence

$$\mathcal{D}(N\backslash G/N)^{T\times T,w} \simeq (\mathcal{D}(N\backslash G/N)^{Z,w})^{T\overset{Z}{\times} T,w}$$

and the category $\mathcal{D}(N\backslash G/N)^Z$ is canonically equivalent to $\mathcal{D}(N\backslash G/N) \otimes \mathrm{Rep}(Z)$!

(1/3/2022) Today I learned a way to twist sheaves of E-spectra for any E via using $E^\times$-gerbes, and computed the classification for it for $E := KU$ over the complex numbers. Specifically, one can identify $(KU \otimes \mathbb{C})^\times$ with the group $\{p(\beta) \in \mathbb{C}[\beta] : p(0) \neq 0\}$ with β in cohomological degree 2. I also learned a more general fact for connective rings R–namely, one can read off the nonzero homotopy groups of $\mathrm{GL}_1(R)$ from the homotopy groups of R itself! This largely follows from the definition of $\mathrm{GL}_1(R)$.

(1/4/2022) Today I learned a fun fact of Bill Goldman which says that if π is the fundamental group of some surface and G is some Lie group, then the Zariski tangent space of the character variety $\mathrm{Hom}_{Gp}(\pi, G)/G$ gives rise to the first group cohomology $H^1(G, \mathfrak{g})$ where the action is given by postcomposing the given representation by the adjoint action.

(1/5/2022) Today I learned an important subtlety in the definition of a reductive group over a nonalgebraically closed field. Specifically, an algebraic group over some field k is by definition is reductive if and only if the base change to the algebraic closure has the property that the unipotent radical is trivial. I also learned that for a *perfect* field, the reductiveness of an algebraic group over k as above is equivalent to every smooth connected unipotent normal subgroup being trivial, but in general this latter property is called being *pseudo-reductive*. (In Springer's book on Linear Algebraic Groups, pseudo-reductive groups are called F-reductive groups, and usual reductive groups over a field F are called reductive F-groups.)

(1/6/2022) Today I learned the notion of a *pin group*, two of the four double covers of the group O_n, and in particular an extension of O_n by $\mathbb{Z}/2\mathbb{Z}$. This group has the property that the pullback via the map $\mathrm{SO}_n \to O_n$ yields the spin group, so that given any surface with a Pin structure, we can take the oriented doulbe cover and use this pullback square to put a Spin structure on the manifold.

(1/7/2022) Today I learned some of the basics in the representation theory of real Lie groups. Specifically, I learned that given a complex algebraic group G, the collection of real forms are in canonical bijection with involutions given by the *based root datum*, i.e. the root datum with choice of simple roots. Furthermore, I learned in this case that there's a result of Nadler which gives rise to an associated dual group to this real form and an equivalence of tensor categories of equivariant perverse sheaves on the associated affine Grassmannian and the representations of this subgroup of $G^\vee$. This affine Grassmannian can be defined as the fixed points of the natural conjugation action given by the real form on Gr_G or can be identified with $G_{\mathbb{R}}(K_{\mathbb{R}})/G_{\mathbb{R}}(\mathcal{O}_{\mathbb{R}})$. (This is largely the beginning of Nadler's paper 'Perverse Sheaves on Real Loop Grassmannians.')

(1/8/2022) Today I learned a theorem of Beraldo on *anti-tempered* $\mathcal{D}$-modules on the category of bi-$G(\mathcal{O})$-equivariant sheaves on the affine Grassmannian. Specifically, the derived Satake theorem identifies these sheaves with $\text{IndCoh}_{\text{Nilp}}(\mathfrak{g}^\vee[1]/G)$, and the *tempered* category is that which corresponds to the 0-singular support, i.e. the quasicoherent subcategory. Then one can define the *antitempered* $\mathcal{D}$-modules to be those sheaves which are right orthogonal to the tempered ones, i.e. if you take homs from any tempered object to your object, it's zero. Then Beraldo showed that the dualizing sheaf as a bi-$G(\mathcal{O})$-equivariant sheaves on the affine Grassmannian is anti-tempered.

(1/10/2022) Today I learned the notion of the *Frobenius morphism*, a morphism on affine $\mathbb{F}_p$-schemes is given by the ring map $(-)^p$ and can be defined more generally via using the Frobenius twist on a k-algebra for k any perfect field of characteristic p. For $\mathbb{A}^1_{\mathbb{F}_p}$, for example, this Frobenius pushes forward the trivial line bundle to a trivial vector bundle of rank p, and one can further show that this same setup works the pushforward of the Frobenius of the line bundle $\mathcal{O}(p-1)$, which also yields the direct sum of p copies of the trivial line bundle.

(1/11/2022) Today I learned some useful facts about the cohomology of the classifying space of some compact connected Lie group G. Specifically, I learned that if G is simply connected, then $H^4(BG, \mathbb{Z})$ is a free rank one group. Furthermore, I learned that if A is the ring $\mathbb{Z}$ after inverting the primes dividing $|W|$, then one has a canonical isomorphism of $H^*(BG, A) \cong \text{Sym}_A(\mathfrak{t})^W$. The inverting of the primes comes from the fiber sequence $W \to B(N_G(T)) \to T$.

(1/12/2022) Today I learned that the ind-completion of a usual abelian category remains an abelian category, using the 'higher categorical' definition of the ind-completion where we identify the ind-completion as the full subcategory of *presheaves* on the category $\mathcal{C}$, i.e. the category $\text{Fun}(\mathcal{C}^{op}, \text{Spc})$, containing the representable objects and closed under filtered colimit. Under this definition, the claim follows from the fact that the representable objects all map to sets and the fact that the category of sets is closed under filtered colimits in spaces.

(1/13/2022) Today I learned that if R commutes with cofiltered limits, is left t-exact and has bounded cohomological amplitude on the right, then R preserves objects which are their own left-completions. The key computation is

$$\text{lim}_m \tau^{\geq m}(R(\mathcal{F})) \simeq \text{lim}_m \tau^{\geq m}(R(\tau^{\geq m-c}(\mathcal{F}))) \simeq \text{lim}_m R(\tau^{\geq m-c}(\mathcal{F}))$$

where the first equivalence uses the fact that R maps $\mathcal{C}^{<m-c}$ to $\mathcal{C}^{<m}$ and the second equivalence uses the fact that R is right t-exact and that $c \geq 0$.

(1/15/2021) Today I learned the full fine print of the definiton of locally almost of finite type prestack. Specifically, one can define the n-truncated prestacks as those prestacks which are determined by their values on the rings associated to n-truncated spaces, and furthermore defined the n-truncated locally finite type prestack as those determined by their value on the finite type rings associated to n truncated spaces. With this in mind, we can define a prestack as *locally almost* *of finite type* if each truncation (which is, essentially by definition, an n-truncated prestack) is a locally of finite type n-truncated prestack and furthermore the prestack $\mathcal{Y}$ is *convergent*, i.e. the natural map $\mathcal{Y}(S) \to \lim_n \mathcal{Y}(\tau^{\leq n} S)$ is an equivalence.

(1/16/2022) Today I learned a theorem of Grothendieck, which states that schemes as functors from rings to sets satisfy fpqc descent. In particular, in my setup this implies that, since $\mathfrak{t}^* \to \mathfrak{t}^*//W$ is fpqc, we can detect maps into $\mathfrak{t}^*//W$ by locally checking everything on a cover and via descent data!

(1/17/2022) Today I learned a theorem known as the *Borsuk-Ulam theorem*, which says that for any continuous map of topological space $S^n \to \mathbb{R}^n$, there exists two antipodal poitns which give rise to the same value. Using this in the case $n = 2$, one can apparently show (according to Hatcher) that given three compact sets in $\mathbb{R}^2$ that there exists some plane which divides each set into equal volume on both sides.

(1/18/2022) Today I learned that there is a bijective correspondence between *immersed* connected closed Lie subgroups of a (real) Lie group and Lie subalgebras of the associated Lie algebras. I also learned that there all path connected subgroups are Lie subgroups, while there exist connected subgroups that are not Lie subgroups. I also learned a theorem of Cartan, which says that any closed subgroup of the Lie group is in fact a Lie subgroup, i.e. it's in particular a smoothly embedded submanifold.

(1/19/2022) Today I learned that some of the theory of Cartier duality and 1-shifted Cartier duality. Specifically, Lurie approaches Cartier duality via functors from commutative algebra objects in some symmetric monoidal category to $\mathbb{E}_\infty$-spaces. With this setup, one can also define the Cartier dual of a symmetric monoidal category or DG category . In any of these cases, there is a notion of the picard groupoid and the associated smash product (given by the smash product of $\mathbb{E}_\infty$-spaces) is closed, i.e. the tensor product with an element admits a left and right adjoint. Therefore, we can define the (one-shifted) Cartier dual as the right adjoint to this smashing.

(1/20/2022) Today I learned some foundations in the setup of real Lie groups, and in particular the connection to hyperbolic space. Specifically, one can define the form on $\mathbb{R}^{n+1}$ which has signature $(n, 1)$, and, using this, one can define the hyperbolic plane $\mathbb{H}^n$ by the vectors in $\mathbb{R}^{n+1}$ which pair with themselves to -1 and have positive 'last' coordinate. This defines the hyperbolic plane, and one has the orthogonal group $O(n, 1)$ as the group of matrices preserving this form. This group has *four* connected components; they split into two based on the determinant of the matrix (which, by the same argument, is ± 1) and two based on which sheet of the above $\mathbb{H}^n$ the 'last' vector is sent to. The identity component switches the last coordinate of $\mathbb{H}^n$, so the connected component of the identity is known as $\mathrm{SO}_0(n, 1)$, which is the also the group of orientation-preserving isometries of $\mathbb{H}^n$. (The orientation of some vectors of the tangent vectors $v_1, ..., v_n$ at some $x \in \mathbb{H}^n$ is positively oriented if and only if $(x, v_1, ..., v_n)$ is positively oriented as a subset of $\mathbb{R}^{n+1}$.

(1/21/2022) Today I learned some points regarding t-structures and its relationship to the Fourier-Mukai transform. First off, I learned that the Fourier-Mukai transform is not t-exact for any abelian variety of positive dimension. Second off, I learned that the two distinct t-structures that one can put on the category of smooth k-varieties actually agree up to cohomological shift.

(1/22/2022) Today I learned that the Fourier-Mukai transform is functorial in the sense that if we have a morphism $f : A_1 \to A_2$ of functors from ordinary rings to 1-truncated spaces, then the associated pullback $f^!$ intertwines (via the Fourier-Mukai transform) with the pushforward of the dual morphism (a result of Laumon). The proof here is entirely formal in the sense that it only uses base changes, projection formulas, and an identification of the two possible pullbacks of Poincaré line bundles to $A_1 \times A_2'$, where the $'$ denotes the Cartier dual.

(1/23/2022) Today I learned an interesting fact about why the notion of 1-shifted Cartier duality became prevalent before the general Cartier duality story (phrased in the language of higher categories). Specifically, if one restricts functors from classical commutative rings (or classical $\mathbb{Q}$-algebras, etc) to one-truncated spaces, i.e. the spaces with no higher than 1 homotopy groups, then the mapping space between that object and $B\mathbb{G}_m$ is 1-truncated (where the unsheafified version of $B\mathbb{G}_m$ is 0-coconnective 0-truncated since it is a colimit of such things, and the fact that the sheafified variant is 0-truncated follows from the fact that inclusion of sets into spaces is a right adjoint and thus preserves limits. In particular, the category of sets in spaces is closed under limits.

(1/24/2022) Today I learned that one can prove Serre's criterion for affineness using the Tannakian formalism of categories. I also learned that the complement of an open subgroup is open, which just follows since the complement is a union of cosets, all of which are themselves open!

(1/25/2022) Today I learned that if $f : \Gamma \to S = \text{Spec}(A)$ is a finite flat Gorenstein map of classical schemes such that S is an affine, smooth, irreducible k-scheme of dimension d and such that Γ has pure dimension d. Then $f^! : \text{IndCoh}(S) \to \text{IndCoh}(\Gamma)$ is t-exact. The right t-exactness comes from a fact of Gaitsgory which says that, for Gorenstein morphisms, the !-pullback agrees with the $*$-pullback up to tensoring with the relative dualizing sheaf $f^{!,\text{QCoh}}(\mathcal{O})$ (and using the action of QCoh on IndCoh). Since one can compute this relative dualizing sheaf is an ungraded line bundle, the claim essentially follows.

(1/26/2022) Today I learned that the map from the union of graphs of the affine Weyl group $\Gamma \xrightarrow{s} \mathfrak{t}^*$ has t-exact !-pullback if and only if there exists some colimit presentation of Γ, say $\text{colim}_n \Gamma_n$ such that the dualizing complex of each Γ_n is in the heart after shifting rightward by $\dim(\mathfrak{t}^*)$. To see this, note that the fact that s is finite flat implies that $s^!$ is left t-exact and, to show right t-exactness given such a colimit presentation, we note that

$$s^!(\mathcal{O}_{\mathfrak{t}^*}) \simeq s^!(\omega_{\mathfrak{t}^*})[-\dim(\mathfrak{t}^*)] \simeq \omega_\Gamma[-\dim(\mathfrak{t}^*)] \simeq \text{colim}_n(i_*^{\text{IndCoh}}(\omega_{\Gamma_n})[-\dim(\mathfrak{t}^*)])$$

and so the fact that the t-structure on an ind-scheme is compatible with filtered colimits implies that if our colimit presentation has the property that all objects are in the heart, and so the colimit is.

(1/27/2022) Today I learned a result of Lurie which says that the Tannaka duality can be used to reconstruct Serre's criterion that a scheme is affine if and only if its global sections functor is t-exact. To see the if direction, one constructs a natural map of schemes $f : X \to \text{Spec}(\Gamma(\mathcal{O})X))$ and applies Tannaka duality to the symmetric monoidal functor f^*, and then (the more difficult part), shows that this particular object is a compact generator.

(1/29/2022) Today I learned a way to construct a t-structure on $\text{IndCoh}(\mathfrak{t}^*//\tilde{W}^{\text{aff}})$ such that both functors to and from $\text{IndCoh}(\mathfrak{t}^*)$ is t-exact. Specifically, one can define the t-structure via the requirement that the category $\text{IndCoh}(\mathfrak{t}^*//\tilde{W}^{\text{aff}})^{\leq 0}$ is the full (non-stable) subcategory containing the pushforward of $\mathcal{O}_{\mathfrak{t}^*}$.

(1/31/2022) Today I learned a fact about connected, smooth varieties which are homogeneous in the sense that the group scheme of automorphisms (which actually is a group scheme!) acts transitively on S-points for every S. In this setup, the variety is isomorphic to the automorphism group modulo a stabilizer, as in usual manifold theory.

December 2021

(12/1/2021) Today I learned a conceptual reason for why, given a simple object L_μ labeled by some $\mu \in \mathfrak{t}^*(k)$, one can compute that the symplectic Fourier transform labeled by some simple root α for which $F_{s_\alpha}(L_\mu)$ lies in the heart if $\langle \alpha^\vee, \mu \rangle \notin \mathbb{Z}$. The reason for this is the cleanness of the extension into the vector bundle indexed by the simple root. Specifically, the inclusion is $\mathbb{G}_m^\alpha$-equivariant, and any map from something supported at the zero section is necessarily a map from something which is $\mathbb{G}_m^\alpha$-monodromic. The assumption $\langle \alpha^\vee, \mu \rangle \notin \mathbb{Z}$ immediately implies that L_μ is monodromic for some other non-integral point of the respective $\mathbb{G}_m^\alpha$, so there are no maps to or from things pushed forward from the zero section, which implies the vanishing $i^*j_*(\mathcal{F})$ and thus the cleanness of the extension. I also learned that one can compute the Slodowy slice of the nilpotent element 0 as $\mathfrak{g}$, since it's the kernel of the map ad_0!

(12/2/2021) Today I learned an interesting construction in Berkovich geometry which is of use in the mirror symmetry program. Specifically, given some variety X over a field k equipped with a valuation, one can define its *analytification* X^{an}, whose points are defined to be the points equipped with a valuation on the associated residue field which extends the given norm on k. One particular advantage of this construction is that $X(k)$, a space which is totally disconnected if the norm is non-Archimedian, embeds into a space X^{an}, a space which turns out to be path connected! I also learned that the mirror of the open Bruhat cell of a reductive algebraic group is the open Bruhat cell of the Langlands dual group.

(12/3/2021) Today I learned a construction in homotopy theory (which is applicable to homological algebra!) known as *shearing*. Specifically, given a filtered object in, say, A-modules, we can equivalently realize it as a *graded* spectrum equipped with the data of an action of a free polynomial algebra in S with a degree 1-operator. We can then define the *shearing functor*, which takes the n^{th}-graded piece $M(n)$ and sends it to $\sigma^{2n}M(n)$, taking our endomorphism above to a weight 2, degree 1 endomorphism. This is often times an equivalence, and, if our underlying A is periodic, this does nothing to the underlying module.

(12/4/2021) Today I learned a useful fact about the specifics of the convergence of Morava K-theory to $\mathbb{F}_p$-cohomology. Specifically, there is a spectral sequence which compares Morava K-theory at a fixed prime p to $\mathbb{F}_p$-cohomology, and, assuming the manifold is of dimension less than $2p^n-2$, this spectral sequence collapses. In particular, the ranks of the associated Morava K-theory and $\mathbb{F}_p$ cohomologies agree.

(12/6/2021) Today I learned the fact that, for any separable extension of fields, the sheaf of relative Kähler differentials vanishes. This is because any object in our upper field a has the propery that $df(a) = 0$, where f is the minimal polynomial over the ground field, and by the product rule this implies that $f'(a)da = 0$. Separability and minimality then gives that $f'(a)$ is nonzero, and so in particular we see that $da = 0$.

(12/7/2021) Today I learned that the identification $\Omega^\infty(KU) \simeq BU \times \mathbb{Z}$ works at the level of spaces and $\mathbb{E}_1$-rings, but can't be promoted to an equivalence of $\mathbb{E}_\infty$-ring spectra.

(12/8/2021) Today I learned a useful trick for computing the Tate fixed points of an S^1-action on a complex vector space. Specifically, to compute this in characteristic zero, one first forms the homotopy fixed points of the associated vector space, which acquires an action of $H^*(BS^1) \simeq \mathbb{C}[\beta]$, and one forms the Tate fixed points by inverting this S^1-action. This idea can at least be used to show that $HH(BG/\mathbb{C}) \simeq \Gamma(G/G)[\beta^{-1}]$.

(12/9/2021) Today I learned the Tate valued Frobenius computation and some examples. Explicitly, one can show that there is a functorial construction of $\mathbb{E}_\infty$-ring spectra known as the *Tate diagonal* $R \to (R^{\otimes p})^{tC_p}$, where the superscript denotes the Tate construction (i.e. the cofiber of the canonical map from coinvariants to invariants) composed with the Tate construction applied to the multiplication map. Furthermore, I learned this Tate construction reduces chromatic height (known as the *blueshift* conjecture) and that, in particular, the Tate construction on KU or E_1 is rational.

(12/10/2021) Today I learned a fun way to construct the Frobenius morphism X, defined over an algebraically closed field k of characteristic p, to its Frobenius twist $X^{(1)}$. This Frobenius twist is defined via the p^{th}-power map $kk \xrightarrow{(-)^p} kk$, and, given any k-algebra A we let $A^{(-1)}$ denote the k-

algebra whose underlying ring is A and new k-algebra structure is given by $k \xrightarrow{(-)^p} k$. We may then define the *Frobenius twist* of a k-scheme X as $X^{(1)} := X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k^{(1)}) = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k)$, where the structural map is given by the p^{th}-power map $k \to k$. The projection map onto the ring $k^{(1)} = k$ makes $X^{(1)}$ into a k-scheme.

Then the relevant observation is that, given k-schemes $\mathrm{Spec}(A)$ and X, maps of k-schemes $\mathrm{Spec}(A) \to X^{(1)}$ are equivalently maps of k-schemes $\mathrm{Spec}(A^{(-1)}) \to X$, which can be checked affine locally and, if $X = \mathrm{Spec}(B)$, is literally given by the definition of the Frobenius twist; a map from $\mathrm{Spec}(A)$ to $X^{(1)}$ is equivalently a ring map $k \otimes_k B \to A$, which is equivalently a map $B \to A$ ('what it does to $1 \otimes b$) such that for any $x \in k$, $\phi(xb) = x^p\phi(b)$, which is equivalently a map of k-algebras $B \to A^{(1)}$. Given this, we can construct the map $X \to X^{(1)}$ via Yoneda–on points, it is given by pulling back by the morphism $\mathrm{Spec}(A^{(-1)}) \to \mathrm{Spec(A)}$ which is given by the p^{th} power map in the other direction (which is a map of k-algebras by construction).

For example, one can take the Frobenius twist of the group $\mathbb{G}_m$ over an algebraically closed field of characteristic p. The k-algebra map $(-)^p : A \to A^{(-1)}$ (which induces a map of rings $A \to A$) given by raising to the p^{th} power then induces the map $\mathbb{G}_m(A) \to \mathbb{G}_m(A^{(-1)}) = \mathbb{G}_m(A)$ given by sending an $x \in A^\times$ to x^p in $A^{(-1)} = A$ (as rings). In particular, the Frobenius twist is defined with these conventions so that, at the level of functor of points, the map $G \to G^{(1)}$ raises the A-points to the p^{th} power.

(12/12/2021) Today I learned that in the ind-completion of the BGG category $\mathcal{O}$ (and more generally if $\mathcal{O}$ is any abelian category with enough injectives and for which all injectives are sums of finitely many indecomposable projectives and whose objects are closed in their ind-completion under subquotients), for any injective object I in the ind-completion, there exists some injective M in the ind-completion for which one can write $I \times M$ as a direct product of indecomposable injective objects in the abelian category. This is because you can use the fact our I is a union of objects of $\mathcal{O}$ and choose an injective resolution for each and show that I injects into the direct sum of all the injectives. By injectivity of I, this injection splits, and so we obtain $I \times M$ can be written as a product of indecomposible injectives for some subobject M. However, a product is injective if and only if each term is, so M is also injective.

(12/13/2021) Today I learned that the nondegenerate subcategory of $\mathcal{D}(G/N)$ can be described as either the full $\mathcal{D}(N\backslash G/N)$-subcategory generated under colimits by the right Q_α-monodromic objects or the eventually coconnective ones. This is because the categories $\mathcal{D}(G/Q_\alpha)$ are compactly generated, and so too are the categories $\mathcal{D}(G)^{Q_\alpha\text{-mon}}$, and compact objects of this category have bounded cohomology. Therefore, every Q_α-monodromic object is colimit of eventually coconnective objects, since every compact object is eventually coconnective and every Q_α-monodromic object is a colimit of compacts.

(12/14/2021) Today I learned that the essential image of the functor Av_*^N on the G-category $\mathcal{D}(G/_\lambda B)$ has the property that all objects are the limit of their $\geq m$ truncations. This is because the functor is t-exact and the category $\mathcal{D}(G/_\lambda B)^{N^-,\psi} \simeq \mathrm{Vect}$ is left-complete and Av_*^N is a right adjoint so it commutes with limits.

(12/15/2021) Today I learned how to classify the essential image of pulling back by the map $\mathrm{Spec}(C_\lambda)/W_\lambda \to *$. Specifically, the classification is given by those objects with a left-complete property for which the pullback by the unique closed point to $\mathrm{Rep}(W_\lambda)$ maps to the trivial representation. Specifically, the missing piece I had was showing that the adjoint to the pullback functor is conservative on the essential image, and this follows because you can show that for cohomologically bounded $M \in \mathrm{IndCoh}(\mathrm{Spec}(C_\lambda)/W_\lambda)$ that the truncated pullback $H^0 i^!$ doesn't vanish, and therefore will admit a subobject which doesn't vanish under the t-exact averaging functor.

(12/16/2021) Today I learned a classification theorem for the homotopy groups of the Tate construction of complex oriented spectra. Specifically, given such a spectra E such that the p-series $[p](t)$ isn't a zero divisor in $E^*(\mathbb{C}p^\infty) = E^*(*)[[t]]$, then one can prove an isomorphism $\pi_{-*}(E^{tC_p}) \cong E^*(*)((t))//[p](t)$. In particular, one can compute the p-series explicitly for the multiplicative group and prove that the ideal given by quotienting out by p is the whole ring, so that, in particular, p is invertible in $\pi_{-*}(E^{tC_p})$ (the motivation, as I understand it, for the blueshift conjecture).

(12/17/2021) Today I learned that one can identify the cotangent bundle of SL_n/N as a Hamiltonion reduction as follows. Namely, we can take the vector space $V := \oplus_{k=1}^n Hom(\mathbb{C}^i, \mathbb{C}^{i+1})$ and equip it with an action of the group $H := \prod_{k=2}^n \mathrm{SL}_k$ where each factor acts by conjugation where you might expect. Then T^*V acquires a canonical Hamiltonian G-action, and it's a theorem that the affine closure of $T^*(\mathrm{SL}_n/N)$ can be obtained as the Hamiltonian reduction of H acting on $T^*(V)$.

(12/18/2021) Today I learned a convenient fact, which says that the data of a group W action on the category A (i.e. the data expressing $A \in \mathrm{AssocAlg}(\mathrm{Rep}(W))$) for some ring A is equivalent to the data of a group action on the category of A-modules along with a W-equivariance on the forgetful functor $\mathrm{A} - mod \to \mathrm{Vect}$.

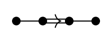
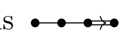
(12/19/2021) Today I learned a largely tautological property about the mapping stack. Specificially, given two prestacks $\mathcal{X}, \mathcal{Y}$, one can define their *mapping prestack* $\underline{\mathrm{Map}}(\mathcal{X}, \mathcal{Y})$ as the prestack $\underline{\mathrm{Map}}(\mathcal{X}, \mathcal{Y}) : \mathrm{cdga}^{\leq 0} \to \mathrm{Spc}$ given by declaring the mapping stack to be the unique thing satisfying $\mathrm{Map}(S, \underline{\mathrm{Map}}(\mathcal{X}, \mathcal{Y})) \simeq \mathrm{Map}(\mathcal{X} \times S, \mathcal{Y})$. From this definition, it follows that if $\mathcal{Y}$ is a sheaf with respect to some Grothendieck topology, then so too is the mapping stack because, if one has a cover S as the colimit of the Cech nerve of some covering map, then one can use the universal property, use the fact that products commute with sifted colimits, and use the fact that $\mathcal{Y}$ is a sheaf to pull it out.

(12/20/2021) Today I learned the most of a proof of the claim that any étale closed embedding $X \to Y$ to an integral scheme Y is an equivalence if X is nonempty. This follows because we can check if a scheme is an equivalence on affine open subsets, and both properties are preserved by base change. Therefore, by the fact our map is a closed embedding and thus in particular affine, we may assume our map is a map of affine schemes induced by some ring map $B \to B/I$. Furthermore, if $i \in I$, then, by the connectedness of Y, we have that the multiplication map $i : B \to B$ is nonzero if i is nonzero, and therefore gives a projective resolution of the pullback. Applying $B/I \otimes_B -$, we then see that there is a nonzero tor group except in the case where either B/I is itself zero (so the ideal is everything). Thus the flatness given by the étale property gives the equivalence. In fact, I think this proves that any flat closed embedding of connected nonempty schemes is an equivalence.

(12/21/2021) Today I learned the explicit homotopy groups of $\pi_*(KU^{tC_p})$, which are given as zero in odd degrees, and the field $\mathbb{Q}_p(\zeta_p)$ in even degrees. This follows from a standard computation more generally, which says that for any complex-oriented cohomology theory E for which the p-series of the associated formal group is nonzero, then one can compute $\pi^*(E^{tC_p})$ via $E^*((t))/[p](t)$.

(12/22/2021) Today I learned that there is a quasicoherent sheaf on an (affine!) scheme whose fiber is zero at every point but the sheaf is nonzero. For example, consider the $k[x]$-module $k(x)/k[x]$. Then, by unique factorization, to any closed point $k[x] \to k$, we have that $1 \otimes \frac{p(x)}{q(x)} = 1 \otimes \frac{r(x)p(x)}{r(x)q(x)} = \phi(r) \otimes \frac{p(x)}{r(x)q(x)} = 0$, where r is a nonzero polynomial in the kernel of the evaluation map. Moreover, if we take the other field $k[x] \hookrightarrow k(x)$, then $\frac{p(x)}{q(x)} \otimes m = \frac{1}{q(x)} \otimes p(x)m = 0$.

(12/23/2021) Today I learned a version of the *Kodaira vanishing theorem*, which says that, given a smooth projective variety X with a canonical bundle K_X (the top exterior power of the cotangent bundle), then for any ample line bundle $\mathcal{L}$, we have $H^q(X, \omega_X \otimes \mathcal{L})$ vanishes for $q > 0$.

(12/24/2021) Today I learned some intereseting fine print in the Borel de-Siebenthal theory classifying all closed subroot systems of a given root system. Namely, for $p > 5$ (or, really, p of good characteristic for the fixed, let's say simple, G), if we fix an irreducible root system, we can obtain all possible closed subroot systems, up to conjugation by the associated Weyl group, by simply choosing some subset of roots of the Dynkin diagram (this is written in Component Groups of Unipotent Centralizers in Good Characteristic). However, simply removing one vertex may not always give a *maximal* closed subroot system. This is because, for example, in the Dynkin diagram corresponding to F_4, i.e. , if we remove the third vertex, the associated simple root appears four times in the sum of the longest root. Therefore, the Borel de-Siebenthal algorithm for telling us how to classify the *maximal* closed subroot systems tells us this is not maximal. And this is true; we can instead first remove the last vertex and obtain the Dynkin diagram for B_2 (because the coefficient indexed by the last vertex is prime, so the closed subroot system is ). The algorithm says then that a subroot system of this is gotten by removing the third root, where we have to add the affine simple root because the coefficient on the highest root is prime. Therefore, in conclusion, we have applied the 'maximal' Borel de-Siebenthal twice and obtained the same result as removing the third vertex of F_4, consistent with the fact that the 'maximal' Borel de-Siebenthal algorithm only allows us to remove the vertices 'labelled' by prime integers. Furthermore, if you delete a vertex labeled by a 1 and *don't* remove the affine root, you get the same root system back.

(12/25/2021) Today I learned a useful and fun fact about the structure theory of reductive groups. Specifically, recall that any reductive group G admits a central isogeny $\tilde{G} \times Z(G)^\circ \to G$, where $\tilde{G}$ is the simply connected cover of $[G, G]$. Any central isogeny is in particular surjective, and therefore the map induces a bijection both on the split tori of the group and of the parabolic subgroups of the group!

(12/26/2021) Today I learned the computation of the equivariant K-theory of a point. Specifically, if G is some group, then the complex, equivariant K-theory of a point is given by the representation ring of G, i.e. the Grothendieck ring of the symmetric monoidal category of representations. I also learned that this ring is generated, modulo lower order terms, by the representations indexed by the fundamental weights since each V_λ, for dominant $\lambda = n_1\omega_1 + ... + n_r\omega_r$ appears as the subrep of $V_{\omega_1}^{\otimes n_1} \otimes ... \otimes V_{\omega_r}^{\otimes n_r}$ since this representation only has highest weight λ. In particular, the representation ring (over $\mathbb{C}$) is generated, as a ring, by the fundamental weights!

(12/27/2021) Today I learned interesting and fun facts about the word 'set theoretic containment.' Specifically, I learned that if we are given a schematic map $f : X \to Y$ and a field-valued point $y \in Y(L)$, then we can take the fiber product and if there is a map from some $\mathrm{Spec}(A)$ to the fiber product which factors through $\mathrm{Spec}(L) \xrightarrow{y} Y$, then the map from the reduction $\mathrm{Spec}(A/\mathcal{N}(A))$ does as well. I also learned a property of the Eilenberg-MacLane space $K(G, n)$, which says that this space has the property that homotopy classes of based maps from a space X into a $K(G, n)$ are in canonical bijection with $H^n(X, G)$.

(12/28/2021) Today I learned the notion of a spin structure. Specifically, if we are given a oriented manifold (real or complex) X, the orientation gives rise to a map $X \to BSO(n)$. A spin structure is a lift of this map to $B\mathrm{Spin}(n)$.

(12/29/2021) Today I learned some of the basics of Arthur's conjecture, which I'll state as over $\mathbb{R}$ right now. Fix a split reductive group G over $\mathbb{R}$. Then you can take the Weil group, which is $L_{\mathbb{R}} = C * \rtimes Z/2Z$. Then you can define Arthur parameters, which are continuous homomorphisms $L_R x \mathrm{SL}_2(\mathbb{C}) - > G^\vee$ whose restriction to the $\mathrm{SL}_2(\mathbb{C})$-factor is an algebraic representation and another condition I'm not as sure about, namely that the restriction to L_R needs to be a Langlands parameter. The conjecture is that, associated to this character, there are finitely many irreducible admissibible representations of $G(\mathbb{R})$ associated to it which satisfy some conditions.

(12/30/2021) Today I learned that pullback via the natural group map $T_{\mathrm{SL}_2} \to T_{\mathrm{PGL}_2}$ (i.e. the map $\mathbb{G}_m \to \mathbb{G}_m$ defined via $x \mapsto x^2$) sends non-monodromic objects of $\mathcal{D}(T_{\mathrm{PGL}_2})$ to monodromic objects of $\mathcal{D}(T_{\mathrm{SL}_2})$. Specifically, we can take the $\mathcal{D}$-module on T_{PGL_2} associated to the character $\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \mapsto \alpha^{1/2}$ (more formally, this is the character $\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \mapsto \alpha$ $\otimes$-ed with $1/2$). This pulls back to an integral character, namely $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \mapsto \alpha$.

(12/31/2021) Today I learned a conceptual framework on perverse sheaves that kind of solidifies why the definitions were made in the way that they were. Assume that we are given a constructible sheaf which is constructible with respect to the trivial stratification, i.e. a local system. There, assuming that the scheme in question is smooth of pure dimension d, we can define a pretty trivial t-structure so that objects of the form $\mathcal{E}[d]$ are in the heart, and *this* is motivated by the fact that objects in that degree (cohomological degree $-d$) are precisely the ones preserved by Verdier duality. Then, for a general stratification by smooth of pure dimension, you can define the condition of perversity literally so that the functor $\ell^!$ for each locally closed embedding is forced to be left t-exact (as we know if ℓ is an open embedding, $\ell^!$ must be t-exact and if ℓ is closed, then ℓ_* must be t-exact so that $\ell^!$ must be left t-exact). The other 'half' of the t-structure is literally just enforcing that ℓ^* (always defined for constructible sheaves) is right t-exact.

November 2021

(11/1/2021) Today I learned that the *co*tangent bundle of the flag variety at a point is *canonically* the unipotent radical of that Borel. To see this, choose a Borel, which identifies the flag variety with G/B. Then, the *tangent* bundle at G/B, by the PBW theorem, can be identified with $\mathfrak{n}^-$. However, when applying duality with respect to the Killing form, which identifies $(\mathfrak{n}^-)^*$ with $\mathfrak{n}$, we obtain our claim. I also learned one of the consequences of Bezrukavnikov's local geometric Langlands duality theorem, which says that one can identify Iwahori-equivariant $\mathcal{D}$-modules on the affine flag variety with $G^\vee$-equivariant sheaves on the (derived) product $\tilde{\mathcal{N}}^\vee \times_{\mathfrak{g}^\vee} \tilde{\mathcal{N}}^\vee$.

(11/2/2021) Today I learned some facts about the correspondence between representations of W and nilpotent orbits. Specifically, one can take the Springer sheaf as a sheaf on the nilpotent cone modulo G and note that, by the decomposition theorem, this sheaf splits as a direct sum of simples labeled by the representations of the Weyl group, possibly up to multiplicity. Therefore, this assignment gives a map wherein one takes a representation of the Weyl group and takes the support of the associated Springer sheaf and assigns it to the nilpotent orbit given by its support. This map need not be injective, but it is in general surjective and is bijective for SL_n.

(11/3/2021) Today I learned some normalizations about the affine Weyl group which are worth writing down. Specifically, one can take the affine Weyl group (often called the affine Weyl group for the Langlands dual group) $\Lambda_r \rtimes W$, where Λ_r is the root lattice. In this case, one can take the alcove, the set of all $\mu \in \mathfrak{t}^*$ such that μ pairs with any coroot to be in $[0, 1]$. In the case of $G = \mathrm{SL}_2$, this group is in particular $2\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ with fundamental alcove $[0, 1]$, and in the case of $G := \mathrm{PGL}_2$, this group is $\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ with fundamental alcove $[0, \frac{1}{2}]$. In the $G := \mathrm{SL}_2$ case, one can also take the (distinct) extended affine Weyl group $\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$. The key difference here is that for SL_2, this group is no longer generated by reflections in the walls of the fundamental alcove, and this group does not act simply transitively on the alcoves (because $\tau_1 r_0$ fixes the interval $[0, 1]$). On the other hand, the group $2\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ (the affine Weyl group for SL_2) is generated by reflections in the walls of the alcoves, since we can write $r_1 = r_0\tau_{-2}$.

(11/4/2021) Today I learned an enhancement of the Mostow rigidity theorem that Florian Stecker told me about. Specifically, given a semisimple Lie group with a lattice (which are a class of subgroups which include all discrete subgroups with compact quotient), this semisimple Lie group is entirely determined by the quotient by this lattice.

(11/5/2021) Today I learned the proof of the following statement, where we fix some real point $x \in \mathfrak{t}^*(\mathbb{R})$. The following subgroups of W are identical:

1. The image $\overline{W_x^{\text{aff}}}$ of the stabilizer W_x^{aff} of x under the W^{aff}-action on $\mathfrak{t}^*$ under the quotient map $W^{\text{aff}} \to W^{\text{aff}}/\mathbb{Z}\Phi \cong W$.
2. The subgroup $W_{[x]} := \{w \in W : wx - x \in \mathbb{Z}\Phi\}$.
3. The subgroup $W_{[x]}^{\bullet} := \{w \in W : w \cdot x - x \in \mathbb{Z}\Phi\}$.
4. The subgroup $W^x := \langle s_\alpha : \langle x, \alpha^\vee \rangle \in \mathbb{Z}\}$, where α varies over the set of roots Φ.

(11/6/2021) Today I learned that the above groups of yesterday can also explicitly be realized as the stabilizer of the given x of the affine Weyl group. The point is that to each $w \in W_{[x]}$, there exists one and only one μ in the root lattice such that (τ_μ, x) fixes x. The key insight I was missing yesterday is that the integral translations in the affine Weyl group only affect the 'rational part' of x.

(11/7/2021) Today I learned a point of quantum groups. Specifically, one can show that the category of representations of a semisimple Lie algebra $\mathfrak{g}$, simply defined to be modules for $H := U\mathfrak{g}$, has a monoidal struture. Where does this monoidal structure come from? One place it can be said to come from is the *Hopf algebra* structure on H, which is in particular a bi-algebra structure (which, for instance, comes with the data of a comultiplication map which is compatible with the multiplication map; the latter statement implicitly uses the braided monoidal structure on Vect) with another map called the *antipode* which allows one to discuss the underlying vector space of a category of representations axiomatically and discuss left/right duals of objects of representations of H, i.e. modules of H. Because Hopf algebras are in particular bi-algebras, there is a notion of commutative Hopf algebras (:= underlying algebra structure is commutative) and dually a co-commutative structure. The Hopf algebra H above is not commutative, but it is co-commutative. The algebra of functions of an algebraic group is commutative but not co-commutative. Quantum groups are a deformation of H which makes the Hopf algebra structure also no longer co-commutative.

(11/8/2021) Today I learned a new perspective on how to view the fundamental group of an associated group from the root datum. Specifically, at least in the complex analytic story, one can view each element in the lattice of maps $\mathbb{G}_m \to T$ as giving an element in π_1 of the group (after all, $\mathbb{G}_m(\mathbb{C})$ is homotopic as a topological space to a circle). Then, lifting this to a map from SL_2 to your group implies it is contractible, because SL_2 is simply connected. This gives the heuristic that the fundamental group of G is given by the lattice $\mathbb{G}_m \to T$ modulo coroot lattice.

(11/9/2021) Today I learned a point about duality for abelian varieties. Specifically, for any abelian group, one can define the one shifted Cartier dual as the mapping stack from the abelian group to $B\mathbb{G}_m$. However, the key point that singles out abelian varieties is that, for these particular varieties, there exists a scheme which represents this functor! Given this, one can define a Fourier-Mukai transform by a 'pull-tensor-push' with tautological line bundle on $A \times A^\vee[1]$ given by the fact that there is a canonical map $A \times A^\vee[1] \to B\mathbb{G}_m$.

(11/10/2021) Today I learned more about what an R-matrix is. Specifically, the R-matrix is an endomorphism for each pair vector space which captures the failure of the forgetful functor to be graded. I also learned that for each triple of matrices, such an R-matrix satisfies the *Yang-Baxeter equation*, which looks a lot like a braid relation on the twistings. I also learned one way to write the Yang-Baxeter equation. Specifically, write down all the transposition in S_3, with the longest one in the middle. Then the associated equality is gotten by taking those endomorphisms and conjugating by $(1,3)$ to get the permutations appearing on the other side.

(11/11/2021) Today I learned that, if one fixes a K-point of $\mathfrak{t}^*$, there is a canonical isomorphism $\mathrm{Spec}(K)\times_{\mathfrak{t}^*}\Gamma_{\tilde{W}^{\mathrm{aff}}}/X^\bullet(T)$ with the coinvariant algebra. I also learned the proof:

We first note that the inclusion $W^{\mathrm{aff}}\hookrightarrow\tilde{W}^{\mathrm{aff}}$ induces an isomorphism $\mathbb{Z}\Phi\backslash W^{\mathrm{aff}}\xrightarrow{\sim}X^\bullet(T)\backslash\tilde{W}^{\mathrm{aff}}$. Therefore we obtain canonical isomorphisms

$$X^\bullet(T)\backslash\Gamma_{\tilde{W}^{\mathrm{aff}}}:=X^\bullet(T)\backslash\tilde{W}^{\mathrm{aff}}\overset{W^{\mathrm{aff}}}{\times}\Gamma_{W^{\mathrm{aff}}}\xleftarrow{\sim}\mathbb{Z}\Phi\backslash W^{\mathrm{aff}}\overset{W^{\mathrm{aff}}}{\times}\Gamma_{W^{\mathrm{aff}}}\simeq\mathbb{Z}\Phi\backslash\Gamma_{W^{\mathrm{aff}}}$$

over $\mathfrak{t}^*$ with respect to the (right) source map. Furthermore, by the lemma in my paper, we see:

$$\mathbb{Z}\Phi\backslash\Gamma_{W^{\mathrm{aff}}}\times_{\mathfrak{t}^*}\mathrm{Spec}(K)\xleftarrow{\sim}\mathbb{Z}\Phi\backslash W^{\mathrm{aff}}\overset{W_\lambda^{\mathrm{aff}}}{\times}(\Gamma_{W_\lambda^{\mathrm{aff}}}\times_{\mathfrak{t}^*}\mathrm{Spec}(K))$$

so that, composing with the quotient map $\mathbb{Z}\Phi\backslash W^{\mathrm{aff}}\xrightarrow{\sim}W$, we see the quotient map induces an isomorphism

$$\mathbb{Z}\Phi\backslash W^{\mathrm{aff}}\overset{W_\lambda^{\mathrm{aff}}}{\times}(\Gamma_{W_\lambda^{\mathrm{aff}}}\times_{\mathfrak{t}^*}\mathrm{Spec}(K))\xrightarrow{\sim}W\overset{W_{[\lambda]}}{\times}\Gamma_{W_\lambda^{\mathrm{aff}}}\times_{\mathfrak{t}^*}\mathrm{Spec}(K)$$

where we identify the image of the stabilizer W_λ^{aff} with the integral Weyl group as in the definition in my paper.

(11/12/2021) Today I learned that one can assign a Lie-algebra-like object to any unipotent group. Specifically, given a unipotent group U, the tower of nilradicals terminates. Therefore, one can define the associated Lie algebra whose underlying abelian group is the associated graded of the filtration given by this filtration, and furthermore one can define a Lie bracket by lifting and taking commutators of brackets, which are well defined modulo lower order terms.

(11/13/2021) Today I finally wrote out the proof (and so solidly learned!) that the degenerate objects of $\mathcal{D}(\mathrm{SL}_2/N)^N$ are precisely the monodromic objects. This essentially follows from the fact that the degenerate category is generated by objects which have bounded cohomological dimension, and there we can apply a recollement sequence to reduce to the heart. Objects in the heart are either in the kernel of $\mathrm{Av}_!^\psi$ xor have L_{-2} as a subobject. (Also this is like one of the first times I get to use xor. Super cool!)

(11/14/2021) Today I learned a different way to argue, for example, that the nondegenerate category of $\mathcal{D}(G/B)^N$ can be identified with ind-coherent sheaves on the Chevalley fiber at zero, i.e. ind-coherent sheaves on the spec of the coinvariant algebra. Specifically, one can use the fact that $\mathbb{V}$ is conservative on the bounded by below category to compute the endomorphisms of the compact generator and identify it with the compact generator of the ind-coherent category given by the pushforward from the unique closed point.

(11/15/2021) Today I learned a neat theorem of Lusztig which gives an integral form of the Hall algebra associated to representations of a quiver. Specifically, one can define a Hall algebra associated to a quiver as a convolution diagram associated to some space attached to all the arrows of the quiver for a fixed dimension vector v, modulo the GL_v action. This gives a convolution diagram where the dimension vectors are allowed to vary and sum together, and this gives a structure of a Hall algebra H_Q associated to a quiver Q, which is in particular a $\mathbb{Z}[v^{\pm 1}]$-algebra. Lusztig's theorem states that the map from the positive part of the quantum group determined by sending each simple root to the associated simple representation of the quiver $U_q(\mathfrak{g})^+\to H_Q\otimes_{\mathbb{Z}[v^{\pm 1}]}\mathbb{Q}(v)$ is an isomorphism, giving an integral form for the positive part of the quantum group.

(11/16/2021) Today I learned a fact that, for a split semisimple group G, the Hecke eigenfunctions which can be realized as compactly supported complex functions on the space of $\mathbb{F}_q$ points of $\mathrm{Bun}_G(\mathbb{F}_q)$ on some curve X are precisely the cuspidal eigenfuntions.

(11/17/2021) Today I learned the *orbit lemma*, which is (1.66) in Milne. This says that any G-orbit acting on an algebraic variety X, a priori only locally closed, is itself closed if the orbit has minimal dimension. This in turn implies that each G-orbit has a closed orbit since we can ask for the dimension of each orbit.

(11/19/2021) Today I learned that $\mathrm{Bun}_{\mathbb{G}_m}(\mathbb{P}^1)$ can be identified, as a stack, with $\mathbb{Z} \times B\mathbb{G}_m$. This follows because the degree map shows that the isomorphism classes of line bundles are given by n (they are the bundles $\mathcal{O}(n)$) and each line bundle has endomorphsisms given by $\mathbb{G}_m$ (at a point, they are k, but in families, these global sections are allowed to vary). Similarly, one can identify the moduli stack of local systems $\mathrm{LocSys}_{\mathbb{G}_m}(\mathbb{P}^1)$ with $B\mathbb{G} \times \Omega(\mathbb{A}^1)$, in part because there is one rank one local system up to isomorphism (for example, because local systems correspond to representations of the fundamental group, which is trivial in this case).

(11/20/2021) Today I learned how to show that the category of representations of an affine algebraic group of finite k-dimension, where k is a field of characteristic zero, has finite ext dimension (where here finite ext dimension means the largest nonzero number a nonzero ext superscript can have). Specifically, one can use the fact that the map between two representations is given by the fixed points of another representation itself, and note that the fixed point functor has bounded dimension because there is an explicit resolution of the fixed point functor, which in charactersitic zero can be resolved by finitely many copies of the adjoint representation of the group on its Lie algebra, a finite length projective resolution of the trivial representation. Therefore, roughly, assuming group and Lie algebra cohomologies agree, we've got the claim.

(11/21/2021) Today I learned that every principal GL_n bundle is locally trivial, and that every principal G-bundle for a semisimple G on a curve is trivializable if you remove a single point from the curve.

(11/23/2021) Today I learned some of the basics of Deligne's theory of weights. Specifically, I learned that the ≥ 0 weights are preserved by the $*$-pushforward and dually the ≤ 0 weights are preserved by the !-pushforward, and that this how Ginzburg shows that the associated cohomology functor is fully faithful–namely, inductively on the cells of the flag variety, Ginzburg shows the connecting map between any open cell and the shift of the closed maps weight precisely zero into weights ≥ 1.

(11/24/2021) Today I learned a new alternative characterization of a smooth morphism. Specifically, a smooth morphism is a locally of finite presentation morphism which is flat and all fibers of geometric points are smooth. I also learned an example where the flatness fails and why it's essential; namely, one can take the union of $\mathbb{Z}$ many points inside $\mathbb{A}^1$. The intuition is that smooth morphisms 'look like fibrations' and this one doesn't.

(11/25/2021) Today I learned some of the basics of the general theory of (B, N)-pairs. Most importantly, today I finally learned that the N stands for normalizer! Given a normalizer and a Borel, one can extract the Cartan $B \cap N$ and the associated Weyl group $N/(B \cap N)$ and choose a set of simple reflections. In this general setup, one can obtain a Bruhat decomposition. For one thing, this works for a parabolic subgroup as B and its normalizer, which gives a relative Bruhat decomposition associated to a smaller Weyl group associated to the parabolic subgroup. This is distinct from the other parabolic Bruhat decomposition which decomposes by minimal elements.

(11/26/2021) Today I learned a cool trick which applies to lots of representation theoretic contexts, and, in particular, representations of the quantum group over an algebraically closed field. Specifically, let A be an algebra, and let M be a simple A-module. Then Schur's lemma says that $\mathrm{End}_{A\text{-mod}}(M)$ is a division algebra (i.e. a field without the commutativity of multiplication axiom) and furthermore if M is finite dimensional over an algebraically closed field this endomorphism ring can be identified with scalars of the identity. In particular, if $z \in Z(A)$, then z acts on a simple representation by one of these scalars. This helps one, for instance, diagonalize a representation of the Kac-De Concini quantum group since, for example, E^ℓ is in the center if our quantum parameter is set to a primitive ℓ^{th} root of unity for quantum SL_2.

(11/27/2021) Today I learned a theorem of Gaitsgory and Drinfeld which says that, for any algebraic stack, its associated category of $\mathcal{D}$-modules is compactly generated. I also (re?)-learned a neat proof of complete reducibility of representations of semisimple Lie algebras. Specifically, semisimple Lie algebras (by one definition) are those for which you can find a nondegenerate form on all representations similar to the Killing form. Using this, given a representation V of a semisimple Lie algebra, you can construct a Casimir element by choosing the Killing form and taking the sum of elements of the form $e_i\kappa^{-1}(e_i^*)$ (i.e. basis times dual basis, which is what the Casimir is in the first place). It's not too hard to compute the trace of this matrix is the dimension of the semisimple Lie algebra, and then you can show that this Casimir at least separates codimension one irreducible equivariant subspaces and splits them (this Casimir must act by zero for a semisimple Lie algebra on a one dimensional representation and must scale by something else so as to make the trace nonzero). You can reduce to this case from non-irreducible equivariant subspaces by induction and modding out by a subspace, and for this case, you can use the restriction map induced by inclusion and Schur's lemma to argue that there exists some map which restricts to the identity.

(11/29/2021) Today I learned a general paradigm to show whether a category $\mathcal{C}$ equipped with a t-structure is equivalent to the bounded by below derived category of its heart in a t-exact way. It's necessary to have a continuous functor that the t-structure on $\mathcal{C}$ be right-complete, so we assume this. Otherwise, given this, the statement is that this equivalence $\mathcal{D}^+(\mathcal{C}) \xrightarrow{\sim} \mathcal{C}^+$ if and only if for any object $\mathcal{F} \in \mathcal{C}^\heartsuit$ which is injective, then $\underline{\mathrm{Hom}}(\mathcal{G}, \mathcal{F})$ is concentrated in degree zero. While this map of vector spaces has no negative exts, it may have positive ones–for example, consider the category of constructable sheaves on S^2. The heart of this category is a given constructible sheaf, which is equivalently a representation of the associated fundamental group, which in fact vanishes. Then the constant sheaf is an object of the heart, which is equivalent to Vect, but it has higher exts(!) because Homs out from it compute cohomology.

(11/30/2021) Today I learned a proof that $\mathbb{G}_a$ invariants and coinvariants disagree. Specifically, consider the exact sequence of $\mathbb{G}_a$-representations given by $1 \to k \to k^2 \to k \to 1$, where the $\mathbb{G}_a$-representation is given by the endomorphism assigning x acting by the 2 x 2 matrix with ones everywhere except for the 2,1 slot where it's zero. Then, taking invariants, one can see that this sequence does not remain exact, so we can't have that $(-)^N$ is the left adjoint to the forgetful functor, since the forgetful functor is t-exact that would imply that $(-)^N$ is right-exact, beign the left adjoint to a t-exact functor.

October 2021

(10/1/2021) Today I learned lots of cool facts about the *wonderful compactification* of a reductive algebraic group G. One starting place for this discussion starts from the fact that, via the Peter-Weyl theorem, one can identify global algebraic functions on G as a direct sum of each irreducible representation tensored with its dual, but the algebra structure placed on it naturally is not an algebra isomorphism. However, the algebra still is filtered. Therefore, one can define *Vinberg semigroup* given by taking the analogue of the Rees construction (over T instead of $\mathbb{G}_m$). To obtain the wonderful compactification, one takes this Vinberg semigroup and takes the GIT quotient via the 'extra' action of T. I also learned that, for SL_2, the Vinberg semigroup is the space of 2×2 matrices, whose GIT quotient by $\mathbb{G}_m$ yields $\mathbb{P}_3$.

(10/2/2021) Today I learned that if $\mathcal{A}$ is a cocomplete abelian category with the property that every object is a union of its compact subobjects, and assume that $\mathcal{A}'$ is a abelian subcategory closed under subobjects such that the inclusion functor $\iota : \mathcal{A}' \hookrightarrow \mathcal{A}$ preserves colimits. Then ι preserves compact objects. I also learned the proof: Let $C \in \mathcal{A}'$ be an object which is compact in $\mathcal{A}'$. By assumption, $\iota(C)$ can be written as a union $\iota(C) \cong \cup_i C_i$ for C_i compact objects in $\mathcal{A}$. By assumption, each $C_i \in \mathcal{A}'$. Therefore, because ι commutes with colimits, we see that

$$\mathrm{id}_{\iota(C)} \in \mathrm{End}_{\mathcal{A}}(\iota(C)) \cong \mathrm{End}_{\mathcal{A}'}(C) \cong \mathrm{Hom}_{\mathcal{A}'}(C, \cup_i C_i) \cong \cup_i \mathrm{Hom}_{\mathcal{A}'}(C, C_i)$$

where the last step uses the compactness of C in $\mathcal{A}'$. In particular, we see that the identity morphism factors through some inclusion map $C_i \hookrightarrow C$, so that this inclusion is also a surjection and therefore an isomorphism. Thus $\iota(C) \cong C_i$ is compact. I also learned that a generalization of this for triangulated categories was proven in a paper of Amnon Neeman whose title starts with 'The Connection Between the K-Theory Localization Theorem of Thomason.'

(10/4/2021) Today I learned a fun trick to show that any object in the full subcategory generated by all colimits under some objects admits a nontrivial map from one of the objects. A sketch of this is as follows: one can consider the left orthogonal of the right orthogonal of the category. An object has no maps if and only if it's in the right orthogonal, and if the object is in the left orthogonal as well, this implies that the identity map is zero.

(10/5/2021) Today I learned a useful identity regarding the de Rham prestack. Specifically, the map $X \to X_{dR}$ for any X induces an isomorphism ${}^{\mathrm{red}}X \to^{\mathrm{red}} (X_{dR})$. This is because the reduction functor is computed by first restricting to classical affine reduced schemes, where X and X_{dR} are equivalent, by definition, and then by applying a Kan extension.

(10/6/2021) Today I learned ideas regarding Frobenius twisted conjugacy classes on the loop group over the closure of a field of characteristic > 0. Specifically, I learned that for any fixed element w of the affine Weyl group, there exists some associated maximal conjugacy class such that there is an Iwahori coset mapping into the Iwahori double coset labelled by w. I also learned a result of Xuhua He, which says that one can explicitly compute the dimension of this, and, if the Weyl group element is *straight*, i.e. the length of the n^{th}-powers is n times the original length, then the dimension of this associated variety is simply the length of the element.

(10/7/2021) Today I learned a proof of the fact that the (T, w) Hecke category $\mathcal{D}(N\backslash G/N)^{T\times T,w}$ is rigid monoidal. This follows from the fact that the monoidal unit is compact, and what is essentially proved by Ben-Zvi and Nadler, which says that the category $\mathrm{IndCoh}(B\backslash G/B)$ is rigid monoidal. There is a monoidal functor $\mathrm{IndCoh}(B\backslash G/B) \simeq \mathrm{IndCoh}(N\backslash G/N)^{T\times T,w} \to \mathcal{D}(N\backslash G/N)^{T\times T,w}$ which admits a conservative right adjoint, which formally implies this rigid monoidality.

(10/8/2021) Today I learned that one can take a spherical variety H and, constructing a dense open B-orbit, show that it admits an open embedding of a group G/H via an orbit-stabilizer construction. One can then use the notion of *colored fans* associated to a G and H to give rise to the spherical varieties which can arise in some fashion.

(10/10/2021) Today I learned some interesting facts about the Dold-Kan correspondence. Specifically, I learned that one can define the category of simplicial objects of some $\mathcal{C}$ as the category of functors from the opposite category to the simplex category, Δ^{op}, to the category $\mathcal{C}$. With this terminology, the Dold-Kan correspondence can be stated pretty succinctly (which I just have copied essentially from higher algebra) as saying that the category of simplicial objects in a stable ∞-category $\mathcal{C}$ is equivalent to the category of functors from $\mathbb{Z}_{\geq 0}$, i.e. the opposite diagrams made from towers i.e. a bounded to the right of 0 chain complex.

(10/11/2021) Today I learned a proof of the fact that every representation in defining characteristic of a finite p-group has a fixed vector. To see this, you can induct on the order of the p-group. Every p-group G has a nontrivial center Z, and any representation of an abelian p-group Z has a nontrivial fixed vector by the Jordan-Holder decomposition, which merely requires the eigenvalues (which are necessarily p^{th}-power roots of unity in the field, i.e. 1) lie in the field. Consider the G-span of this nonzero fixed vector; this G-rep naturally is a G/Z-rep, and so our inductive hypothesis applies.

(10/12/2021) Today I learned some facts about *Heisenberg groups*. Specifically, I learned their definition, which, for a symplectic vector space V, is given by the semidirect product $V \ltimes (V^\vee \times \mathbb{G}_a)$. I learned that it's a theorem that there exists a unique unitary irrep on the (analytic) version of this grou psuch that the exponential acts as the identity.

(10/13/2021) Today I learned that the Satake isomorphisms, as a general principle, intertwine hyperbolic localization on the A-side with restriction to a Levi on the B-side. I also learned a theorem of Feng-Gaitsgory, which says that the derived commuting variety for GL_n agrees with the associated (derived) commuting variety on the torus, assuming the derived Satake of Arinkin-Bezrukavnikov.

(10/14/2021) Today I learned a way to phrase a bunch of things in terms of the notion of an *operad*. Specifically, I learned that one can regard an operad as a multi-category, i.e. a category in which source and targets are allowed to have multiple maps. In this way, one can define the associative algebra operad (for example) as the collection where you have n-ary multiplication from an object to itself.

(10/15/2021) Today I learned more details into a general isomorphism of Goresky, Kottwitz, and MacPherson. Specifically, I learned that, given any complex space with an action of a torus which has an isolated set of T-fixed points, the natural map from cohomology to the set of fixed points (which maps to a direct sum of equivariant cohomology rings of a point, i.e. of $\mathrm{Sym}(\mathfrak{t})$) is an injection, and one can explicitly identify the image as the functions agreeing at each fixed point for which you can translate 'from 0 to ∞' to on the associated $\mathbb{C}P^1$.

(10/16/2021) Today I leaned that spherical varieties with a fixed dense open B orbit and corresponding fixed G-orbit G/H are in bijective correspondence with *colored fans*, which are generalizations of fans which incorporate the notion of *colors*, which are B-stable divisors on G/H which are not G-stable (so, when $T = G$, there are no colors).

(10/17/2021) Today I learned the notion of a *Thom spectrum*, which is obtained by taking a vector bundle over a space and crushing the zero section. For example, over the trivial vector bundle on a point, taking the vector bundle and crushing the non-zero section yields $\mathbb{A}^1/\mathbb{G}_m$, which, in the world of $\mathbb{A}^1$ homotopy theory, yields $*/\mathbb{G}_m \simeq \mathbb{P}^1$, since $\mathbb{P}^1 \simeq \mathbb{A}^1 \times_{\mathbb{G}_m} \mathbb{A}^1$.

(10/18/2021) Today I learned that one can identify the maps $\mathrm{Map}(\mathbb{G}_m, B\mathbb{G}_m)$ corresponds to the usual Betti S^1 stack. This follows largely formally from the fact that $B\mathbb{G}_m$ is a co-affine stack; in particular, over a field it is isomorphic to $\mathrm{Sym}^*(k[-1])$. From this, the fact that the maps correspond to $k[\mathbb{Z}][1] \simeq k[B\mathbb{Z}]$ follows largely formally.

(10/20/2021) Today I learned (cleared up?) some important terminological distinctions in derived algebraic geometry. Specifically, I learned that you can truncate any prestack to take its underlying 'classical prestack' $\mathcal{Y} \mapsto {}^{\mathrm{cl}}\mathcal{Y}$, and that this truncation is a right adjoint (its left adjoint is the left Kan extension functor along the inclusion of classical rings into all derived rings). However, this classical truncation need not, for example, be a scheme. For example, because this cl functor is a right adjoint, we can apply it to $\mathbb{G}_m \simeq * \times_{B\mathbb{G}_m} *$ and use the fact that right adjoints commute with limits to obtain the expression, in classical prestacks, that $\mathbb{G}_m \simeq * \times_{{}^{\mathrm{cl}}B\mathbb{G}_m} *$, which at the very least shows that the underlying classical prestack of $B\mathbb{G}_m$ is not a scheme. I believe, but have not literally proved, that $B\mathbb{G}_m$ itself is not a classical prestack. I also learned that the property of a morphism being proper (along with other things which are discussed in EGA IV.2.2) is fpqc (faithfully flat quasicompact) local on target.

(10/21/2021) Today I learned (and typed up!) the proof of the following fact. Let $\mathcal{C}$ and $\mathcal{D}$ be DG categories equipped with t-structures, and let $F : \mathcal{C} \to \mathcal{D}$ be a (continuous) functor between them which is t-exact and conservative. Then if $\mathcal{D}$ is right-complete with respect to its t-structure, then $\mathcal{C}$ is also right-complete with respect to its t-structure, and if $\mathcal{D}$ is left-complete with respect to its t-structure (and not necessarily right-complete), then if F commutes with (small) limits, $\mathcal{C}$ is also left-complete with respect to its t-structure.

The proof is: Assume $C \in \mathcal{C}$, and consider the map $C \xrightarrow{\phi} \mathrm{colim}_n \iota^{\leq n}\tau^{\leq n}(C)$ in $\mathcal{C}$. By the conservativity of F, it suffices to show that $F(\phi)$ is an isomorphism. However, we have:

$$F(\mathrm{colim}_n \iota^{\leq n}\tau^{\leq n}(C)) \simeq \mathrm{colim}_n F(\iota^{\leq n}\tau^{\leq n}(C)) \simeq \mathrm{colim}_n \iota^{\leq n}\tau^{\leq n}(F(\iota^{\leq n}\tau^{\leq n}(C))) \simeq \mathrm{colim}_n F(\iota^{\leq n}\tau^{\leq n}(C))$$

where the first equivalence uses the continuity of F, the second step uses the right t-exactness of F, and the third step uses the left t-exactness of F. By the left-completeness of $\mathcal{D}$, this composite map is an equivalence. However, we can identify this composite map with $F(\phi)$, so we see that $F(\phi)$ is an isomorphism, as desired. An argument dual to the above gives (2). Specifically, if $C \in \mathcal{C}$, then we have equivalences:

$$F(\lim_m \iota^{\geq m}\tau^{\geq m}(C)) \simeq \lim_m F(\iota^{\geq m}\tau^{\geq m}(C)) \simeq \lim_m \iota^{\geq m}\tau^{\geq m} F(\iota^{\geq m}\tau^{\geq m}(C)) \simeq \lim_m \iota^{\geq m}\tau^{\geq m} F(C)$$

where the first step uses the assumption that F commutes with small limits, the second step uses the fact that F is left t-exact, and the third uses the right t-exactness of F. Therefore, as above, the conservativity of F gives the t-structure on $\mathcal{C}$ is left-complete provided the t-structure on $\mathcal{D}$ is.

(10/22/2021) Today I learned a theorem of Gaitsgory-Nadler which identifes the 'dual group' associated to an affine G-spherical variety, which is a subgroup of the Langlands dual group. This group is constructed via a Tannakian formalism; specifically, Gaitsgory-Nadler define the notion of *quasimaps*, a finite dimensional model for loops on the space, and define a category of perverse sheaves on it and show it is equivalent to some subgroup of the Langlands dual group of G via the Tannakian formalism.

(10/23/2021) Today I learned a theorem which states that the parity sheaves on the affine Grassmannian associated to a group for a prime $p > n$ (and, usually better) are precisely the tilting objects under the geometric Satake theorem. I also learned an explicit example of a complex which is nonzero in $\mathrm{IndCoh}(\mathrm{Spec}(k[\epsilon]/(\epsilon^2)))$ but which is only in cohomological degree $-\infty$. Specfcially, because in $\mathrm{Coh}(\mathrm{Spec}(k[\epsilon]/(\epsilon^2)))$ there is a map $k \to k[1]$ (its fiber is $k[\epsilon]/(\epsilon^2)$, because there is a fiber sequence $k \to k[\epsilon]/(\epsilon^2) \to k\epsilon \cong k$) and therefore, by definition in IndCoh (which, for this particular example, is ind of Coh) the colimit $\mathrm{colim}(k \to k[1] \to k[2] \to ...)$ is nonzero. However, since $\tau^{\geq m}$ is a left adjoint/by definition of the t-structure, it commutes with colimits and so we see this is a colimit of objects which are eventually zero and therefore it is zero!

(10/24/2021) Today I learned that a finite graded commutative k-algebra for k any field is Gorenstein if and only if the associated ring satisfies Poincaré duality. In particular, for the rings A are the coinvariant algebra, we obtain that the associated dualizing sheaf is trivial, because it is a line bundle (by Gorenstein-ness) on a topological space with one point.

(10/25/2021) Today I learned the *Atiyah-Bott theorem*, which says that the cohomology $H^*(\mathrm{Bun}_G, k)$ is a free commutative graded algebra.

(10/26/2021) Today I learned that one can relax the notion of invariance for a Coxeter group W acting on a vector space by reflections to obtain the notion of quasi-invariants, which only requires the reflections to agree up to the power of some ideal cut out by the hyperplanes. It's a theorem of Yuri Berest that this map from the affine space itself is a cuspidal quotient morphism, and so that in particular the space is Cohen-Macaulay (in fact, it's Gorenstein) and the category of $\mathcal{D}$-modules on this singular scheme is equivalent to the $\mathcal{D}$-modules on the usual affine scheme.

(10/27/2021) Today I learned a few interesting results on the cohomology of Schubert varieties of the flag variety. For example, I learned a result of ALP which states that, given a prinipal nilpotent element of $\mathfrak{n} \subseteq \mathfrak{gl}_n$, one can define a vector bundle on G/B, and then for any Schubert variety Y, one can take the coordinate ring of this Schubert variety with the intersection of this zero set of the associated vector bundle and the associated ring of functions is isomorphic to the cohomology of Y, as a quotient of the cohomology of the flag variety. The grading parameter comes from the fact that the associated zero sets have a $\mathbb{G}_m$ action by scaling.

(10/28/2021) Today I learned that if one takes the derived Satake equivalence $\mathcal{D}(Gr_G)^{G(\mathcal{O})} \simeq \text{IndCoh}_{\text{Nilp}}((* \times_{\mathfrak{g}^\vee} *)/G^\vee)$ and takes the long Whittaker averaging functor to $\mathcal{D}(Gr_G)^{G(\mathcal{O})}$, this factors through the category of quasicoherent sheaves on the B-side of the Satake equivalence.

(10/30/2021) Today I learned two notions of the *Mostow rigidity theorem*, which says that the existence of an isomorphism of lattices in $\text{PO}(n,1)$ implies the lattices are conjugate by an element of the group. In fact, I learned that this theorem holds for all lattices in simple Lie groups which are not isomorphic to $\text{SL}_2(\mathbb{R})$.

September 2021

(9/1/2021) Today I learned that if μ is an antidominant weight, then there is some simple root α and some minimal IC sheaf $\mathcal{I}(w) \in \mathcal{H}_{w\cdot\mu,-\mu}$ such that $\mathcal{I}(w) \star L(\mu) \cong L(w \cdot \mu)$, $\langle \alpha^\vee, w \cdot \mu\rangle \in \mathbb{Z}^{\geq 0}$.

I also learned the proof: any antidominant weight μ, by definition, has the property that $\Xi_{\mu}^{\geq 0} := \{\beta \in \Phi^+ : \langle \beta^\vee, \mu\rangle \in \mathbb{Z}^{\geq 0}\}$ is nonempty. We induct on the minimal height of an element in $\Xi_{\mu}^{\geq 0}$. For the base case, note that if there is an element of $\Xi_{\mu}^{\geq 0}$ of height 1, then by definition there is a simple root $\alpha \in \Xi_{\mu}^{\geq 0}$, and so (3) holds with $w = 1$. Now assume the minimal height is larger than one, and let β be a minimal root in $\Xi_{\mu}^{\geq 0}$. By assumption on the minimal height, β is not simple, and therefore, the theory of root systems gives that there exists a simple root α such that $\langle \beta, \alpha^\vee\rangle > 0$, so that $s_\alpha(\beta) \in \Phi^+$ and the height of $s_\alpha(\beta)$ is smaller than the height of β. Note that this in particular implies that $\beta - \alpha$ is a positive root, and therefore, by the minimality of β, we see that $\langle \alpha^\vee, \mu\rangle$ is not an integer.

(9/2/2021) Today I learned (computed?) a new way to show that the nondegenerate category, defined to be the kernel of the averaging functor, is closed under the Hecke action (on the remarkable side). Specifically, one can use the averaging formalism and the explicit description of the essential image of $\text{Av}_!^\psi$ to show that it suffices to show that the kernel is merely closed under the action of $\text{IndCoh}(\mathfrak{t}^*)$. However, this is true at each point (where we use the $\text{IndCoh}(\mathfrak{t}^*)$ action on the other side) because Soergel's theorem can be reinterpreted to say that the action of $\text{Sym}(\mathfrak{t})$ on $\mathcal{O}_\lambda$ factors through a finite length subscheme.

(9/3/2021) Today I learned some of the basic results on symmetric spaces. Specifically, one can define a *symmetric space* as a Riemannian manifold such that for each x in the manifold there exists an inversion which changes the orientation of any geodesic through that point. Except for Euclidean spaces, all symmetric spaces can be realized as quotients of the form G/K for G a Lie group with K the fixed points of some involution.

(9/4/2021) Today I learned some facts about Grothendieck abelian categories and AB5 categories. Specifically, I learned that the Serre quotient of any Grothendieck abelian category remains Grothendieck abelian, and any abelian category closed under direct sums which admits an exact conservative functor to an AB5 category is AB5.

(9/5/2021) Today I learned that if Λ is any discrete set of points, the canonical map $\Lambda \to \Lambda_{dR}$ is an equivalence. This follows immediately from the fact that, as a prestack, Λ is 0-truncated, and therefore the maps from any affine scheme are determined by the truncation to the underlying classical scheme. Therefore, you can show that maps from affine schemes into Λ factor through some finite subscheme, and therefore since any finite subscheme of points is isomorphic to its de

Rham prestack, the claim follows.

(9/6/2021) Today I learned a useful base change lemma which says that $\mathcal{D}(X) \otimes_{\mathrm{Vect}_(k)} \mathrm{Vect}_(L)$ for a field L is canonically isomorphic to $\mathcal{D}(X_L)$ i.e. on the base changed variety, for X a smooth classical scheme. This follows because since $\mathcal{D}(X)$ is smooth, $\mathcal{D}(X)$ is canonically isomorphic to quasicoherent sheaves on its de Rham prestack, and therefore the claim follows because affine schemes are in particular passable prestacks, and therefore quasicoherent sheaves on the tensor product is the tensor product of quasicoherent sheaves.

(9/7/2021) Today I learned a useful categorical fact from Germán Stefanich. Specificially, assume that $\mathcal{C}$ is a symmetric monoidal category closed under colimits such that the tensor preserves colimits in each variable, and that $A \to B$ is a map of commutative algebra objects of $\mathcal{C}$. Then the category B is naturally itself a commutative algebra object A-mod$(\mathcal{C})$, and the canonical map B-mod$(A$-mod$(\mathcal{C})) \to B$-mod$(\mathcal{C})$ is an equivalence.

(9/8/2021) Today I learned motivation for the Nisnevich topology, which is a topology of which all Zariski covers are Nisnevich and all Nisnevich covers are étale. Specifically, the Nisnevich topology is defined so that, after sheafifying for it, one obtains a 'Mayer-Vietoris' sequence in motivic homotopy theory.

(9/9/2021) Today I learned a proof that the right adjoint to an additive functor L is left t-exact. Specifically, let L be the left adjoint. Given a short exact sequence, we can take $\mathrm{Hom}(LX, -)$ of it, which is left exact (as a functor of abelian categories), so it sends our short exact sequence to a left exact sequence. We then apply the adjoint property, and then use Yoneda's lemma and the fact that X is arbitrary, to see that the sequence obtained after applying R remains left exact, and dually for right exactness.

(9/10/2021) Today I learned that the delta sheaf $\delta_1 \in \mathcal{D}(T)$ is W-equivariant, where W acts on $\mathcal{D}(T)$ via the $W, \cdot$ action (the action determined by the usual W action on functions and the $W, \cdot$ action on differential operators). This turns out to be a direct computation, using the fact that this W-action is still t-exact.

(9/12/2021) Today I learned that the heart of any DG category whose t-structure is closed under filtered colimits is AB5. I also wrote out the full proof: By assumption, the category $\mathcal{C}^{\geq 0}$ is closed under filtered colimits, and the category $\mathcal{C}^{\leq 0}$ is closed under *all* colimits because $\tau^{>0}$ is a left adjoint and therefore commutes with all colimits. Therefore we see that $\mathcal{C}^{\geq 0}$ and $\mathcal{C}^{\leq 0}$ are closed under arbitrary direct sums, which are in particular filtered colimits, so $\mathcal{C}^{\heartsuit}$ is cocomplete.

Now let $f : C \to D$ be any map in $\mathcal{C}^{\heartsuit}$. Note that f is surjective (respectively, injective) if and only if $\mathrm{cofib}(f) \in \mathcal{C}$ lies in $\mathcal{C}^{<0}$ (respectively, $\mathcal{C}^{\geq 0}$). Furthermore, for any colimit of maps $f_i : C_i \to D_i$ whose colimit is $f : C \to D$, we have an equivalence $\mathrm{cofib}(f) \simeq \mathrm{colim}(\mathrm{cofib} f_i)$, since colimits commute with colimits. Therefore, if each f_i is a map of objects in $\mathcal{C}^{\heartsuit}$ which is a surjection (respectively, injection), we see that $\mathrm{cofib}(f_i) \in \mathcal{C}^{<0}$ (respectively $\mathrm{cofib}(f_i) \in \mathcal{C}^{\geq 0}$) and because $\mathcal{C}^{<0}$ is closed under all colimits (respectively, $\mathcal{C}^{\geq 0}$ is closed under filtered colimits), that therefore $\mathrm{cofib}(f) \in \mathcal{C}^{<0}$ (respectively, $\mathrm{cofib}(f) \in \mathcal{C}^{\geq 0}$).

(9/13/2021) Today I learned a possible new way to match the left and right nondegenerate notions (the ones given by being in the kernel of the left and right Whittaker averaging respectively). Specifically, the functor of Whittaker averaging the other direction is likely conservative on $\mathcal{D}(T)$, so these are both likely equivalent to (at least on eventually coconnective categories) being in the kernel of bi-Whittaker averaging. I also learned the proof of the (9/12) thing is also a remark in Higher Algebra.

(9/14/2021) Today I learned a way to show that an equivalent way to form the category $\mathcal{D}(N\backslash G/N)_{\mathrm{deg}}$ is to ask that the *bi*-Whittaker averaging functor $\mathrm{Av}_!^{\psi \times -\psi}$ vanishes. This is because the functor $\mathrm{Av}_!^{\psi} : \mathcal{D}(T) \to \mathcal{H}_{\psi}$ is conservative, which follows from the application of a base change lemma and the fact that if you right average an object in $\mathcal{D}(T)$ to an object of $\mathcal{D}(N\backslash G/N)$, it has the same object in $\mathcal{D}(T)$ as a subobject.

(9/15/2021) Today I learned that any ∞-category is a localization of an ordinary category, but this ordinary category is *not* the usual homotopy category of the ∞-category.

(9/16/2021) Today I learned a new interpretation of the various functorialities one can get with groups acting on categories. In particular, one can define, for a *discrete* group W, the space BW, and define a group acting on a DG-category via a map Maps(BW, DGCat). The nice part is, the equivalence here is totally formal–you can write BW as a colimit of a certain simplicial object, and the point is the category in question and the pullback is to maps$(W, \text{DGCat}$, and the limit provides identifications of pulling back via the group action map and the projection map, which individually provides isomorphisms for the category with each w acting on it. The (homotopy) fixed points then correspond to pushing forward via the terminal map $BW \to *$. This is 2.3.4 of Spectra Are Your Friends, an all around great resource.

(9/17/2021) Today I learned a theorem of Mac Lane which says that any monoidal (1,1)-category is equivalent, as a monoidal category, to a *strict monoidal category*, i.e. a monoidal category where the right unitor, left unitor, and associator are all the identity map.

(9/18/2021) Today I learned how to compute the underlying discrete space of endomorphisms of the monoidal unit in the nondegenerate category $\mathcal{D}(N\backslash G/N)$. Specifically, one can use the fact that the functor $I^!$ shifts cohomological degree up by two and the fact that $J^!$ is exact to show the induced map on endomorphisms is an isomorphism, and so the endomorphisms of the monoidal unit have H^0 isomorhpic to the discrete space k.

(9/19/2021) Today I learned a slick proof that the affine Weyl group acts on the T, w invariants of any category with a $T \rtimes W$ action. Specifically, this largely follows from an equivalence of categories $\underline{\text{End}}_{T\rtimes W}(\mathcal{D}(T \rtimes W)^{T,w}) \xrightarrow{\sim} \underline{\text{End}}_{\mathcal{D}(T)^W}(\mathcal{D}(T)^{T,w}) \simeq \underline{\text{End}}_{\text{IndCoh}(\mathfrak{t}^*/\tilde{W}^{\text{aff}})}(\text{IndCoh}(\mathfrak{t}^*))$, where the first equivalence is given by Ben-Zvi Gunningham Orem and the second is the Mellin transform.

(9/20/2021) Today I learned that there is an equivalence of 2-categories given by a correspondence of the 2-singularity category of $\mathbb{P}V$ (for V a vector space) with the 2-singularity category of $\mathbb{P}V^*$. This equivalence is given by a correspondence which is analogous to the Radon transform.

(9/21/2021) Today I learned some basic facts in the theory of the input of semifields into split reductive groups over $\mathbb{C}$. Specifically, I learned that there is a semifield with one element $\{1\}$, and for any reductive group G, we can identify $G\{1\}$ with the set $W \times W$ as a monoid (but not a group). This implies that, since any semifield admits a map to $\{1\}$, that any semifield splits as a disjoint union over two copies of the Weyl group.

(9/22/2021) Today I learned that in mirror symmetry, the 2-category of boundary conditions on the A-side is given by the category of all G-categories. I also learned a possible new way to show the W-action on $\mathcal{D}(G/B)_{\text{nondeg}}$. Using my monoidal equivalence of categories, one can show that the induced functor on N-invariants $\text{Av}_!^{\psi}$ is W-equivariant and maps to the trivial W-action at the point 0. Since the endomorphisms of the monoidal unit are k (at least at the level of spaces!) this gives the claim.

(9/23/2021) Today I learned about the notion of *(Weil) restriction of scalars*. This is specifically defined as follows–given an algebraic group G defined over a ring R and a ring map $S \to R$, we can define a new group scheme over S defined via $A \mapsto G(A \otimes_S R)$. This sends a point to a point and it's a theorem for group schemes defined over a large field L/k that $\text{Res}_k^L(G)$ is a product of all the Galois conjugates of G.

(9/24/2021) Today I learned a lot of fine print. For example, I learned that one can show that, in the parabolic Bruhat decomposition associated to some index set I, that there is some subvariety U^w for each minimal w in W/W^I such that the map given by multiplication exhibits $U^w \times P$ as the associated cell. I also learned the fine print of the definition of an open embedding of (derived) affine schemes. Specifically, one must hard code in the definition that this morphism is flat, which in particular means that it is affine schematic and that the pullback map preserves the heart of quasicoherent sheaves. A flat morphism is a Zariski open embedding if and only if, for each classical scheme which maps into it, the map is a map of classical schemes (by definition).

(9/27/2021) Today I learned the details of a general result in the theory of reductive groups. Specifically, given any reductive group G, one can take the simply connected cover of the adjoint quotient G', and then G fits into a short exact sequence of groups $1 \to Z(G') \to G' \to G \to G/\mathrm{im}(G') \to 1$ such that $G/\mathrm{im}(G')$ is a torus. In particular, every reductive group is an extension of a torus by a semisimple group. Similarly, any reductive group G can be realized as the cokernel of the map $Z(G') \to G' \times Z(G)$ (given by $z \mapsto (z, \overline{z}^{-1})$ using the fact that the map $G' \to G$ maps $Z(G')$ to $Z(G)$ (possibly up to quotienting).

(9/28/2021) Today I learned that any (connected) split reductive group G can be written as the quotient $(S \times T)/Z$ for S a semisimple, simply connected group, T a split torus, and Z a finite subscheme. I also learned the proof, with the help of Milne's algebraic groups book. By 19.30, we may write G as the quotient $(G' \times Z(G))/\tilde{F}$, where G' is the simply connected cover of the adjoint quotient of G and $\tilde{F}$ is the finite group scheme given by the scheme theoretic image of the map $Z(G') \to G' \times Z(G)$ defined by $\xi \mapsto (\xi, \xi^{-1})$. Now let F denote the scheme theoretic image of the closed subscheme $Z(G') \times_{Z(G)} Z(G)^{\circ}$ under the above map to $G' \times Z(G)$. Then the map $G' \times Z(G)^{\circ} \to (G' \times Z(G))/\tilde{F}$ induces an isomorphism $(G' \times Z(G)^{\circ})/F \xrightarrow{\sim} (G' \times Z(G))/\tilde{F}$ (see theorem 5.82 with $H := G' \times Z(G)^{\circ}$ and $N := \tilde{F}$).

Set $S := G'$ and $T := Z(G)^{\circ}$. Then T is smooth since k has characteristic zero and connected by assumption. Furthermore, T is diagonalizable since $Z(G)$ is (see proposition 21.8) and subgroups are diagonalizable groups are diagonalizable (Chapter 12, section e of ibid). Therefore, we see that T is a smooth connected diagonalizable group, and thus a split torus via loc cit.

(9/29/2021) Today I learned that character sheaves are expected to be related to mass deformations of certain three dimensional $\mathcal{N} = 4$ Superconformal Field Theories known as $T^{\rho}[G]$-*theories*.

(9/30/2021) Today I learned the notion of a *quasismooth* closed embedding, which is a closed embedding for which the relative cotangent complex is perfect and concentrated in cohomological degrees -1 and 0. I also learned that such quasismooth embeddings can be realized Zariski locally as the set cutting out the zero of some function on the codomain.

August 2021

(8/1/2021) Today I learned some more categorical descriptions for certain representations of the quantum group. Specifically, one can take the mixed version by regarding it as relative Drinfeld center of the category of representations of the Hopf algebra of the Kac-De Concini quantum universal enveloping algebra for N^- (relative to the fact that we defined that as a Hopf algebra in the braided monoidal category $Rep_q(T)$, so categorically you need the natural morphisms you can get from that to agree). The formalism of the center of the representations of a Hopf algebra says a representation of a quantum group (and the fact that Lusztig and KD quantum universal enveloping algebras are dual) says that a mixed representation of the quantum group has a compatible action of the KD quantum enveloping algebra for N^- and the Lusztig one for N^+, but, for the Lusztig one, the action has to be locally nilpotent.

(8/2/2021) Today I learned some useful facts to prove that the Whittaker sheaf in $\mathcal{D}(G/N)$ is equivariant with respect to the Gelfand-Graev action. Specifically, one can use the fact that the action is G-equivariant to show that the Whittaker sheaf goes to a sheaf which is also Whittaker invaraint. This allows one to write out the explicit diagrams and compute base change with respect to the big cell of both copies of G/N.

(8/3/2021) Today I learned a lemma of Sam Raskin and David Yang, which says that if one has a functor of two G-categories with t-structures which are compatible with the G-action, then one may check whether a functor is t-exact if the functor is t-exact on the N-invariant subcategories.

(8/4/2021) Today I learned the full statement of the Jacobson-Morozov theorem. Specifically, the theorem says that given any nilpotent element, there exists an $\mathfrak{sl}_2$ triple which takes h to a semisimple element and e to that nilpotent element, but, furthermore, I learned a bonus result of Kostant which says that this embedding is unique up to choice of element in the unipotent radical of the centralizer of the nilpotent element you chose. Therefore, as a sanity check, we can see that if we chose the nilpotent element zero in a reductive group, since reductive groups have no unipotent radical by definition, the $\mathfrak{sl}_2$ embedding is unique, and if we choose a regular nilpotent element, the centralizer is abelian, and so in particular is also reductive and so the embedding is unique.

(8/5/2021) Today I learned a possible more intrinsic definition of degenerate and nondegenerate G-categories. Specifically, we can view G-categories by assumption as having a commuting action of $\mathcal{D}(G/G)$, and we have a monoidal functor $\mathcal{D}(G/G) \to \mathcal{H}_\psi$. In particular, we can define the degenerate G-categories as those G-categories $\mathcal{C}$ for which $\mathcal{C} \otimes_{\mathcal{D}(G/G)} \mathcal{H}_\psi \simeq 0$, and hopefully it turns out that the more naive $[P, P]$-vanishing definition for $\mathcal{C}^N$ agrees.

(8/6/2021) Today I learned the above potential alternate definition might not deal with the t-structures incredibly well, and a better alternate definition may be to require that the horocycle/Harish-Chandra functor vanishes upon averaging to some Q (instead of N). This allows still for the fact that the N-invariants of a given G-category have the same nondegenerate category as before, while maintains that nondegenerate G-categories are intrinsically G-categories themselves, i.e. not by force.

(8/8/2021) Today I learned the results of some quantum Hamiltonian reduction computations. Specifically, I learned that one can identify the $T \times T$-fixed points of global differential operators on $N\backslash G/N$ with $\text{Sym}(\mathfrak{t}^*) \otimes_{Z\mathfrak{g}} \text{Sym}(\mathfrak{t}^*)$.

(8/9/2021) Today I learned a result of Ginzburg's which states that the isomorphism $\mathcal{D}(G/N)^{N^-,\psi} \xrightarrow{\sim} \mathcal{D}(T)$ is compatible with the Gelfand-Graev action! The proof uses an isomorphism of the respective rings, which we can use since every quasicoherent sheaf in the category $\mathcal{D}(G/N)^{N^-,\psi}$ is globally generated (since $N^- \times B$ is affine).

(8/10/2021) Today I learned that the fact that the compact objects of the category of ind-coherent sheaves on an ind-scheme can be written as the pushforwards of some compact object from some closed subscheme is actually a more general categorical fact, and that for any two objects in the closed subscheme you can explicitly describe the maps between them as a colimit of the 'inclusions beyond'.

(8/11/2021) Today I learned a way to define a t-structure on the nondegenerate category, finally. Specifically, assuming that I can show that on the *heart* of the t-structure that the only objects in the kernel are the monodromic objects, the kernel of a t-exact functor by design kills subobjects, and the condition on $J_*J^!$ being exact only requires a condition that the *heart* is closed under subobjects.

(8/12/2021) Today I learned that the compact objects in the category IndCoh(Γ) on the union of graphs of the affine Weyl group Γ has an (alternative) explicit description of the compact objects. Specifically, since Γ admits an ind-proper surjection $\coprod_{w \in W^{\text{aff}}} \mathfrak{t}^* \to \Gamma$, we can realize the sheaf as a totalization of the cosimplicial object. Using a convenient lemma about limits and colimits of categories, this is also the colimit over the category of right adjoints, and Gaitsgory and Drinfeld showed that any filtered colimit has compact objects given by the 'pushforward' of the compacts in the diagramatic categories.

(8/13/2021) Today I learned that holonomic $\mathcal{D}$-modules are Artinian, so that in particular one may find a Jordan normal form of them. I also learned of the existence of a condition of when a localization of a category gives a t-structure on it in Higher Algebra.

(8/14/2021) Today I learned why the nondegenerate category, defined as the subcategory cut out by a bunch of kernels of the appropriate averaging maps, is actually closed under the braid group action. Of course, it suffices to check this for the generators, and the key fact comes from the fact that any nondegenerate sheaf in $\mathcal{D}(N\backslash G/N)$ is also nondegenerate on the other side too. This is because you can reduce to showing that the averaging associated to a simple refleciton s vanishes on each left s coset, and you can use the (right) symplectic Fourier transform to bring that coset all the way down to the coset $\{1, s\}$ for free.

(8/16/2021) Today I learned a very explicit statement about how compact objects generate. Specifically, given a set of compact objects in a compactly generated category, then all compact objects can be obtained by finite colimits and retracts (which, in the world of DG categories, are direct summands) of the compact generating objects, even if this is an infinite set of generators.

(8/17/2021) Today I learned a heuristic explanation for the abstract Cartan that I like. Specifically, it is known that any two choices of Borel in a semisimple Lie group are conjugate. One might ask–must this isomorphism respect the choice of tori? The answer is yes, because once you choose the tori in the respective Borels, the isomorphism gives an identification of both tori with the respective 'abstract Cartan', and thus identifies them with no extra choices. In particular, in the Lie algebra case one can identify the $\mathfrak{sl}_2$ embeddings, since these are determined by the weights up to scalar multiple.

(8/18/2021) Today I learned that there is a partially defined left adjoint to the symplectic Fourier transform given by the symplectic Fourier transform for $j_!$. Therefore, we may equivalently show that our usual symplectic Fourier transformations preserve the nondegenerate subcategory by arguing that their associated left adjoints preserve the category of degenerate objects, i.e. send the various $\omega_{Q_\alpha/N}$ to the structure sheaves.

(8/19/2021) Today I learned a way to possibly compute what object $\text{Av}_*^N(\omega_{\mathfrak{t}^*})$ is at each central character without using the fact that the W action gives an equivalence. Specifically, if we could identify the symplectic Fourier transformations on $\mathcal{D}^\lambda(N\backslash G/B)$ with the Arkhipov twisting functors (where both are acting on the left!) then we could show that the functor $\text{Av}_!^\psi$ sends no minimal simple object to zero. This is because the Arkhipov twisting functor for w sends the dual Verma for an antidominant weight λ to the dual Verma indexed by $w\lambda$. In particular, if $w\lambda$ is antidominant, then it agrees with the simple, and the fact that the Gelfand-Graev action is W-equivariant says that we can show that $\text{Av}_!^\psi$ is isomorphic to any $\text{Av}_!^\psi F_w$ for any composite of symplectic Fourier transformations F_w.

(8/20/2021) Today I learned a neat computational tool about the sth symplectic Fourier transformation. Specifically, given a sheaf entirely supported on the minimal s cosets such that there is no nontrivial extension to the line (or, indeed, any sheaf for which the $j_!$ and the j_* agree with respect to inclusion of $\mathbb{A}^2\backslash 0 \xhookrightarrow{j} \mathbb{A}^2$ has the property that the two a priori distinct symplectic Fourier transformations are the same.

(8/21/2021) Today I learned that one can associate to each field-valued λ, say $\mathfrak{t}^*$, a character sheaf on T, say $\mathcal{L}_\lambda$ such that the associated twisted equivariance given by λ and $\mathcal{L}_\lambda$ gives rise to an equivalence of categories $\mathcal{D}(G/_{\mathcal{L}_\lambda}B) \simeq \mathcal{D}^\lambda(G/B)$.

(8/22/2021) Today I learned a cool new verison of a localization theorem. Specifically, one can show that the functor $\mathcal{D}(G/_\psi N^-) \xrightarrow{\Gamma} \mathfrak{g}\text{-Mod} \xrightarrow{i^!} \mathfrak{g}\text{-Mod}_{-\rho}$ for a simple Lie algebra identifies the full subcategory of $\mathcal{D}(G/_\psi N^-)$ generated by the B-monodromic objects with $\mathfrak{g}\text{-Mod}_{-\rho}$.

(8/23/2021) Today I learned an analogy for the parabolic Beilinson-Bernstein localization theorem, at least at an integral central character. Specifically, being at an integral (regular) central character means that your global sections is $\mathfrak{t}^* \times_{\mathfrak{t}^*//W} *$, whereas for paraboics, the central character is replaced with $\mathfrak{t}^*//W_P \times_{\mathfrak{t}^*//W} *$ (which has a less heuristic description at the ring level).

(8/24/2021) Today I learned an important example in Lie theory about why closed subroot systems are so important and why Soergel's theory is so interesting. Specifically, consider the root system associated to $\mathfrak{sp}_4$, which is a root system of type C_2. One can check that there is a subroot system corresponding to the four short coroots. Furthermore, this subroot system comes up as the associated root system for the blocks of category $\mathcal{O}$ of central character of 1/2 times the short coroot, see Exercise 3.4 in Humphreys. Therefore, the representation theory of this block (with four elements in the W-dot orbit of this coroot) is controlled by the root system of the four short roots, and the two positive roots are perpendicular. However, *there is no embedding of* $\mathfrak{sl}_2 \times \mathfrak{sl}_2$ *which maps to these coroots.* This is because one can check that the two positive e associated to the two positive short roots bracket to something nontrivial, whereas in the product of the two $\mathfrak{sl}_2$'s, they bracket to something trivial.

(8/26/2021) Today I learned a similar fun fact as yesterday. Specifically, with λ in the associated dual Cartan to $\mathfrak{sp}_4$ given by (in the notation of Fulton and Harris) $L_1 - 1/2L_2$, then λ is *not* antidominant and it does not arise from *any* parabolic category $\mathcal{O}$.

(8/27/2021) Today I learned that the category of modules over the finite W algebra with nilpotent parameter 0 can be identified with the category of $U\mathfrak{g}$-modules which integrate to an N-action.

(8/28/2021) Today I learned a small piece of terminology which allows one to talk about positive roots given an integral root system. Specifically, given a collection of roots, a *facet* of the collection is a nonempty(!) subset determined by a partition of roots into the ρ-positive, ρ-zero, and ρ-negative parts. This terminology allows one to say, for example, when simples with respect to certain translation functors vanish.

(8/30/2021) Today I learned a fun fact about Lie algebroids. Specifically, one can view a groupoid as a group which lies over multiple points via a source and target map. For a given groupoid, one can therefore take the diagonal map which is the associated point at the identity. In certain contexts, one can take the tangent space of this identity and obtain, for the case of a groupoid over a point (i.e. a group) a Lie algebra, and for other contexts when the basepoint is, for example, a manifold, one obtains a vector bundle on the manifold, which may be nontrivial!

(8/31/2021) Today I learned the fact that any map from an affine scheme to the union of graphs of the affine Weyl group is necessarily affine schematic. This is because one can show that any map must factor through some union of some finitely many graphs, and so any two maps agree at the union of the finitely many graphs (which is also an affine scheme) and so the product over the union of all graphs agrees with the product over an affine scheme, which is affine.

July 2021

(7/1/2021) Today I learned the notion of *hyperbolic localization*, which is the generalization of the following fact. Given a variety with a $\mathbb{G}_m$ action, one can consider the inclusion of the fixed points or the 'projection' functor onto the fixed points given by 'taking the limit' if one further has an *attracting* $\mathbb{G}_m$ action (an action which extends to an action of $\mathbb{A}^1$ as a monoid). Then one can show that if a sheaf has weakly $\mathbb{G}_m$ equivariant structure, then the canonical maps one can make between this limit map and the restriction is an isomorphism. Furthermore, hyperbolic localization is about mixing attracting and repelling $\mathbb{G}_m$ actions. I also learned there is a decomposition of the affine closure of the basic affine space G/N into a union of $G/[P,P]$ for the various parabolics P. It was a good day for learning!

(7/2/2021) Today I learned an interpretation by Ben-Zvi and Raskin on the action of local systems on category of constructable sheaves on the affine Grassmannian. Specifically, this category monoidally acts and can be viewed as a monoidal subcategory of the affine Grassmannian, and one can ask what this action corresponds to on the B side of derived Satake. The action corresponds to the inclusion of the group scheme of regular centralizers (possibly shifted?) which lives in $\mathfrak{g}[2]/G$!

(7/4/2021) Today I learned a fun fact about the derived category of an abelian category. Specifically, assume that you have a property which holds for the heart of the t-structure of the category and is closed under distinguished triangles. Then the property holds for all bounded complexes! The reason for this is an inductive argument on the length of the complex–if $-m$ is the lowest cohomology group on the complex, then one is tautologically allowed to consider the distinguished triangle $\tau^{\leq -m}F \to F \to \tau^{\geq -m+1}\mathcal{F}$ where the heart assumption shows the condition holds for $\tau^{\leq -m}F \simeq H^{-m}(F)$ and induction shows it for $\tau^{\geq -m+1}\mathcal{F}$.

(7/5/2021) Today I learned an explicit description for the ring given by the affine closure of the basic affine space when $G = SL_3$. Specifically, I learned that the ring is given by six variables given some arbitrary choice of pairing off, that the product of all those pairs must be equal to zero. This is because the product of the ring in the basic affine space is the direct sum of all of the representations, and the product of each weight for SL_3 either goes to a distinct highest weight vector or (the remaining three) go to the zero eigenspace of the adjoint representation.

(7/6/2021) Today I learned some pretty strong evidence to show that, in general for a given reductive group, the ring of global functions on G/N is not Cohen-Macaulay. This is because in the paper Contractions of Actions of Algebraic Groups, Popov defines the notion of a stable property of local rings, which is an open property which is stable under invariants of reductive groups and tensoring by the ring of global functions of reductive groups modulo its Levi. Popov then goes on to say other previously mentioned open properties are (for example, the property of integrality or having rational singularities) are stable, but omits the previously mentioned CM property. Since invariants of reductive groups are CM and CM is an open property, its likely that the latter condition doesn't hold. (Update: This turned out to be false and G/N is Cohen-Macaulay. I wrote the summary on MathOverflow.)

(7/7/2021) Today I learned the notion of a *rational singularity* on Y. Specifically, this is a singularity for which there exists a birational map $f : X \to Y$ from a regular scheme which induces an isomorphism on (derived) global sections $\mathcal{O}_Y \to f_*\mathcal{O}_X$. In particular, I learned that these rational singularities are also Cohen-Macaulay.

(7/8/2021) Today I learned that, assuming we have an affine algebraic variety X which has rational singularities, the $d+1$ local cohomology with respect to the diagonal can be computed as the d^{th} cohomology of the pulled back open subset to the respective open subset of the smooth resolution of singularities.

(7/9/2021) Today I learned that Chern-Simons theory (a 3D TFT) assigns the representations of the small quantum group to a point. This is one reason to justify that the representations of the small quantum group is a modular tensor category. I also learned one feature of the phrase 'modular tensor category' today–specifically, every object canonically comes with a *twist*, which is an endomorphism which plays well with the identity. The other unfamiliar part of the definition of modular tensor category I also (kind of) learned today–the modularity part says that the simples generate the endomorphisms of the identity.

(7/10/2021) Today I learned that, when $G := \mathrm{Sp}_4$ and A is the ring of global functions on G/U, we have an isomorphism

$$A \cong k[a,b,c,d,e,w,x,y,z]/(ad-bc+2e^2, ex-2cy+2aw, bx-ey+2az, dx-ew+2cw, dy-bw+ez).$$

Here, the variables a,b,c,d,e are the five one-dimensional weight spaces of the representation $\Gamma_{0,1}$ associated to the long fundamental weight, with a is a highest weight vector, and the first relation corresponds to the projection

$$\Gamma_{0,1} \otimes \Gamma_{0,1} \to \mathrm{Sym}^2(\Gamma_{0,1}) \cong \Gamma_{0,2} \oplus \Gamma_{0,0} \to \Gamma_{0,2}.$$

The variables x,y,z,w correspond to the four one-dimensional weight spaces of the representation $\Gamma_{1,0}$ associated to the short fundamental weight, with x a highest weight vector. The last four relations follow from the isomorphism $\Gamma_{1,0} \otimes \Gamma_{0,1} \cong \Gamma_{1,1} \oplus \Gamma_{1,0} \to \Gamma_{1,1}$. (Since $\mathrm{Sym}^2(\Gamma_{1,0})$ is irreducible, the only relation given by $\Gamma_{1,0} \otimes \Gamma_{1,0}$ is that the variables x,y,z,w commute.) This follows just as above, using computations carried out in 16.2 of Fulton and Harris.

(7/12/2021) Today I learned that for κ sufficiently negative, one can take the global sections functor $\mathcal{D}_{\kappa,I\text{-Mon}}(\tilde{Fl}) \to \hat{\mathfrak{g}}_\kappa$-Mod and, according to a theorem of Kashiwara and Tanisaki, this functor is fully faithful and t-exact, and the essential image of this map is given by explicit blocks of the category (including the regular block).

(7/13/2021) Today I learned a bit of the terminology used in quantum field theory. For example, I learned that the N in $N=?$ is a statement about how much 'supersymmetry' one has in the field theory. I also learned that, given a quantum field theory with supersymmetry, one can 'twist' the theory to obtain a different family. Furthermore, I learned an overview of mirror symmetry, which says that for certain theories attached to a space (loosely speaking) X, there is a mirror space Y such that a certain theory on X agrees with the twisted theory on Y.

(7/14/2021) Today I learned a neat idea in (topological/conformal?) field theory. Specifically, I learned the notion of an *interface*, which is just the name of a morphism of a boundary theory, and a *boundary theory*, which is a map from the trivial theory. Assuming some sort of dualizability, this is equivalently a map to the trivial theory. Therefore given two boundary theories, one can construct an interface from the trivial theory to itself. In particular, since this is given by a quantum field theory one dimension lower, we have a way to reduce dimensions given two boundary theories!

(7/15/2021) Today I learned why the convolution of $\mathrm{IndCoh}(\mathfrak{t}^* \times_{\mathfrak{t}^*//W^{ext}} \mathfrak{t}^*)$ preserves the $+$-subcategory! Specifically, it follows because pullback by a closed embedding is left t-exact (because the right adjoint ot a t-exact functor is left t-exact and all ind-affine embeddings of ind-schemes are t-exact) and the pushforward by an ind-finite (and thus ind-affine) scheme.

(7/16/2021) Today I learned a proof of a fact I probably should have known a long time ago. Any finite dimensional representation of any group splits uniquely (up to reordering) into a direct sum of its indecomposable objects. This is because if you have two decompositions of, say, V_i's and W_j's, then you can intersect both sides with W_1 (say) to see that $W_1 \cong \oplus V_i \cap W_1$ and so that, by indecomposability, there is exactly one V_i for which $V_i \cap W_1 = W_1$.

(7/17/2021) Today I learned that a left adjoint is an equivalence of categories if and only if it is conservative and the associated right adjoint is fully faithful (and the dual statement). Here is how to show that L is fully faithful from this information: It suffices to show that the unit $X \to RL(X)$ is an equivalence for all X. By conservativity of L, it suffices to show $LX \to LRL(X)$ is an equivalence, but by definition of adjoints, the composition $LX \to LRL(X) \to LX$ is the identity and the right hand map is given by $L(\text{counit})$ which is an isomorphism since R is fully faithful.

(7/18/2021) Today I learned that both eventually coconnective subcategories of the respective functors in the equivalence are preserved under convolution because convolution is left t-exact. This follows because one of the functors involved in the definition for each is t-exact, and the other is a right adjoint to a t-exact functor and therefore left t-exact.

(7/19/2021) Today I learned that the cofiber of a sequence of towers is the tower of the cofibers. In particular, the nilpotence of the associated sequence of cokernels is equivalent to the convergence of an effective limit.

(7/20/2021) Today I learned some ideas behind the notion of a Coulumb branch! Specifically, one can take the moduli space which parametrizes G-bundles on the disk with trivialization on the punctured disk, and this parametrizes the affine Grassmannian. Now, if one adds the additional data to the moduli space of a section of the associated bundle with respect to a given representation, one acquires a new moduli space, and the equivarariant BM homology ring is commutative and finitely generated, and therefore one can take its spectrum and obtain the Coulumb branch! (When the representation is zero, the section is canonically trivial, so the Coulumb branch at zero is the $G(\mathcal{O})$-equivariant BM homology of the affine Grassmannian.)

(7/23/2021) Today I learned the impications which imply how to get a W action on a category. Specifically, from a braid group action, if one has the data of square identity endomorphisms for each simple reflection which satisfies the natural conditions on the cubics *and* satisfies a commutivity condition for all w which exhibits the cancellation principle, then this action descends to a W action. Furthermore, if these identifications are merely maps (and thus need not be isomorphisms), then we get the data of a lax action from this (by a theorem of Polishchuk).

(7/24/2021) Today I learned that an a priori mysterious morphism given by $\mathbb{G}_m^{R(w)} \to T_w$ can be base changed by the incidence variety $X(w)$ in $G/N \times G/N$ such that the base change yields a certain incidence variety that is very close in nature to a Demazure resolution.

(7/25/2021) Today I learned that the pullback via a closed embedding of some sheaf concentrated in nonnegative degrees whose restriction is nonzero need not have cohomology in degree zero. This can even be tested at the level of $IC_{\mathbb{A}^1}$, because at a closed point of $\mathbb{A}^1$, $i^!(IC_{\mathbb{A}^1}) \simeq i^!p^!(k)[-1] \simeq k[-1]$.

(7/26/2021) Today I learned that a colimit $\text{colim}_\alpha \mathcal{F}_\alpha \in \text{IndCoh}(\mathcal{X})^\heartsuit$ is zero if and only if all structural maps $\mathcal{F}_\alpha \to \text{colim}_\alpha \mathcal{F}_\alpha$ are nullhomotopic. The reason is that since $\tau^{\geq 0}$ commutes with colimits, we see that $\tau^{\geq 0}\text{colim}_\alpha \mathcal{F}_\alpha \simeq \text{colim}_\alpha \tau^{\geq 0}\mathcal{F}_\alpha$, and, by assumption, each $\mathcal{F}_\alpha \in \text{Coh}(X)^{\leq 0}$ for some subscheme $X \hookrightarrow \mathcal{X}$. Now, since the maps $\mathcal{F}_\alpha \to \text{colim}_\alpha \mathcal{F}_\alpha$ are null homotopic, their truncations $\tau^{\geq 0}\mathcal{F}_\alpha \to \text{colim}_\alpha \tau^{\geq 0}\mathcal{F}_\alpha$ are zero. In an abelian category, a colimit is zero if and only if all structural maps are zero, so the claim follows since $\text{colim}_\alpha \mathcal{F}_\alpha \simeq \tau^{\geq 0}\text{colim}_\alpha \mathcal{F}_\alpha$.

(7/28/2021) Today I learned that there is a canonical t-structure on the ind-category for which ind of the heart is the heart of the ind. This in particular says that if you have any abelian category which is closed under quotients that you take the inductive limit of, since the image is a quotient and therefore every object in the ind-category has a subobject of the original category.

(7/29/2021) Today I learned the exact location where one uses the fact that $\mathfrak{t}^*//\text{W}^{ext}$ is given by not the union of graphs of the extended affine Weyl group, but the balanced product of the graphs of the affine Weyl group with the extended affine Weyl group (both for the Langlands dual). Specifically, when $G = \text{SL}_2$, we have that our extended affine Weyl group is given by the weight lattice semidirect product with the usual Weyl group. In particular, this fixes the element $1/2 \in \mathfrak{t}^*$, and so one can compute that the map by this quotient at $1/2$ would look integral.

(7/30/2021) Today I learned (solidly, finally?) that the Mellin transform takes the functors $\mathcal{D}(T) \to D(T)^{T,w} \to \mathcal{D}(T)^T$ to the Mellin dual side of $\text{IndCoh}(\mathfrak{t}^*/\Lambda) \to \text{IndCoh}(\mathfrak{t}^*) \to \text{IndCoh}(*)$, identifying $* \simeq \Lambda/\Lambda$. In particular, this shows the two Mellin dual definitions of a sheaf satisfying Coxeter descent are equivalent, since forgetting λ invariants (i.e. averaging by T, w) is conservative.

June 2021

(6/1/2021) Today I learned that there is a notion of the compactified space of Bun_B, known as *Drinfeld's compactification*, which has the property that it is stratified by dominant weights λ. I also learned that there is another version of compactification for GL_n known as *Laumon's compactification*. Push pull along Drinfeld's compactification yields compactified Eisenstein series.

(6/2/2021) Today I learned a small mistake I made in a draft of my paper. Specifically, I had hoped that the map from ind-coherent sheaves on the coarse quotient maps fully faithfully into the nondegenerate category. However, since it maps fully faithfully to the usual category $\mathcal{D}(N\backslash G/N)^T$ and the essential image contains the delta sheaf at the origin, it's unlikely the quotient functor is fully faithful.

(6/3/2021) Today I learned a notion of the Drinfeld double of a given Hopf algebra. Specifically, one can show that the Drinfeld center of a given (1-)monoidal category has an explicit description of a certain algebra which is as a vector space $A \otimes A^*$, with the commutativity relation given by R-matrices. I also learned these R-matrices are interpreted as *operators* on the category of representations, which are given a priori as an infinite sum but use the fact that the various e and f operate nilpotently.

(6/4/2021) Today I learned that it is a fact that the algebraic K-theory spectrum applied to topological K-theory has chromatic height two, and this is an instance of what is called the redshift conjecture. Furthermore, I learned a bit about the notion of how topological K-theory and algebraic K-theory relate–specifically, topological K-theory is the algebraic K-theory of the set of vector bundles.

(6/5/2021) Today I learned the notion of a *transchromatic character map*, which can be phrased as the existence of a map from Morava E-theory at height n which maps to a formal group of height $n-t$ for some integer t. Given such a t, the target of this map is constructed from the fact that, if $K(n-t)$ denotes an integral lift of Morava K-theory, then the localized $E(n)$ theory splits as a direct sum of some formal group of height $n-t$ and some copies of $\mathbb{Q}_p/Z_p$, and we can project to the former.

(6/6/2021) Today I learned a pretty interesting construction of the spectrum KU. Specifically, first one can construct a functor known as the *J-homomorphism* as follows. Specifically, given a (complex) vector bundle on a space V, one can take its one point compactification S^V and form the infinite suspension spectrum and then take the associated automorphisms as spectra. Since this is functorial in all the ways it needs to be, it constructs a map from the classifying space BU (a colimit!) to the full $(\infty, 0)$ subcategory of Spectra 'cut out' by S itself, written $BGL_1(S)$. One can include this J-homomorphism further into spectra, and take the colimit, and (amazingly!) it yields the spectrum which classifies complex bordism. Go figure!

(6/7/2021) Today I learned a bunch of fun facts about gerbes. For example, I learned that multiplicative gerbes on a group like $G((t))$ are defined to be compatible with the group structure. Furthermore, I learned that such gerbes are classified by maps from BG into the fourfold classifying space of $\mathbb{C}^\times$.

(6/8/2021) Today I actually learned the full fine print definition of the notion of a smooth representation of an affine Kac-Moody algebra, at least by what seems to be most people's conventions. Specifically, one can show that an affine Kac-Moody algebra is given by a canonical central extension of the loop Lie algebra by a central one dimensional basis element with distinguished basis element and the definition requires that this scalar acts by the identity. Further, since the cocycle which determines the central extension more or less has to do with the residue, any $t^N\mathfrak{g}[[t]]$ is a Lie subalgebra of an affine Kac-Moody algebra, and we require that some t^N acts trivially for the definition of smoothness.

(6/10/2021) Today I learned a fun fact about L-functions. Specifically, I learned that (at least at the conjectural level) that the L-function associated to any representation of the Galois group which doesn't have a trivial factor is entire. Furthermore, I learned that this is insensitive to throwing out finitely many factors, so that we don't need to think about ramified primes to prove this conjecture.

(6/11/2021) Today I learned a few facts about the tilting character formula proof. Specifically, I learned that the quotient functor to the Smith-Treumann category of sheaves on the μ_p fixed points of the affine Grassmannian preserves indecomposible parity complexes more or less *because* the coefficients are in characteristic p. Specifically, one can use the fact that the action of loops which factor through $\mathbb{G}_m/\mu_p$ with coefficients in a field of characteristic p is trivial and therefore inverting the Bott parameter is usually harmless.

(6/12/2021) Today I learned a really cool fact about the KZ equations! Specifically, these are certain differential equations depending on a level and a choice of Casimir which is a differential equation on the power of a curve minus a strong diagonal. It turns out that this differential equation is integrable, and so it it determined by an associated representation of the fundamental group (via its monodromy), and this monodromy is given by the R matrix as in the quantum group! I also finally learned what the p-series height is, and learned that, given a formal grou E, one can find a formal group law via $E^*(BS^1)$ and the map given by n-fold loop rotation $E^*(BS^1) \to E^*(BS^1)$ is the formal group law given by the n series $[n]$. Super exciting!

(6/13/2021) Today I learned some of the nitty gritty on how to get the various sheaves of interest on the affine Grassmannian. Specifically, one can take the Iwahori orbits labeled by an element of the affine Weyl group which are minimal with respect to their W orbit. This allows the closure to actually support a rank one local system on the Iwahori orbit, and then we may push this forward by one of the three closed embeddings to get the sheaves of interest.

(6/15/2021) Today I learned a basic form of Koszul duality, which says that if β is in degree 2, we have an equivalence of categories given by $k[\beta]$-Mod$\simeq$ IndCoh$(\Omega(\mathbb{A}^1))$. In particular, the quasicoherent part is entirely cut out by the part by which β acts nilpotently.

(6/16/2021) Today I learned how to de-conflate two things that I thought were the same. Specifically, although the affine Grassmannian for G is given as twofold loops on BG as a topological space, this is not the same algebro-geometric object as the ind-scheme. Specifically, the μ_p fixed points are given by a group scheme in the algebro-geometric story and a $\mathbb{Z}/p\mathbb{Z}$ in the honest loops story.

(6/17/2021) Today I learned that there are actually two variations on the category of $\mathcal{D}$-modules on $B\mathbb{G}_m$. Specifically, the renormalized version declares that the constant sheaf is compact, and this is the one for which $\mathcal{D}(B\mathbb{G}_m) \simeq k[\beta]$-Mod where β is a parameter in degree 2.

(6/18/2021) Today I learned that mimicing the formula of Riche-Williamson in computing the character formula via the affine Grassmannian will likely give me a similar character formula for the ℓ canonical basis with the 0 canonical basis!

(6/20/2021) Today I learned a roadmap to showing that the functor Av_*^{ψ} is conservative on the bounded by below subcategory. Specifically, since Av_*^{ψ} is t-exact, one can show that it suffices to show that a subquotient of the functor doesn't die to show that the object doesn't die.

(6/21/2021) Today I learned that there are different kinds of parity sheaves for different kinds of parity functions. This is how parity sheaves can be realized as tilting objects (say, on the affine Grassmannian) and can also be realized as IC complexes.

(6/22/2021) Today I learned that I didn't use the fact that my sheaves were nondegenerate in proving that the symplectic Fourier transform drops the Bruhat cell 'support'. I also learned that the $\mathbb{G}_m$ action given by loop rotation and the gerbe translations are distinct $\mathbb{G}_m$ actions.

(6/23/2021) Today I learned a parametrization of Nakajima quiver varieties, which are given by a choice of dominant weight. For each choice of dominant weight, there is a map from the (modified) universal enveloping algebra to the top Borel Moore homology of the Steinberg variety associated to the Nakajima quiver variety of that weight. I also learned that all representations can be obtained this way (i.e. given a representation, you can find a large enough dominant weight where it is in the image of this map... I'm a bit fuzzy on the details right now.)

(6/24/2021) Today I learned a fun way to define the n-fold based loop space on a topological space X. Specifically, the way one can do this is to realize the n-fold based loop space as maps from $\mathbb{R}^n$ to your space which are trivial outside a disk. With this definition, the notion of the $\mathbb{E}_n$ structure on this space becomes clear, because if we are given a few such maps, we can put them all in a bigger box to get another map.

(6/25/2021) Today I learned a bit of the technical details as to how to show that $\mathcal{D}(G) \to \mathcal{D}(G)_n$ can be made to be t-exact for a choice of the t-structure for the right category. Specifically, Sam argued in his affine Beilinson-Bernstein paper that you can do this for the invariants of a single subgroup, and so one can define for a general G category the notion of quotienting by monodromic objects (Sam also showed this notion is closed under subobjects so we maintain our t-structure).

(6/28/2021) Today I learned the fact that, while the symplectic Fourier transform applied to the delta function doesn't give a T equivariant sheaf for the usual adjoint T action, it does give one for the twisted T action where on both sides the multiplication is given by multiplying by the same scalar. I believe this is a reflection of a way by which we can construct that $T \rtimes W$ acts on $\mathcal{D}(G/N)_{\text{nondeg}}$.

(6/29/2021) Today I learned that Verma modules in category $\mathcal{O}$ and Weyl modules have something known as a *Jantzen filtration*. Specifically, this means that there is a decreasing filtration which eventually terminates for which each subquotient admits a contravariant form and the sum of the characters of the various subquotients is the same as the sum of the simple Verma/Weyl modules which are labeled by a simple root which lower the weight away.

(6/30/2021) Today I learned that groups acting on categories can be equivalently realized as sheaves of categories over the classifying space of the group (or its de Rham prestack). With this interpretation, it may be easy to give what it means for a free group on n generators to act on a category–it's just specifying what the generators are, which is similar to (at least in the algebro-topological story) the fact that the classifying space of a free product of $\mathbb{Z}$'s is a wedge product of circles.

May 2021

(5/1/2021) Today I learned a technical point about the groupoid that Lonergan uses to construct his Fourier dual description to sheaves on the bi-Whittaker quotient of $\mathcal{D}(G)$. Specifically, Lonergan does not use the union of all graphs of the (partially) extended affine Weyl group, but instead makes a copy of the affine Weyl group many graphs for each element of the fundamental group of the Langlands dual group. This solves a lot of technical problems, including the fact that I can only currently prove that the union of graphs for the affine Weyl group is ind-flat over $\mathfrak{t}$.

(5/2/2021) Today I learned a heuristic to explain how one could show that pullback via the map map $\pi : \mathfrak{t}/W^{\text{aff}} \to \mathfrak{t}//W^{\text{aff}}$ factors through those things which give trivial W representation structure. Specifically, assuming that I can show that any element of classical rings in $\mathfrak{t}//W^{\text{aff}}$ factors through discrete spaces, one can show that the restriction maps on the stacky points factor through the underlying scheme, and pullback by the underlying scheme corresponds (on a point where the action is trivial) to the trivial representation!

(5/3/2021) Today I learned a quick example to argue why, even given a finite flat map, the pullback functor on quasicoherent sheaves need not be fully faithful. Specifically, this fails for the map from two points to one, given that pullback preserves the structure sheaf and a functor is fully faithful only if the associated ring of global functions agrees!

(5/4/2021) Today I learned that you can combine the equivalences of $\mathrm{Rep}_q(G^\vee)$ with its Kac-Moody equivalent (due to Kazhan-Lusztig, representations of the affine Lie algebra which are integrable with respect to the loop group) with the newer equivalence by Campbell-Dhillon-Raskin which states that this Kac-Moody category is also Whittaker equivariant $\mathcal{D}$-modules on the affine Grassmannian. This gives a constructable description of representations of quantum groups.

(5/5/2021) Today I learned an interpretation from Gaitsgory's paper on a conjectural extension of the Kazhdan-Lusztig equivalence. Specifically, I learned that the representations of $\mathrm{Rep}_q(G^\vee)$ being equivalent to the $G(O)$ equivariant representations at negative level can be thought of as a constructable-$\mathcal{D}$ module sort of equivalence, like Riemann-Hilbert. I hope that one can move the FLE (for the parabolic $G(O)$) to a constructable side involving something like K(1).

(5/6/2021) Today I learned a bunch of details about the theory of quantum groups. Specifically, one can define, given a G-invariant bilinear form on the torus, the *quantized enveloping algebra*, which is a certain $\mathbb{Q}(v^{\pm 1})$ subject to certain relations given by the certain bilinear form (requiring all the roots pair to integers like 2). From this, one can take the divided powers so to speak (at least when we specialize the parameter to be a root of unity) to get an integral form of this quantized enveloping algebra, and from this, given any ring and any unit q of the ring, we can take a tensor product over $\mathbb{Z}[\pm v]$, we can take the corresponding tensor product and get the associated specialized algebra.

(5/7/2021) Today I learned a bit of the difference between K_1 and $K(1)$, or, said better, I learned an easier way to access something close to $K(1)$, written K_1, which is a direct sum of $K(1)$'s as a spectra. Specifically, one can take the complex K-theory spectrum KU (which, a theorem states, is obtained by inverting the Bott element of the suspension spectra of $\mathbb{C}P^\infty$) and taking the cofiber of multiplication by p.

(5/8/2021) Today I learned a source for the Kirillov model, which is a subcategory of sheaves which is isomorphic to the Whittaker subcategory when such a thing is defined. Specifically, this is defined by the kernel to the $\mathbb{G}_a$ action when it extends to a $\mathbb{G}_m \rtimes \mathbb{G}_a$ action.

(5/9/2021) Today I learned how to work with representations of the quantum group of the torus! Specifically, the underlying category, but to any nondegenerate bilinear form on the torus, we change the symmetric monoidal structure to a braided monoidal structure. In particular, the category $Rep_q(T)$ is still indexed by constructable sheaves valued in a field for the affine Grassmannian of the torus, although it can be more or less contorted to make that claim true!

(5/10/2021) Today I learned a fun way to construct the braided monoidal structure on 'quantum category $\mathcal{O}$'. Specifically, one can define a global Lie algebra and a global analogue of the affine Kac-Moody algebra, and with this, we can view the tensor product as taking a certain coinvariant algebra of the respective trivialization. This is the braided monoidal structure matching the one for representations of the quantum group.

(5/11/2021) Today I learned a sketch of a proof of why pulling back by the quotient map $\mathfrak{t}/W^{\text{aff}} \to \mathfrak{t}//W^{\text{aff}}$ is fully faithful. Specifically, since the quotient map $\mathfrak{t} \to \mathfrak{t}//W^{\text{aff}}$ is ind-proper itself, one can use Barr-Beck to show the category of ind-coherent sheaves on the coarse quotient is modules for the dualizing complex on (more or less) the union of graphs. In particular, one can check the fully faithfulness of the pullback functor on this particular object. There, by the conservativity of the covering map from $\mathfrak{t}$, one can check it as a map of quasicoherent sheaves on $\mathfrak{t}$. This can be checked pointwise, which follows from the fact that the pullback of the map $\mathfrak{t}/W^{\text{aff}} \to \mathfrak{t}//W^{\text{aff}}$ by some k-point $[\lambda]$ of $\mathfrak{t}//W^{\text{aff}}$ is given by $*/W_{[\lambda]}$.

(5/12/2021) Today I learned a more refined statement about the Betti geometric Langlands conjecture. Specifically, this says that given a curve with a finite subset of points, we can obtain the category of constructable sheaves with nilpotent singular support on the moduli space of G-bundles on the curve with reduction to N at the points labeled by S, and this is supposed to be equivalent to indcoherent sheaves with nilpotent singular support on LS of the Langlands dual group with a $B^\vee$ reduction to all points. The N reduction for a torus corresponds to reducing from a point, so it matters even in the torus!

(5/13/2021) Today I learned an interesting conjecture by Henning Haahr Andersen, which says that for a given dominant weight in the lowest p^2 alcove, the tilting characters for a quantum group are identical to those given by the associated representations of characteristic p quantum groups.

(5/14/2021) Today I learned the notion of a *gerbe* and a *multiplicative gerbe*, at least for the latter on the affine Grassmannian. Specifically, a gerbe with values in some ring R is a map from $\Omega^{-2}R^\times$, and a multiplicative gerbe is one given by realizing the affine Grassmannian as double loops on BG and requiring the map is a map of double loop algebras.

(5/15/2021) Today I learned a perspective on the thing I proved (and something I didn't!). Specifically, I learned that even though my original task was to prove that $\mathcal{H}_\psi \to \mathcal{D}(T)^W$ was fully faithful, the method I proved it by using a universal case argument shows a more general functor is fully faithful. Then, I can also show that the left adjoint is comonadic, which is analogous to classifying the essential image.

(5/17/2021) Today I learned what a $\mathbb{C}^\times$ gerbe buys you. Specifically, running through some of the standard homotopical arguments, one can see that equivalently a gerbe gets you a map to $B^3\mathbb{Z}$.

(5/18/2021) Today I learned that there's a lot known about the Morava K-theory applied to a finite group. Specifically, for a fixed p and finite group H with an abelian maximal p-subgroup T, the rank of $K(n)_*(BH)$ computes the number of orbits of the normalizer acting on the p group T^n.

(5/19/2021) Today I learned one of the facts I actually have to show if I want to argue the explicit description of H_ψ in terms of sheaves on $\mathfrak{t}^*//W^{\text{aff}}$, i.e. the extended affine Weyl group. Specifically, there are two maps from H_ψ to $QCoh(\mathfrak{t}^*)$–one is given by forgetting one side's Whittaker invariance and then taking the N, T, w average (which factors to W invariants), and the other side is given by averaging to G, w and then forgetting down to the B, w equivariance and then averaging up via the Beilinson-Bernstein map. The agreement of these corresponds to the agreement of the two different pullbacks on the indcoh side of $\mathfrak{t}^*/W \to \mathfrak{t}^*//W^{\text{aff}}$.

(5/20/2021) Today I learned I made a mistake in my proof earlier that the pullback functor on the GIT quotient is fully faithful. Specifically, I learned that I assumed the pullback of the generator is one dimensional.

(5/21/2021) Today I corrected that mistake! Wooh! The main idea is that I can check that a map in $IndCoh(\mathfrak{t}^*//W^{\text{aff}})$ is an equivalence by checking it on each k-point. On each k point, you can explicitly compute the fiber product $* \times_{\mathfrak{t}^*//W^{\text{aff}}} \mathfrak{t}^*/W^{\text{aff}}$ as the fiber of the usual characteristic polynomial map, and therefore you can obtain fully faithfulness and the essential image.

(5/22/2021) Today I learned the notion of type n (p-local) spectra. Specifically, I learned a useful proposition which says that if $K(n)_*(X)$ vanishes for some p-local spectrum X, then so too does $K(n-1)_*(X)$, and so in particular we may define the *type* of X to be the minimal such X such that $K(n)_*(X)$ doesn't vanish.

(5/23/2021) Today I learned a bit about the quantum Frobenius. Specifically, the quantum Frobenius on type one representations (for ℓ an odd prime, say) maps into the universal enveloping algebra of the usual Lie algebra over $\mathbb{C}$, and kills the usual E's and F's, but not the divided powers or the K's!

(5/24/2021) Today I learned an alternative condition of satisfying Coxeter descent, I think. Specifically, I think the Mellin transform of the condition on W^{aff} equivariant sheaves allows you to forget the Λ equivariant structure (following Lonergan's remark paper) so you can check descent by just averaging and checking the associated representation is trivial. This is word salad I think, but hopefully I have a paper coming out with this soon!

(5/25/2021) Today I learned that one can define a pairing on the Kazhdan-Lusztig category at negative level κ which integrates the objects to live as $\mathcal{D}$ modules on Bun_G (or possibly the dual category) and then tensors them with some sheaf and then pushes the tensor forward to Vect. I also learned that this factors through the small quantum group if the sheaf you tensor with is a Hecke eigensheaf.

(5/26/2021) Today I learned a universal property of the braided monoidal structure on quantum category $\mathcal{O}$. Specifically, one can define the notion of *coinvariants* of the tensor product of many $\mathfrak{g}_\kappa$ representations after one chooses trivializations at each of the finitely many points we removed. We may then define a braided monoidal structure such that maps into its dual from some space are the coinvariants of the three points. This, at least, to me, gives me a conceptual insight into the braided monoidal structure, since we can imagine two of those three points varying.

(5/27/2021) Today I learned an explicit expression for Hochschild homology. Specifically, one can view the 2-category of $k-$algebras with morphisms given by bimodules. Then, all objects are dualizable and self dual, and the associated trace map maps from the unit k and yields Hochschild homology as the associated trace. In fact, I learned there is a natural map $\tau : A \to HH_*(A)$ which has the property that $\tau(ab) = \tau(ba)$ and that $HH_*(A)$ is universal with respect to this property.

(5/28/2021) Today I finally learned a reasonable interpretation for myself as to why a gerbe, defined to me as a BG torsor for some discrete (say) abelian group G, can be used to give a twisted sheaf. Specifically, one can note that since $* \to BG \to *$ yields the fact that sheaves on any space X can be identified as sheaves on BG of the form $k_{BG} \boxtimes \mathcal{F}$, one can define gerbes to be sheaves on a gerbe (i.e. a BG torsor) whose restriction to the second factor locally is isomorphic to the trivial sheaf. I guess actually in the source I looked up it's not the constant sheaf, but rather the standard representation of $\mathbb{G}_m$ corresponding to the identity. That makes sense too, you can decompose it into a sum of graded pieces and therefore ask that your sheaf be entirely concentrated in degree one (not zero).

(5/29/2021) Today I learned an un-sexy simplifying step in the proof of the comonadicity of the various plus categories of the two categories I want to show is equivalent. Specifically, noting that evaluation and the forgetful functor are conservative and conservative functors preserve effective limits, I was able to simplify the proof that the evaluation map $\text{End}_{\mathfrak{t}^*//W^{\text{ext}}}(IndCoh(\mathfrak{t}^*))$ is comonadic.

(5/30/2021) Today I learned enough to put the words together to make the mostly tautological claim that $\mathfrak{t}^*//W^{ext}$ is a 0-truncated locally finite type prestack. This is because this condition is closed under colimits, and so it's true for $\mathfrak{t}^*$ and $\Gamma_{W^{\text{aff}}}$ (the latter using the closed under colimit condition again) and therefore the colimit of this diagram is.

(5/31/2021) Today I learned a very fun fact called the blueshift theorem which states that the Tate valued Frobenius lowers chromatic level. This is supposed to be inverse to the *redshift* conjecture which states that K theory raises chromatic level by one, and indeed, I learned informally that this idea was used to prove cases of the redshift conjecture by using the (true) blueshift conjecture.

April 2021

. (4/1/2021) Today I learned one fact related to the Tannakian formalism for quantum groups. Specifically, one can recover the monoidal category of $Rep_q(G)$ via its forgetful functor, but the forgetful functor to vector spaces is not monoidal. This is captured by the notion of R-matrices.

(4/2/2021) Today I learned the utility of the twisted arrow category! Specifically, the twisted arrow category of a category is defined as the category whose objects are maps of two objects and the morphisms are those whose domain is covariant and codomain is contravariant (kind of hasty, but look it up). The morphisms of the twisted arrow category are defined such that any map of functors is given by a limit of maps of the twisted arrow category! I also learned that, in the Moy-Prasad filtration of a reductive group, the quotient of a group at a certain depth by the 'next' group is abelian.

(4/3/2021) Today I learned that the union of graphs is flat in type A2, A3, B2, B3, and G2. This was also the first time I seriously used Macaualy2 in a more programmer style version to show this. I also learned the notion of BRST reduction, and that this gives a $\mathcal{W}$ representation of a representation of an affine Lie algebra, and that Feigin-Frenkel conjectured in the 90's that any simple module will either be killed or go to a simple module. I learned that this is proven in progress by Gurbir Dhillon!

(4/4/2021) Today I learned of a fun open problem–the open problem is: is there a cube such that the lengths of all diagonals are integers? This was solved by Euler if you don't require the long diagonal to be integers.

(4/5/2021) Today I learned a rough analogue on how to define things like curvature in algebraic geometry. Specifically, given some function h on a space X, we may define a twisted connection determined by the rule $\nabla(fs) = f\nabla(s) + h\partial fs$, obtaining a short exact sequence equivalent to $0 \to End(\mathcal{E}) \to At(\mathcal{E}) \to O_X \to 0$ (where $\mathcal{E}$ is the underlying vector bundle), and a connection is a splitting of this sequence, and a choice of extension class in that short exact sequence determines curvature.

(4/6/2021) Today I learned a normalization condition one can use to sort of get the bearings on Koszul Duality. Specifically, one can realize that $k[\epsilon]/(\epsilon^2)$ is identified with the exterior algebra of k as a $k[x]$ module, and then upgrade this to an equivalence of derived categories of graded algebras which sends the trivial $k[\epsilon]$ module to k. Then, one can check by specific computations that this sends a shift of $k[\epsilon]$ to the Tate twist of the shift!

(4/8/2021) Today I learned a bit more about K3 surfaces. For example, a smooth projective geometrically irreducible surface is K3 if the dualizing sheaf is the structure sheaf (up to shift) and there is no h^1. I also learned the definition of the Brauer group, which is the H^2 group cohomology with coefficients in $\mathbb{G}_m$.

(4/9/2021) Today I learned a soft fact about why, to obtain information about the union of graphs inside of $\mathfrak{t} \times \mathfrak{t}$, I may work in the product of two $Spec(C)$, where C denotes the coinvariant algebra. This is an explicit computation and makes the approach of showing the increase of degrees of the ideals amenable to finite dimensional k-vector space arguments, hopefully.

(4/10/2021) Today I learned the definition of a Hilbert series of a graded module over a polynomial algebra! It's actually a pretty easy definition–it's just the sum of the dimension of the various graded components. This is very nice because this told me (via Macaualy2) that if I take a closed subset with d length ℓ elements in it, the associated quotient algebra I will get by quotienting out by those graphs of the Weyl group has the expected Poincare series, i.e. in particular having d length ℓ elements!

(4/12/2021) Today I learned a minor point in the most general version of the Geometric Satake, specifically something about what happens when one tries to apply it over the sphere spectrum. The idea is that there is a natural $\mathbb{Z}/2\mathbb{Z}$ torsor on the affine Grassmannian, and the 'true' Geometric Satake incorporates this twist.

(4/14/2021) Today I (re)-learned the definition of a Zastiva space on a curve! Specifically, one can ask for maps from a curve into a quotient stack $G/(N^-B)$ which generically land in the point. This also has motivation from the affine Grassmannian, for the same kind of idea for the comultiplication of the isomorphism $U\mathfrak{n}^\vee \simeq \oplus_{\mu\in\Lambda} H_c^{\text{top}}(S_\mu \cap T_0; \mathbb{C})$ of Braverman Finkelberg Gaitsgory and Mirkovic.

(4/15/2021) Today I learned a theorem of Lubin-Tate, which classifies deformations of a formal group law. Specifically, given an infinitesimal thickening A of a perfect field of characteristic p κ, Lubin-Tate classifies *deformations* of a formal group law over κ, i.e. a formal group law whose restriction to $Spec(\kappa)$ is the fixed one. More specifically, Lubin-Tate theory says that such deformations are classified by κ-linear ring maps from $W(R)[[v_1, ..., v_{n-1}] \to A$. This universal law is given by taking any lift of the canonical ring map $W(R)[[v_i]] \to \kappa$ killing the v_i and p to R.

(4/16/2021) Today I learned the difference between two morphisms I was confusing, the *geometric Frobenius* and the absolute Frobenius. The geometric Frobenius is defined on schemes of the form $Y_0 \times_{Spec(\mathbb{F}_q)} Spec(\overline{\mathbb{F}_q})$, and is given by $\Phi \times id$ where Φ takes an $\mathbb{F}_q$ algebra point $\phi : \text{Spec}(A) \to Y_0$ to $\text{Spec}(A) \to \text{Spec}(A) \to Y_0$ where the first arrow is induced by the Frobenius ring map. The advatage of this 'geometric Frobenius' is that it's a map of $\overline{\mathbb{F}_q}$ schemes, which is the realm where 'geometric' algebraic geometry occurs.

(4/17/2021) Today I learned the notion of a *Slodowy slice* for a nilpotent element in the Lie algebra $\mathfrak{g}$. Specifically, given such a nilpotent element e, one can construct an associated $\mathfrak{sl}_2$ triple, and define the slice as the slice $e + ker(ad(f))$ for an associated $\mathfrak{sl}_2$ triple to e.

(4/18/2021) Today I learned a possible strategy will work to show that objects of the Whittaker-Hecke category centralize $\mathcal{D}(G)$. Specifically, given our functor $\mathcal{H}_\psi \to \mathcal{Z}(\mathcal{H}_{N,\text{nondeg}})$, assuming we can show that there is a family of generators of the category $\mathcal{H}_N$ for which the quotient functor to the nondegenerate category is fully faithful on the heart, we can by hand construct a functor $\pi_{\leq 1}\mathcal{H}_\psi \to \mathcal{Z}_{\text{Dr}}(\pi \leq 1(\mathcal{H}_N) \simeq \pi_{\leq 1}\mathcal{Z}(\mathcal{H}_N)$. Here's hoping, anyway. (Although other people seem to think this is false so this is unlikely...)

(4/19/2021) Today I learned how, given a generalized Cartan matrix, one can construct a Kac-Moody algebra as sort of the Lie algebra with the appropriate generators and relations of the associated GCM. The associated algebra to a positive semidefinite matrix gives rise to an affine Lie algebra!

(4/20/2021) Today I learned the definition of the word endo-trivial representations of a finite group over a field of characteristic p. Specifically, these are the representations whose associated endomorphism representation splits as a direct sum of the trivial module (associated to the identity map) plus a projective module.

(4/21/2021) Today I learned a general overview of how Beilinson, Gelfand, and Gelfand constructed a top degree function on the ring of cohomology. Specifically, they choose a generic element $h \in \mathfrak{t}$ and obtain a polynomial Q for which $Q(w_0 h) \neq 0$ while $Q(wh) = 0$ for other $w \in W$. Using this and the fact that one can write Q as a sum $Q = \sum_w g_w D_w(\rho^\ell)$ for invariant g_w, one can get an associated Q of degree ℓ via taking the polynomial $Q = \sum_w g_w(\rho) D_w(\rho^\ell)$.

(4/22/2021) Today I learned a construction of Morava K-theory. Specifically, one can take the complex bordism spectrum MU and kill all the generators except for the indexed one and then invert the final generator! I also learned there are uncountably many associative algebra structures on these, but the underlying spectrum is equivalent.

(4/23/2021) Today I learned a theorem about the *moduli stack of oriented Formal groups*, which parametrizes the data of a formal group over A with an identification of the canonical sheaf on the group with the two shifted loops on that group. Specifically, this stack is equivalently a stack which returns a point if the derived ring is complex oriented and the empty set otherwise.

(4/24/2021) Today I learned a possible approach to proving the flatness of the union of graphs of the Weyl group in $\mathfrak{t} \times \mathfrak{t}$. Specifically, I hope to copy the approach in BGG view the h from above as a formal variable, and use some kind of extension property to rig the polynomial to work for me.

I also learned that the quotient of $k[x, y]$ by the *diagonal* $\mathbb{Z}/2\mathbb{Z}$ invariants is three dimensional!

(4/26/2021) Today I learned the notion of a *Grothendieck spectral sequence*. This essentially is the sequence you get from noting that you can't recover the composite of two derived functors from their cohomology, but of course, they talk to each other. Specifically, the spectral sequence of two functors F, G has second page $E_2^{p,q} = RF^p \circ RG^q(A) \implies R^{p+q}(F \circ G)$. I also learned that the critical notion to define the derived functor of F is F acylicity. I bet I sort of knew that already, but it was good to hear out loud.

(4/27/2021) Today I (hopefully) learned a way to prove that there's a function on $Sym(\mathfrak{t} \times \mathfrak{t})$ which vanishes on all graphs except the graph of w_0 and is nonzero on the coinvariant algebra. The key new idea was to notice that, given an old polynomial you could apply the above machine to, you can always divide out by the $f(x, w_0x)$.

(4/28/2021) Today I learned the technical difference between an affine Kac-Moody algebra, an affine Lie algebra, and a loop algebra. Specifically, one can obtain the affine Lie algebra by adding an element corresponding to a central charge to the (analogue of the) Cartan, and one can obtain the affine Kac-Moody algebra by taking the semidirect product of the affine Lie algebra with a derivation.

(4/29/2021) Today I learned (okay, I've learned this before, but now all the words make sense) what homological mirror symmetry is saying. Specifically, one can take a symplectic manifold X and a 'mirror' Y. Then homological mirror symmetry says that the Fukaya category of the X matches the bounded derived category of the coherent sheaves on the mirror.

March 2021

(3/1/2021) Today I learned the analogue of the punctured disk in the setting of a local number field. Specifically, for a function field, but given a perfectoid space instead of a perfect $\mathbb{F}_q$ algebra (because, at least heuristically in the paper I'm looking at, we need to replace perfect algebras with perfectoid ones so that we have continuous sections of bundles on which to do things like Hecke modifications), they are locally of the form Spa$(R, R+)$, where R is a perfect topological $\overline{\mathbb{F}_q}$-algebra with a *pseudouniformizer* $\omega \in R$, i.e. a topologically nilpotent element making it a Banach algebra over $\overline{\mathbb{F}_q}((\omega))$. Well, okay, I learned the definition of perfectoid Tate $\overline{\mathbb{F}_q}$ and the fact that you can characterize the punctured disk for a function field in such a way that the same definition for the punctured disk for such an algebra works over any local number field.

(3/2/2021) Today I learned a theorem which says that the quivers whose category of representations has finitely many indecomposible objects are precisely those quivers which are simply laced Dynkin diagrams (with the arrows pointing in an arbitrary dimension).

(3/3/2021) Today I *actually* learned the analogue of the punctured disk in the setting of a local number field. Specifically, given the field of Laurent series in $\mathbb{F}_q$, one can characterize the punctured disk as those elements for which the uniformizer t acts invertibly. When considering R families of a punctured disk, where R is a perfectoid $\overline{F}_q$ Tate algebra, one also needs to require that the pseudouniformizer acts by a unit. This definition can be applied to the Witt vectors $W_{\mathcal{O}_E}(R^+)$, where R^+ is the given subring of the perfectoid ring (R, R^+).

(3/4/2021) Today I learned the content of Artin-Schrier theory (possibly again), which classifies the degree p Galois extensions of a field of characteristic p. Specifically, any such extension is given by a polynomial of the form $x^p - x - 1$, and conversely!

(3/5/2021) Today I learned a way you can argue that the stabilizer of any element of $\mathfrak{t}$ is generated by reflections. Specifically, given an x in the stabilizer, you can pick the alcove that x is in and draw a path to the fundamental alcove, and move it a little such that it doesn't hit the intersection of any two hyperplanes.

(3/6/2021) Today I learned that, given a collection of a closed subroot system of a root system, one can define a *Coxeter element* as the product of all of the associated simple reflections (in order!). It turns out that all of these Coxeter elements do not fix any element of $\mathfrak{t}$.

(3/8/2021) Today I learned the full proof why if you add a new hyperplane associated to the graph of an element in the affine Weyl group, it will intersect a component in codimension $dim(\mathfrak{t})-1$. Specifically, one can index all of the points not by $(\lambda, w\lambda)$, but, rather, $(\lambda, w\lambda')$ where λ' is the translation via the affine Weyl group orbit to the fundamental alcove.

(3/9/2021) Today I learned a statement of geometric class field theory! Specifically, one can show that the character sheaves on the picard group of a smooth projective algebraic curve (in other words, maps from the curve to $B\mathbb{G}_m$) can be identified with the moduli space of local systems. The idea of the proof is this: one can define a map from the 'symmetric power' of the curve to the picard group (via identifying line bundles with their class group) which ends up being a torsor for projective space. In particular, the π_1 of both spaces are identified, and so given a local system on the curve labeled by a divisor, you can use this idea to descend.

(3/10/2021) Today I learned the general framework of groupoid objects. Specifically, once you get a groupoid object in a category, if you can take the geometric realization, you get all the relevant diagrams you want to commute. However, the only way you can canonically get a groupoid object in a category is to start with a *single* morphism. In other words, you can't declare what the groupoid is upfront.

(3/11/2021) Today I learned (at least at a sketchy level) the reason that sometimes stacks are required to have a diagonal affine morphism. Specifically, as far as I understand it, for Artin-esque stacks, this makes the stacks 'Tannakian affine' in the sense that the canonical map $X \to \text{Spec}(\text{QCoh}(X))$ is an equivalence.

(3/12/2021) Today I learned the actual full definition of the singular support of a sheaf on a space X. Roughly, you can characterize it as the subset of the cotangent bundle for which you can find a local function whose derivative is in that codirection and the local sections change (or, at least for a smooth X, where the nearby cycles vanish).

(3/13/2021) Today I learned a combined theorem of Nadler-Yun and Sam Raskin and friends which says that the ind-constructable sheaves on Bun_G for which the Hecke operators (indexed by points on the curve) factor through $\text{Lisse}(X)$ is *precisely* those ind-constructible sheaves with nilpotent singular support on Bun_G. I also learned a more precise definition of this nilpotent singular support condition. Specifically, there is a subset of the cotangent space of Bun_G given by... 'Nilp.'

(3/15/2021) Today I learned a possible method for showing that the map $\Gamma_{W^{\text{aff}}}$ is ind-flat. Specifically, note that this ind-scheme obtains an action of the integral coroot lattice, and the quotient by this lattice should be the union of graphs for the standard W. Therefore, since the map $\Gamma_W \to \mathfrak{t}$ is flat, the composite of these maps should be ind-flat.

(3/16/2021) Today I learned a way to define microlocal sheaves on a given space X. Specifcally, it's the sheafification of the following presheaf: Given an open subset of X, we take all (bounded derived category of) constructible sheaves on X and quotient out by those whose singular support lives entirely on the complement of the open subset.

(3/17/2021) Today I learned a way to characterize where the perverse sheaves on the constructible side of the tamely ramified equivalence of Bezrukavnikov goes. Specifically, given an algebraic stack, one can define the notion of a perversity function on the stack, and define perverse coherent sheaves of middle perversity.

(3/18/2021) Today I learned the full statement of Borel's thesis. Specifically, Borel shows that the cohomology of a flag variety of a semisimple algebraic group is isomoprhic to the coinvariant algebra as a graded algebra which respects the W representation structure! I'm not actually sure if Borel actually proved all that in his thesis himself. However, this doesn't say that there's some square one element in this coinvariant algebra, just by looking at the example of $\mathfrak{sl}_2$, wherein the algebra $k[\epsilon]/(\epsilon^2)$ and $s \in \mathbb{Z}/2\mathbb{Z}$ acts by $\epsilon \mapsto -\epsilon$. There is no square one element in this ring, besides ± 1, but neither of these things send $a + b\epsilon \mapsto a - b\epsilon$!

(3/19/2021) Today I learned a hope/conjecture I have. Specifically, I hope that one can take a w in W and consider the union of graphs of the elements $\leq w$ in the Bruhat order, and that, when I do this, the Borel isomorphism identifying the cohomology of the flag variety G/B with the coinvariant algebra of $\mathrm{Sym}(\mathfrak{t})$ surjecting onto the associated Schubert variety indexed by w corresponds to the union of graphs under w, resricted to zero.

(3/20/2021) Today I learned a few constructions on the loop space! First off, I re-learned the statement of HKR, which says that, over a field of characteristic zero, we may identify the loop space of a (derived) scheme with its (-1)-shifted tangent space. I also learned that the category of sheaves on a loop space is $\mathbb{E}_2$, with monoidal structure given by convolution according to the cobordism given by a pair of pants.

(3/22/2021) Today I learned a really cool fact which says that you can identify the Steinberg variety with loops on the space $B\backslash G/B$. Moreover, the identification is pretty formal–one can push symbols around to see that $B\backslash G/B \simeq \backslash B(* \times_{BG} *)/B$.

(3/23/2021) Today I learned a fun fact about shifted Cartier/Pontriagin duality. Essentially, in many cases for abelian group schemes A, we can identify quasicoherent sheaves on A with QCoh sheaves on its shifted dual $A^{\vee}[1]$. In the case where A is the de Rham space of a vector space, V this shift is self dual (which you can see by resolving A as the kernel of the usual vector space by its formal neighborhood of zero) which yields $\mathcal{D}(V) \simeq \mathrm{QCoh}(A^{\vee}[1]) = \mathrm{QCoh}(Hom(A, B\mathbb{G}_m))$, which is identified as the space of flat connections on A!

(3/24/2021) Today I learned that the two realizations theorem of the affine Hecke algebra has a parabolic version which has been proven by Bezrukavnikov and Losev. The proof uses the parabolic flag variety and the Steinberg variety (specifically, the version whose version for $P = B$ is $\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathcal{N}}$) and the natural functors to the $P = B$ version and identifies the difference between the category and their essential image.

(3/25/2021) Today I learned a technical fact that, because of a theorem of Nadler-Yun and its converse (proven by AGKRVV) which says that any ind-constructible sheaf which has nilpotent singular support has Hecke functors factoring through the natural functor from $Shv(Bun_G) \otimes Lisse(X)$. Specifically, using this and Beilinson's spectral projector, in families, one can define a quotient functor, at least for affine scheme points of the stack of restricted local systems. For affine points, these will be a left adjoint, but for general points, this functor is only a right adjoint.

(3/26/2021) Today I (re?)learned the notion of a shifted symplectic form! Specifically, one can define it as an isomorphism of the tangent and cotangent bundle subject to some alternating condition. In particular, this explains why BG is 2-shifted symplectic–one can compute that the tangent sheaf is $\mathfrak{g}[\pm 1]$ and the cotangent bundle is the opposite, so the identification is literally given by the Killing form!

(3/27/2021) Today I learned (did?) the explicit computations of the Demazure operators on the coinvariant algebra of the torus for $\mathfrak{sl}_3$. Specifically, you can just label the directions given by the simple positive roots as x and y and note $\rho = x + y$. Then you can easily see, for example, that $D_x(\rho^3) = x^2 + 3xy + 3y^2$ by explicit computation. Specifically, the reason that the constant $\frac{1}{\text{length!}}$ appears is, at least in $\mathfrak{sl}_3$ but I suspect in general, so that if you go all the way down to the base field, the constant that appears is 1!

(3/29/2021) Today I learned the explicit coordinates on the *invariant* algebra of $\mathfrak{sl}_n$, which are explicitly given by the invariant functions on vectors pointing in the direction *of* the hyperplanes.

With this, I learned that the ring given by killing the third graded piece of $H^*(SL_3/B)$ is not flat.

(3/30/2021) Today I learned that, given a closed subset of the Weyl group of $\mathfrak{sl}_3$, the corresponding quotients obtained by killing the various Demazure operators applied to $\rho^3/3!$ yield the various unions of the hyperplanes. In turn, I learned that $\mathfrak{t} \times_{\mathfrak{t}//W^{\text{aff}}} \mathfrak{t}$ is probably an ind-scheme for either SL_3 or PGL_3.

(3/31/2021) Today I learned a near tautological once you write it down fact that, allowing yourself coefficients in the field of rational functions of $\mathfrak{t}$, one can write the Demazure operator indexed by w as a linear combination of the operators indexed by those $v \in W$ which are $\leq w$.

February 2021

(2/1/2021) Today I learned that truncation functors commute with all filtered limits in a pro category, essentially by construction. This is, conceptually, because pro categories are formally dual to the ind-completion of the category, and here all truncations commute with filtered colimits.

(2/2/2021) Today I learned that, in addition to the fact that you can identify a group via the automorphisms of its fiber functor, you can also identify the Lie algebra as *end*omorphisms of its fiber functor. I also learned a basic case of Langlands duality–because for an oriented 3 manifold M, one can classify the space of line bundles by $H^2(M, \mathbb{Z})$. Therefore the dual group is maps from the first homology to an abelian group, so equivalently, maps from the first fundamental group to $\mathbb{C}^\times$ which classifies local systems!

(2/3/2021) Today I learned a 'physical' interpretation of why the sperical Hecke algebra is commutative. Specifically, Geometric Langlands is actually a special case of an equivalence of two four dimensional quantum field theories, and the Hecke operators (physically called T'Hooft operators) correspond to line operators on the theory, which in the two dimensional slice associated to Geometric Langlands, are just certain lines which act on the boundary conditions. These lines can be moved around in a four dimensional theory to show they commute, and, since it's 4 and not 3 dimensional, they commute in a canonical way (although the higher homotopies remain!).

(2/4/2021) Today I learned a kind of fun proof of the fact that, for a manifold M, $H^2(M, \mathbb{Z})$ classifies complex line bundles on M. Specifically, one can use the fact that $B^2\mathbb{Z} \cong \mathbb{C}P^\infty$ and how cohomology and looping are related. Very fun!

(2/5/2021) Today I learned this crazy physical fact which can be said as 'light is polarized.' The way this manifests itself in reality is that if you shine light through a vertical slit and shortly thereafter a horizontal slit, none will go through. But if you instead put a bunch of 'interpolating' slits which slowly change the angle, the light will go through. I also learned a bit about electric fields and their interpretation–roughly, they are similar to the force on a pebble in the ocean.

(2/6/2021) Today I learned one reason that the two types of sheaves on $\mathfrak{t}//W^{\text{aff}}$ may agree. Specifically, one can hope that both functors $QCoh^*$ and $IndCoh^!$ send (at least filtered) colimits of schemes to limits of DG categories, and then the fact that Υ is functorial with respect to these will give the desired result, I hope.

(2/8/2021) Today I learned that the above characterization is slightly wrong, sadly. Specifically, in the definition of $\mathfrak{t}//W^{\text{aff}}$, the union of graphs is used, which is in particular not smooth!

(2/9/2021) Today I learned a fact from Elijah Bodish about computing Kazhdan-Lusztig polynomials of certain Schubert varieties. Specifically, if the variety is actually smooth, then the IC cohomology is the same thing as the usual cohomology, and for the usual cohomology, the KL polynomial is the same thing as the Poincare polynomial.

(2/10/2021) Today I learned a general construction of the Hecke eigensheaf! The way this works is this–using the isomorphism of global differential operators on Bun_G with functions on $G^\vee$ opers, given a $G^\vee$ oper, you can localize the associated sheaf to obtain what turns out to be a Hecke eigensheaf!

(2/11/2021) Today I learned an overview of how Gus Lonergan proves his theorem regarding the equivalence of the nil-Hecke algebra modules and the sheaves on $\mathfrak{t}^\vee$ satisfying descent. Specifically, he used the Satake theorem which says that these modules (for a parameter) can be realized as the loop rotation equivariant cohomology on the affine Grassmannian, which, through some manipulations, can be realized as 'close to' the groupoid of the loop rotation equivariant cohomology of the flag variety (roughly, a $W \times W$ cover). One can then use the fact that these are groupoids and can prove some theorems about descent of groupoid (ind-)schemes.

(2/12/2021) Today I learned a theorem about the stack of restricted local systems. Basically, a way to interpret the theorem of the Langlands crew is to say that they've defined a stack of local systems on a curve X where if you pick a point x and rigidify the local system there (to remove the stackiness, roughly) it's reduction is the disjoint union of its k-points.

(2/13/2021) Today I learned about the notion of a convergent prestack, which says that the evaluation on a test affine scheme can be determined by the limit of the various truncations of the scheme. This has technical importance, because I also learned today that I seem to be able to recover the formal equivalence $(\mathfrak{t}^\vee//W^{\text{aff}})_{\overline{\lambda}}$ formally completed with $(\mathfrak{t}^\vee//W_{[\lambda]})$ formally completed.

(2/15/2021) Today I learned that, not only is it possible to snow a lot in Austin, Texas, but also a perscriptive recipe which allows one to show a certain X fitting inside $Y \to X$ where Y admits deformation theory. The conditions are that X satisfies etale descent, the map is locally surjective, Y satisfies deformation theory absolutely and relatively to X, and that Y is formally smooth over X. I hope this works.

(2/16/2021) Today I learned a fact about how the forgetful functor of the center of a monoidal category is realized. Explicitly, realizing the center as the category $\text{Hom}_{\mathcal{C}\otimes\mathcal{C}^{\text{op}}}(\mathcal{C})$, each monoidal category $\mathcal{C}$ has a monoidal unit, and you can realize forgetful functor to $\mathcal{C}$ as evaluating this functor at the monoidal unit. I also learned directly why any such functor F can be realized as saying $F(unit)$ has central structure, you can just use the monoidal identifications to move any object of $\mathcal{C}$ in and out of F.

(2/17/2021) Today I learned one way to view nearby cycles on the degeneration of two lines to a cone. Basically, you can view nearby cycles as a pushforward of a semi-canonical map from a very small ϵ to the zero fiber. On this viewpoint, you can view two complex lines glued at a point as two cones and the degeneration as a tube where you crush a neighborhood of a circle to a point.

(2/18/2021) Today I learned a heursitic for why, given a B monodromic ℓ-adic sheaf on G/N it obtains an action of the completed ring $Sym(\mathfrak{t})$. Specifically, one can show that for any such sheaf, there is an action of $\pi_1(T)$ via monodromy, and it turns out that this action is unipotent. Therefore, it makes sense to take the logarithm of this action, which gives the associated completed action.

(2/19/2021) Today I learned a consequence of Braiden's hyperbolic localization theorem. Specifically, I learned the *second adjointness theorem* for parabolic induction, which says that parabolic induction for a parabolic subgroup P has a left(!) adjoint given by parabolicly restricting to the *opposite* subgroup.

(2/20/2021) Today I learned a proof that $\mathfrak{t}^\vee/W^{\text{aff}}$ is actually a 0-truncated prestack! Specifically, this follows from the more general fact that such prestacks are preserved under colimits, and also because W^{aff} is 0 truncated, because of the first fact!

(2/22/2021) Today I learned a possible way to more easily show that the horocycle/Harish-Chandra functor hc is fully faithful. Specifically, one can determine the W action on the Springer sheaf by taking its Whittaker averaging, which I am pretty sure is conservative on the heart and all the functors are W equivariant.

(2/23/2021) Today I learned an important technical point that the Weyl group semidirect the torus is not the same thing as the normalizer of the group! Specifically, in SL_2, the canonical short exact sequence $1 \to T \to N_G(T) \to W \to 1$ doesn't split! I also learned an interpretation of the reciprocity law. Specifically, one can check that the reciprocity map to the abelianlized Galois group can be interpreted as saying a generators and relations statement. Basically, each point gives an element on both sides and the global reciprocity map matches them, and the statement can be thought of as saying that those relations agree in both groups.

(2/24/2021) Today I learned a theorem which says that you can express the affine Hecke category as a monoidal colimit of the associated Hecke category of the parahorics (at least for simply connected semisimple groups). This follows from a general categorical consideration about when one can lift extensions in a category of words in a braid group.

(2/25/2021) Today I learned a misconception I had a little while ago. Polishchuk's theorem says when an action of a braid *monoid* upgrades to an action of the *monoid* W. I also learned explicitly, though, that the braid *group* acts on the B-monodromic subcategory of the category $\mathcal{D}(G/B)$.

(2/26/2021) Today I learned some of the tools that go into the proof of Tate's thesis. Specifically, one can use the *Poisson summation formula*, which says that, for any (Schwartz?) function f, $\sum_{n\in\mathbb{Z}} f(nt) = /frac1t \sum_{n\in\mathbb{Z}} \hat{f}(n/t)$. In particular, if you could find a function that is Fourier self dual, we'd get an interesting identity. The Gaussian satisfies this property, and this is a main step in Riemann's proof of the analytic continuation of the Riemann ζ (or, really, the Xi) function.

(2/27/2021) Today I learned one way that you can realize that $ch \circ hc(\delta)$ is the Springer sheaf. Specifically, you can write out the diagram computing hc and note that $hc(\delta)$ maps to δ_1. Base changing restriction to $(N\backslash B/N)/T$ we then see that this map can alternatively be given by Res, and it maps to δ_1. Then apply Ind and follow the diagram!

(2/28/2021) Today I learned that it's pretty unlikely that braid group invariants are t-exact. This is because $\mathbb{Z}$ embeds into $\mathbb{G}_a$, a group which is well known to not have exact invariants.

January 2021

(1/1/2021) Today I learned a quick way to argue that on the Whittaker invariants of $\mathcal{D}(G/N)$, the canonical W action is given by the usual W action on the torus. The slick way to argue this, once you have the $T \rtimes W$ action, is to argue that the sheaf $\psi \boxtimes \delta_1$ is fixed by all the various symplectic Fourier transforms, which is easy enough to compute!

(1/2/2021) Today I learned a sketch of the proof that the left adjoint to the G-conjugation averaging functor exists (by Sam Raskin). Specifically, the idea is to use the fact that we know the averaging functor exists for $N \times N$, and then realize that, because every conjugation equivariant sheaf on G can be realized as a direct summand of $ch(\mathcal{F})$ for some $\mathcal{F}$ (specifically, $\mathcal{F} = hc$ of the original sheaf, since $ch \circ hc(\mathcal{F}) \simeq \mathcal{S} \star \mathcal{F}$, where $\mathcal{S}$ denotes the Springer sheaf, and the Springer sheaf admits an identity delta as a direct summand. Finally, one can compute the averaging of an object in the 'image' of ch by realizing that the parts which don't Whittaker-average to zero are supported on the big cell.

(1/3/2021) Today I learned a mistake I was making in the definition of the restriction functor. Specifically, the restriction functor by definition takes the domain of B invariant sheaves on T, acting diagonally. With this definition, one can show that this restriction functor is the restriction to $T/_{ad}B$ of the horocycle stack $(N\backslash G/N)/T$. I also (re?)learned the definition of unipotent nearby cycles on the puncutred disk! Specifically, given a local system on the punctured disk, we can view the unipotent nearby cycles as obtained by the nearby cycles functor as follows. First, we tensor our local system with the universal local system, the pushforward of the constant sheaf of the universal cover of the punctured disk. Then, because this universal local system has a monodromy action, we can separate our sheaf into generalized monodromy eigenvalues. Further, we can recover all such eigenvalues from the eigenvalues at 1 by twisting by a constant sheaf, so we often consider the unpotent nearby cycles at 1!

(1/4/2021) Today I learned a hope to show that the functor $\text{End}_{\text{QCoh}(\mathfrak{t}//W^{aff})}(\text{QCoh}(\mathfrak{t})) \to \text{QCoh}(\mathfrak{t})$ is comonadic. Specifically, one can define the concept of an effective morphism/limit $\mathcal{F} \mapsto ... \to \mathcal{F}_n \to \mathcal{F}_{n-1} \to ... \to \mathcal{F}_0$ as a diagram for which $\text{coker}(\mathcal{F} \to \mathcal{F}_{n+N}) \to \text{coker}(\mathcal{F} \to F_n)$ is an isomorphism. Given such an effective map, then any DG category functor preserves limits in this effective case.

(1/5/2021) Today I learned (at least a totally plausible conjecture!) on where the horocycle sheaf takes the delta sheaf of $G/_{ad}G$. Specifically, one can do the same diagram with merely the N quotients and see that the resulting sheaf will remain in the N coset, which leads to an at least very plausible guess that hc takes the delta sheaf at 1 with its G coinvariance to a similar delta sheaf on $N\backslash G/N$ with its similar (diagonal) T coinvariants. This is at least plausible because the inclusion of the diagonal is $T \rtimes W$ equivariant and the fiber of the diagonal map is the regular W representation, but it needs to be worked out further...

(1/6/2021) Today I learned a sort of obvious fact. Suppose that you are given a monoidal, fully faithful functor hc and a functor F that you want to know is $\mathbb{E}_2$ or not. If the functor is $\mathbb{E}_2$, you can check this by postcomposing with the fully faithful functor and regarding it as a category to the essential image. This is because if the functor is $\mathbb{E}_2$, the essential image of the new functor had better be!

(1/7/2021) Today I learned the proof that the restriction to the diagonal composed with the horocycle functor hc is the restriction functor. Basically, this follows because the pullback of the stack $N\backslash B/N$ inside of the horocycle sheaf (or rather, its T_{ad} equivariance) inside the horocycle stack by the map $G/_{ad}B \to (N\backslash G/N)/T$ is given by the closed substack $B/_{ad}B$. I also learned, following Geordie Williamson's course notes, the definition of the Hecke category used in modern geometric representation theory for a finite Weyl group. Specifically, it's the bounded homotopy category of the full subcategory of (the bounded derived category of) $B \times B$ equivariant sheaves on G generated by the IC sheaves. The shift given by the bounded derived category gives one grading and the shift on the homotopy category gives the other. The grading is what keeps track of the q in, eg, the Hecke algebra.

(1/8/2021) Today I learned a useful fact about computing limits of, say, functors in an invariant category, termwise. Specifically, if one can show that for every morphism in the diagram yielding an invariant category, the limit commutes with the associated functor, then all structural morphisms in the limit preserve the limit.

(1/10/2021) Today I learned how to compute limits in certain Hom categories of functors. Specifically, given two DG categories (with continuous functors between them) $\mathcal{C}, \mathcal{D}$, then the category of maps between them has totalizations (because it is presentable) but these totalizations need not be computed termwise. However, the functor of $ev_C : \text{Hom}(\mathcal{C}, \mathcal{D}) \to \mathcal{D}$ on a *compact* $C \in \mathcal{C}$ does preserve limits.

(1/11/2021) Today I learned the full statement of Barr-Beck. Specifically, one can define the notion of a split simplicial object, and the notion of (given a functor F) an F-split simplicial object. Then, a functor is comonadic if and only if it is conservative and preserves all F split totalizations.

(1/12/2021) Today I learned that, given an elliptic curve over $\mathbb{Q}_p$ (at least one with *good reduction*, meaning that you can realize the elliptic curve as living over $\mathbb{Z}_p$), you can define its l^{th} Tate module as the limit of its l^n torsion, and this can also be realized as the first cohomology group of the $\mathbb{Q}_l$ constant sheaf!

(1/13/2021) Today I learned the actual notion of an effective morphism. Specifically, a map to a diagram $\mathcal{F} \to \mathcal{F}_n$ is effective if for all n there exists $N \gg 0$ such that $coker(\mathcal{F} \to \mathcal{F}_{n+N}) \to coker(\mathcal{F} \to \mathcal{F}_n)$ is nullhomotopic. This condition on these maps imply that $\mathcal{F} \simeq \lim_n(\mathcal{F}_n)$ because, effectively, the cokernel of the limit is the limit of the cokernels and the limit of the cokernels essentially has a bunch of zero maps, so the limit is zero.

(1/14/2021) Today I learned what Erdos's sumset conjecture says–specifically, given a set of positive integers A with positive upper density (meaning the limit $\lim_N \max_M \frac{A\cap[M,M+N]}{N}$) is positive, there are two infinite sets B and C such that $b+c \in A$ for all $b \in B, c \in C$.

(1/15/2021) Today I learned the cleanest statement I currently know on what is called Koszul duality for $\mathcal{O}$. Specifically, one can write $\mathcal{O}_0$ as ungraded modules over the projective generator ring R. But this ring is canonically graded, and so one can define *graded category* $\mathcal{O}_0$ as graded modules over this graded ring, and Koszul duality states that there is an equivalence of this graded category sending projectives to simples.

(1/16/2021) Today I learned that two Morita equivalent monoidal categories yield the same center. This is because the relevant module categories are equivalent categories, and thus have the same Hochschild cohomology, which is only defined in terms of the (dualizable) category and its dual.

(1/18/2021) Today I learned a theorem which says that the generic category $\mathcal{O}$ yields a strongly monoidal functor from the HC category of G to the HC category of a maximal torus lifts to a strongly monoidal functor to the braid group invariants of the HC category!

(1/19/2021) Today I learned that a modular form can be realized as the global section of the dualizing sheaf raised to a certain power on the upper half plane union the rational points of $\mathbb{P}^1$. This also explains why people don't call them modular functions–from this definition (with a certain lattice inside of $SL_2(\mathbb{Z})$), by Liouville's theorem, the only modular functions are the constant functions!

(1/20/2021) Today I learned a sketch of how to compute the connecting map between the degenerate part and the nondegenerate part of $\mathcal{D}(SL_2/B)^N$. Specifically, once one can show that the projective object goes to zero, you can argue that the simple must go to something isomorphic to a shift of itself, and from this, you can compute the complex that it is.

(1/21/2021) Today I learned that the Grothendieck group of the G equivariant Steinberg variety is given by the group algebra of the (extended) affine Weyl group, and that this is something that can be seen by, for example, the Bruhat decomposition.

(1/22/2021) Today I learned that perverse sheaves on $\mathbb{P}^1$ with respect to the usual stratification can be recovered by their nearby cycles and their vanishing cycles, and the canonical maps between the two. Furthermore, these sheaves can be constructed by that data, so long as the unipotent nearby cycles of the vanishing cycles of the unipotent nearby cycles vanish.

(1/23/2021) Today I learned that the inclusion functor of the G-monodromic sheaves on $\mathcal{D}(G/N)^N$ is not a monoidal functor, even though I had been operating under the assumption that it was for too long. The reason for this is that it doesn't send the monoidal unit to the monoidal unit! On the other hand, we have a monoidal functor given by de Rham cohomology because $\mathcal{D}(G/G)$ is a perfectly good bimodule mapping to Vect. I also learned a conjecture of Ben-Zvi-Francis-Nadler, the *Betti Langlands* conjecture, which says that, roughly, that sheaves on the stack of Betti local systems on $X(\mathbb{C})$ are the factorization homology of representations of the Langlands dual group, meaning that the QCoh category is corepresented in symmetric monoidal functors as equivalently having a symmetric monoidal functor from Rep of the Langlands dual to your category $\otimes$ analytic locally constant sheaves.

(1/25/2021) Today I learned an equivalent definition of a stable category with a t structure having finite cohomological dimension. Specifically, one can define this as having the property such that there exists some N such that for any $X \in \mathcal{C}^{\leq 0}$ and $Y \in \mathcal{C}^{>N}$, the maps $Y \to X$ are connected.

(1/26/2021) Today I learned some basic facts about working with a pro category. Specifically, one can define the pro of a DG category as the opposite category of all functors from that category to Vect. It is closed under limits. It also admits a functor from $\mathcal{C}$ via Yoneda, and any object of this pro category can be written as a filtered limit of objects from $\mathcal{C}$. With this formalism, we can define the left adjoint as valued in the pro category, since if $F : \mathcal{C} \to \mathcal{D}$ is some functor, we know that $F^L(D)$ must be the functor sending C to $\mathrm{Hom}_{\mathcal{D}}(D, F(C))$. I also learned that you can write the objects in the pro category as 'limits' of objects in $\mathcal{C}$ equivalently as *co*limits of the functors you get, which makes sense from a contravariance perspective.

(1/27/2021) Today I learned an interpretation of the μ_l roots of unity fixed points of the affine Grassmannian. Specifically, the zero weight corresponds to the affine Grassmannian with t^l instead of t, and for a regular weight, the associated space is a flag variety.

(1/28/2021) Today I learned that, given an oriented cohomology theory, the general Chern class law which says that the Chern class of the tensor product maps to the sum of the Chern classes, does not hold. However, there is a power series which one can make a similar more complicated law holds, and one can use the various properties of tensor products to extract properties about this power series. These properties are defined to be a *formal group law*.

(1/29/2021) Today I learned the actual strict definition of ℓ-adic sheaves. Specifically, they are defined to be an inverse limit over n of the sheaves valued in the $\mathbb{Z}/\ell^n\mathbb{Z}$ coefficients!

(1/30/2021) Today I learned a naming convention that I've been screwing up in the past. Specifically, one can define the *Steinberg variety* as $\tilde{\mathcal{N}} \times_{\mathcal{N}} tilde\mathcal{N}$, *not* as $tilde\mathfrak{g} \times_{\mathfrak{g}} tilde\mathfrak{g}$. I also learned that this latter term is sometimes called the *relative Steinberg variety* and that one of Bezrukavnikov's equivalenves identifies the bounded derived category of coherent sheaves on this relative Steinberg variety with the pro-unipotent completion of I_0 equivariant sheaves on the flag variety.

December 2020

(12/1/2020) Today I learned a theorem of Kazhdan and Lusztig which says that one can represent representations of the quantum group at a paramter q as the $G(O)$ integrable representations of the Kac-Moody algebras of the Lie algebra at the level κ. I also learned that this equivalence is t-exact, which fails in Gaitsgory's modified conjecture, which conjectures that the Iwahori integrable representations as a DG category is equivalent to representations of what Gaitsgory calls the *mixed quantum group*.

(12/2/2020) Today I learned a reason that B bundles for SL_2 (i.e. not just rank two vector bundles!) can be realized as global sections of the tensor square of the bundle, and why that trivialization of the top exterior power of the canonical bundle (and sometimes–a choice of square root) is important. Specifically, when one has a B bundle when the underlying group is SL_2, then if one knows its associated quotiented line bundle, then the sub can be recovered as the dual line bundle.

(12/3/2020) Today I learned why $\tilde{\mathfrak{g}}/G \simeq \mathfrak{b}/B$! Specifically, it becomes a lot easier once one identifies $\tilde{\mathfrak{g}} \simeq G \times_B \mathfrak{b}$. I also learned one heuristic for why the map $\mathfrak{b}//B \to \mathfrak{g}//G$ is a W cover on a certain locus–specifically, for $\mathfrak{sl}_2$, given a matrix with two distinct eigenvalues, the two W orbits form different lifts above.

(12/4/2020) Today I learned that given any regular connection whose leading term is in $\mathfrak{b}$, there's a canonical $g \in G(K)$ such that Gauge_g of that connection is entirely in $\mathfrak{b}[[t]]\frac{dt}{t}$. This uses the fact that you can gauge by the opposite Iwahori to modify and kill one term at a time.

(12/5/2020) Today I learned a semi flaw in the last argument, that sort of comes from using the fact that you can have lifts in *families*, and one element of the family may not be liftable.

(12/6/2020) Today I learned/explicitly computed the fact that any regular local system on the group $B := \mathbb{G}_a \rtimes \mathbb{G}_m$ has a constant monodromy up to integer shift on its eigenvalue. Specifically, one can show that Gauging by a strictly upper triangular matrix cannot change the eigenvalues at all, and gauging by an element of the $\mathbb{G}_m$ changes the -1 coordinate of the diagonal entries by the derivative of the logarithm, which always produces an integer entry in the lowest degree nonzero entry!

(12/7/2020) Today I learned the context for miraculous duality of categories. Specifically, I learned that for more quasicompact stacks, we don't expect the category of sheaves on it to be self dual, roughly owning to the fact that the $\boxtimes$ operation is not an equivalence. This makes it especially interesting that the category of sheaves with nilpotent singular support on Bun_G actually is self dual!

(12/10/2020) Today I learned a sort of hands on way to work with a regular singular connection on a punctured disk. Specifically, after choosing a coordinate, such a connection can be viewed as a filtered morphism of $A((t))^{\oplus ?}$. In turn, the regular singularity induces a map on associated graded which gives a bunch matrices of the form $\Gamma_{-1} + m\text{id}$ for some $m \in \mathbb{Z}$.

(12/11/2020) Today I learned the basic proof of the endomorphismensatz a bit more. Specifically, one can define the usual translation functors and attempt to extend their definitions to the category $\mathcal{O}$. This can be done using a shift operator of the associated $\mathfrak{t}$ action to make it match with what you know it 'has to be.'

(12/13/2020) Today I learned the actual definition of curvature of a connection. It turns out that it's easily defined as the connection squared, suitably interpreted via the signed Leibnitz rule, and it's valued in two forms. You can also similarly define a flat section–it's defined as the sections for which this curvature vanishes!

(12/14/2020) Today I learned that, given a regular singular trivialization of a connection of a vector bundle of a punctured disk, even a $\mathbb{Z}$ regular semisimple one, there may be many $\mathfrak{b}((t))$ which give rise to it (and in particular, more than two). The idea is that, once you fix such a Borel, you can use matrices which are the identity plus t^n in the 2,1 slot for some $n \in \mathbb{N}$.

(12/15/2020) Today I learned one possible perspective in showing the monoidality of the functor. Specifically, this would involve showing that the associated bimodule yielding a map to endomorphisms is fully faithful on compact objects, and this idea basically involves constructing a t-structure on the endomorphisms by considering only those which send the ≥ 0 part of the associated t-structure through the $\geq -n$ part. We will see how this goes!

(12/16/2020) Today I learned some connections between various iterations of $\mathcal{D}(N\backslash G/N)$ which allow me to not need to directly show that the functor Av_*^N is fully faithful when mapping to W invariants! This is because if I can match this 'over $\mathfrak{t}$' then I can argue that the aforementioned averaging matches with the pullback to one of the factors, and that by base change (and the fact that ω generates IndCoh) that the functor is fully faithful.

(12/17/2020) Today I learned that the actual proof I have going for me that $\mathcal{D}(N\backslash G/N)_n$ is IndCoh on the appropriate product actually is predicated on the proposition that the averaging functor to W invariants is fully faithful, and in particular I don't currently have that the fully faithfulness just falls out of the larger claim.

(12/18/2020) Today I learned that one can explicitly also define Soergel bimodules to work simply using the polynomial algebra, graded to be in degree 2! I also learned that in the context of the Soergel bimodules with w of length less than two, the associated Bott-Samuelson bimodule is merely the product of the two length one Bott-Samuelson modules.

(12/19/2020) Today I learned more about the definition of the free monodromic tilting object. Specifically, you can define it as the N-averaging of the sheaf (as in Bezrukavnikov-Yun) of the Whittaker category. I also learned (relearned?) a fact for a local field K, there's a map called the *Artin reciprocity map* taking $K^\times$ to the (abelianization of the) absolute Galois group and whose completion induces an isomorphism and whose valuation map matches with the projection onto the Frobenius component (i.e. the unramified part). Oh and later I learned that the image is the abelianization of the *Weil-Deligne* group, defined to be the full subgroup of the absolute Galois group whose projection to the unramified part projects onto an actual integer.

(12/20/2020) Today I learned a possible outline for proving that the equivalence of categories is monoidal. Specifically, one possibly may be able to show the monoidality by showing the equivalence lives over the appropriate endomorphisms (which is probably above) and that this is fully faithful on compacts by arguing that every sheaf is supported on some scheme, where IndCoh agrees with the appropriate monoidal structure.

(12/21/2020) Today I learned some restrictions on my strategy above. Specifically, since the desired target category is monoidal, the structure that we quotient by must be a groupoid. In particular, this means that we must quotient by some actual group. So the above strategy likely works only if we use the various parahoric W_P.

(12/23/2020) Today I learned that it is not a valid expectation to expect the evaluation map $\text{End}_{H_\psi}(\text{QCoh}(\mathfrak{t}^\vee) \to \text{QCoh}(\mathfrak{t}^\vee$ is t-exact, because we expect the relevant diagram with that endomorphism category to commute and the functor which takes an ind-coherent sheaf and treats it as acting on $\text{QCoh}(\mathfrak{t}^\vee$ is not t-exact because the union of graphs is not smooth.

(12/24/2020) Today I learned more about a different possible outline to argue that the functor from IndCoh to endomorphisms is fully faithful on projectives. I also learned the statement of the local Langlands conjecture, which says that there is a finite to one map which takes in a smooth admissible representation of $G(K)$ for K a local field and returns a 'Frobenius semisimple' representation of the Langlands dual group, along with a nilpotent element which corresponds to the logarithm of the monodromy.

(12/25/2020) Today I learned a fun fact about the Steinberg variety! Specifically, one can recognize the product of two nilpotent cones as the cotangent bundle of $B\backslash G \times G/B$, and one can take the locus given by the Steinberg variety, and one can identify this as the conormal bundle to all the G orbits in this product of flag varieties.

(12/26/2020) Today I learned further of the outline of the characteristic polynomial proof. Specifically, given a regular semisimple element $\xi \in \mathfrak{g}$ and a Borel containing it, one can show that the centralizer inside the Borel forms a Cartan subalgebra. Further, one can show that given a Cartan subalgebra, the Bruhat decomposition for that torus/cartan subalgebra holds. This shows that the Grothendieck-Springer resolution has finite fibers for regular elements.

(12/27/2020) Today I learned an outline of the proof of the affine Beilinson-Bernstein localization theorem for PGL_2. Specifically, using the results of Frenkel-Gaitsgory, which allowed one to show it for the Iwahori invariant vectors, one (=Sam Raskin) can show it for the Whittaker invariants of the category, and then show that any category with an action of loops on PGL_2 is generated by the Whittaker invariants and the Iwahori invariants.

(12/28/2020) Today I learned a slicker way to show that the nondegenerate category $\mathcal{D}(G/N)_n$ is actually invariant under all of these symplectic Fourier transforms! The idea is to recognize it as the orthogonal complement of the full $G \times \mathbb{Z}/2\mathbb{Z}$ subcategory generated by the zero section, and this can be realized as having two easy to work with generators as a G category!

(12/29/2020) Today I learned that there are two different functors one can define $\mathcal{D}(T) \to \mathcal{D}(G)^G$ (with the conjugation action). Specifically, one can either define the usual induction functor, or one can define the horocycle functor on the diagonal T invariants of $\mathcal{D}(N\backslash G/N)^T$.

(12/30/2020) Today I learned one anchor in the hopeful guiding possible-equivalence $\mathcal{D}(G)_n^G \simeq \text{IndCoh}(\mathfrak{t}/\Lambda \times_{\mathfrak{t}//W^{aff}} \mathfrak{t}/\Lambda)^{\Delta_{T \rtimes W}}$. Specifically, note that if you take the W^{aff} off, you see that the horocycle stack can be identified as the diagonal invariants, and thus hope that the W averaging functor is the ch functor. At least some evidence is that both counits are direct sums of functors, one of which is the identity.

November 2020

(11/2/2020) Today I learned a really cool theorem relating the various Geometric Langlands conjectures. Specifically, I learned that you can define a certain sheaf on a certain space of arithmetic Langlands parameters, known as the Drinfeld sheaf, for which the global sections gives compactly supported functions on the stack of G bundles. The fun moral is that this says that this module actually has sheaf structure, and causes a lot of the structure in the Langlands program!

(11/3/2020) Today I learned that is a canonical metric on $\mathbb{P}^n$ known as the *Fubini-Study* (pronounced Fubini-Shtoody!) metric which the unique $U(n+1)$ invariant Kahler metric which integrates to 1. More generally, this construction takes a vector space V with a Hermetian form and gives a Fubini-Study metric on $\mathbb{P}(V)$.

(11/4/2020) Today I learned a general method to show that certain categories are compactly generated from Sam. Specifically, given a monad in a compactly generated category, the associated modules over that monad are compactly generated. This allows one to show that, for instance, $\text{IndCoh}(\mathfrak{t} \times_{\mathfrak{t}//W^{aff}} \mathfrak{t}/\Lambda)$ is compact because ind-coherent sheaves on an ind-scheme are compact.

(11/5/2020) Today I learned a whole new overview of Hodge theory! Specifically, given a Riemannian metric on your space, you get a canonical volume form and therefore a way to integrate on your manifold. Using this gets you an inner product, and you also have an adjoint operator to d, written δ, via this inner product. This gives rise to the *Laplacian* $\Delta := \delta d + d\delta$, and each element in each de Rham cohomology class has a canonical representative where this Laplacian vanishes. In particular, for a Kähler manifold one obtains a metric. Furthermore, for Kähler manifolds, the *harmonic* r have the (p, q) coefficients harmonic, so one can define the cohomology of a Kähler manifold as a direct sum of the p, q classes, yielding the *Hodge diamond*.

(11/6/2020) Today I learned a sheaf theoretic definition of tilting objects for a stratification (specifically, that both associated restrictions for each locally closed embedding are tilting) and I learned a theorem that states that for any sheaf on an open such that both associated pushforwards are perverse (which happens if, say, the open embedding is affine) then there exists a tilting perverse extension.

(11/7/2020) Today I learned a sort of idea which makes the naive approach that IndCoh of the product described before is $\mathcal{D}(N\backslash G/N)_n$. Specifically, one of the problems is that you can ask that this happen over $T \times T$ where the IndCoh factor maps down via embedding into $T \times T$ and the latter maps down by forgetting the diagonal G equivariance and averaging. The problem is, the former is not exact (at least its bi T, w version) and the latter is.

(11/9/2020) Today I learned that you can't recover the compact objects of a category from its bounded by below part–specifically, the quasicoherent and ind-coherent sheaves agree on the bounded by below parts, while the subcategories differ!

(11/10/2020) Today I learned an interpretation of the endomorphismensatz–specifically, in the context of the quotient category setup, one can determine the endomorphismensatz as saying that you can recover the abelian quotient category as modules over the coinvariant algebra.

(11/11/2020) Today I learned the idea of the semi-infinite IC sheaf on the affine Grassmannian. Specifically, one can define a t-structure on the IC sheaves on the affine Grassmannian and consider the $N(K)$-equivariant orbit labeled by 0, and hope that the minimal extension of the standard object gives this to you. However, the standard object is already simple. On the other hand, fixing a curve X and performing this same construction over the Ran space, you obtain a factorizable space, and obtain a conjectural description for something that would play the role of $G(K)/B(K)$. This gives a factorization algebra structure in equivariant $\mathcal{D}$ modules on the affine Grassmannian and is a conjectural (by Gaitsgory) description of the algebro-geometric analogue of $\mathcal{D}(G(K)/B(K))$.

(11/12/2020) Today I learned a semi slick fact of the fact that the socle of any Verma module in $\mathcal{O}_0$ is given by just the simple object. Specifically, you can show it for objects which are only one away from the smallest point pretty easily, and then you can just use the simple parabolics and use the fact that $G/B \to G/P_\alpha$ is a $\mathbb{P}^1$ fibration and apply it to the length one sequence to see that general sheaf computations tell you that any subobject has to be inside something lower, so the theorem follows by induction.

(11/13/2020) Today I learned a rough heuristic for a fun fact that the endomorphisms of the big projective is the cohomology of the flag variety. Specifically, one can use the endomorphismensatz and the map $BB \to BG$ being proper, and using the fact $BB = BT$.

(11/14/2020) Today I learned a way to hint at a part of the Beilinson-Bernstein localization theorem for singular blocks. Specifically, abelian categorically one can define a translation functor from the extended universal enveloping algebra (i.e. tensor the usual one with $\text{Sym}(\mathfrak{t})$ over the center) and then use the translation functor to realize the singular block as a quotient of some regular integral block, and this is one way to summarize what's going on in Wall Crossing and D-Modules.

(11/15/2020) Today I learned an explicit example of the phenomenon above! Specifically, the geometric category at $-\rho$ is equivalent to the usual one (by tensoring with line bundles!) and the translation functors on the representation theory side makes the diagram commute. Using this, you can show that the $(\mathfrak{sl}_2\text{-Mod})_{-\rho}$ category identifies with the quotient of the geometric one as the nondegenerate category and killing the IndCoh thing.

(11/16/2020) Today I learned a lot more about the Kähler package. Specifically, one can view the Kähler package as giving you a collection of graded vector spaces that comes equipped with a finite dimensional $\mathfrak{sl}_2$ representation that allows you to decompose the space, where the 'wedging with the form' is the raising operator. This allows you to prove a bunch of results for just lowest weight vectors in your $\mathfrak{sl}_2$ representation and then you get it for all of the vector spaces. For example, the operator in this case satisfies hard Lefschetz (a version of Poincare duality) and if the operator satisfies the *Hodge-Riemann bilinear relations*, one can show that the form one can define on cohomology groups is definite and that this form is orthogonal with respect to this $\mathfrak{sl}_2$ decomposition.

(11/18/2020) Today I learned some fun facts about $\mathcal{D}(\text{Bun}_G)$. Specifically, one can define two *pseudo-identity functors* $\mathcal{D}(\text{Bun}_G)^\vee \to \mathcal{D}(\text{Bun}_G)$, which by duality is given by the objects $\Delta_!(k)$ and $\Delta_*(\omega)$ (using the fact that the tensor product of two $\mathcal{D}(\text{Bun}_G)$ is $\mathcal{D}$ modules on the product) and a theorem of Drinfeld-Gaitsgory (*miraculous duality*) says that the !-pseudo-identity is an isomorphism.

(11/19/2020) Today I learned the idea of *rational homotopy theory*. Specifically, it considers a map of spaces to be an equivalence if it induces an isomorphism on the rational homotopy groups. For each (nice enough) space X, there exists a rationalization and a map $X \to X_\mathbb{Q}$ such that the rational homotopy group of X is the usual homotopy group of its $X_\mathbb{Q}$.

Given a space, you can construct a cdga out of it (it has a pretty explicit description on simplicies!) and through this one can define an equivalence of simply connected, finite-enough spaces with CDGA's modulo quasiisomorphism. Roughly, homotopy questions then correspond to formality (i.e. whether the cdga of a space is quasiisomorphic to the direct sum of its cohomology groups). Lie groups and symmetric spaces and compact Kähler manifolds are formal.

(11/20/2020) Today I learned what the (+) Virasoro algebra is! Specifically, one can define the Lie algebra of the group of automorphisms of the punctured disk, which has a canonical basis given by certain vector fields $L_{-1}, L_0, L_1, ...$ which commute like $[L_m, L_n] = (m-n)L_{m+n}$. I learned that representations of this algebra when restricted to the L_0, etc, this is an extension of $\mathbb{G}_m$ by a large unipotent sub, and so the irreps are one dimensional. I also learned that there is a universal torsor over this Lie algebra from which, given a representation of the L_0 and above, you can acquire any quasicoherent sheaf, and if you are able to equip the L_{-1}, you have also equipped your bundle with a flat connection!

(11/21/2020) Today I learned one way to compute de Rham cohomology of a local system on a punctured disk. Specifically, one can define it as the (homotopy) kernel of the connection given as a differential module (i.e. a free module with a connection satisfying the Leibnitz rule). I also worked through a few examples and learned that if you have a regular singular connection whose leading term is invertible after adding $N * id$ for almost all $N \in \mathbb{Z}$, your operator is Fredholm via filtration reasons.

(11/22/2020) Today I learned a way to consider the map $Bun_B \to Bun_T$ for the setup with $G = SL_2$ which justifies the slogan 'global sections of line bundles yield this bundle map.' Specifically, one can consider a B bundle to be classified by a short exact sequence given by a fixed line and a projection onto its dual. Therefore you can write B-bundles as a certain ext class. Therefore the fiber over this map is given by a (shifted) global sections, because of duality of lines. Applying this construction in families yields the desired map.

(11/24/2020) Today I learned a subtle thing about vector bundles on, say, Riemann surfaces that I've always overlooked. Specifically, there are two notions of these–*complex* vector bundles, which just require that the transition maps be differentiable, which are classified by the rank and degree on a Riemann surface, and further, *holomorphic* vector bundles, i.e. transition maps are required to be holomorphic.

(11/25/2020) Today I learned (possibly very late in my mathematical career) a heuristic for the Fourier transform. The idea is this–specifically, if you are given an 'orthonormal basis' for the space of functions, and it turns out that the set of periodic functions of all the various periods *do* have these orthonormal properties, then you can simply find the coefficient on that basis by integrating against that periodicity. Thus entails the definition of Fourier series.

(11/26/2020) Today I learned a fun proof from a 3Blue1Brown video that the sum of the recipricols of the squares is $\pi^2/6$. The reason for it is conceptually that you can view the sum as the brightness one sees from a light shining (as this follows a square decay law) and then can, roughly speaking, view the line as a limit of circles with larger and larger radius, and track that as the circle gets larger, the brightness of the resulting light (really, the square distance) remains invariant. This is not a great explanation. Watch the video instead!

I also learned a conceptual explanation to what's going on with Ngo's lift and stuff. The setup is this–you are given a space for which G acts on, say X, in a Hamiltonian way such that one admits a moment map. Then one can postcompose the moment map down to the center and get a bunch of commuting operators on the group, because this is what maps to polynomial rings give, and one can construct a group scheme over this center acting on your space which literally integrate in the sense that the relevant diagram commutes.

(11/27/2020) Today I learned what technically amounts to half of a research project for $\mathfrak{gl}_2$. This is a bit facetious, but I really did learn that given any vector bundle of rank n with a connection, one can take the top exterior power of said vector bundle and obtain a rank one vector bundle with a connection! Assuming this applies to Tate vector spaces, one immediately obtains a map $LS_{GL_n} \to LS_{GL_1}$. This is at least part of the characteristic polynomial map!

(11/28/2020) Today I learned a fun differential equations fact for regular singular connections on a punctured disk. Specifically, it is a classical fact that if you are given a differential equation whose connection is of the form $\Gamma(t)/t$ for some $\Gamma \in GL_n(K)$ with a regular connection, it is gauge equivalent to a matrix with just $\mathbb{C}$ entries! I also learned this sort of fact is not true for groups like SL_n.

(11/29/2020) Today I learned a quick and easy solution to the fact that IndCoh of the union of graphs preserves compact objects–you have a set of compact generators given by the W^{aff} many translates of the structure sheaf, and can check that the convolution of two structure sheaves go to the third, and therefore since convolution preserves compactness in the set of a compact generators, convolution preserves compact objects!

(11/30/2020) Today I learned a way to classify all Weyl group representations in terms of their Springer fibers (finally!). Specifically, you can note that the group itself, and thus the group of connected components, acts on the top cohomology of each nilpotent orbit, and that each of these nilpotent orbits has top cohomology isomorphic to the group ring of the Weyl group. Therefore, since these actions turn out to commute, you can split into the various isotypic components one obtains via a commuting actions, and all Weyl group reps can be obtained this way!

October 2020

(10/1/2020) Today I learned some of the differences one needs to take into account when working over category $\mathcal{O}$ over a field of positive characteristic. Specifically, one can take the center of the universal enveloping algebra, but those representations and the way one can construct category $\mathcal{O}$ from this only gives representations of the first Frobenius kernel. Instead, one may want to take the hyperalgebra associated to G, defined as the algebra of distributions on G supported at the identity. This gives, roughly speaking, a divided power structure to the center.

(10/2/2020) Today I learned why a particular approach for constructing a characteristic polynomial won't work for local systems. Originally, I had hoped to just work with the infinite affine space, written as $\times_{m\in\mathbb{Z}}\mathfrak{t}//W^{(m)}$, with its $\mathbb{Z}$ action induced by translation, and hope that the 'characteristic polynomial' map defined on each integer translate would factor through the Gauge action and that descent would be easier to verify. However, one problem with this (among many) is that the $\mathbb{Z}$ action is not free on this space!

(10/3/2020) Today I learned that one can classify representations of $G(\mathbb{Q}_p)$ via Moy-Prasad filtrations, and that, for large p, these classify all such representations. Furthermore, I learned that every representation appears as a subrepresentation of an induced representation of a supercuspidal representation.

(10/4/2020) Today I learned that there's at least some relationship between the nondegenerate quotient category of $\mathcal{D}(N\backslash G/N)$ and the category of H_ψ equivaraint endomorphisms of $\mathcal{D}(T)$. Specifically, one can construct a T, T equivariant functor to that category, and examine each piece separately. I learned some 'evidence' that this functor may be an equivalence on the fully nonintegral subcategory and maybe fully faithful on all compacts (which can be done in general by just breaking into the various W_λ pieces for $\lambda \in \mathfrak{t}/\Lambda$).

(10/6/2020) Today I learned a fun fact that on an elliptic curve M, there exists indecomposible holomorphic vector bundles of each rank such that any indecomposible vector bundle on the elliptic curve can be realized as that vector bundle pulled back by an automorphism of that elliptic curve.

(10/7/2020) Today I learned an actual definition of the space of opers on the punctured disk! Specifically, one can define G-opers as the space $(f + \mathfrak{b}((t))/LN$, where the quotient acts by the gauge action. Furthermore, one can similarly define the *regular opers* to be those which come from

$\mathfrak{b}[[t]]$. Here, f is a principal nilpotent element. With this in mind, one can descirbe the center of the renormalized $\mathfrak{g}_\kappa$ modules at critical levels as functions on these opers, and the endomorphisms of the vaccum object as the functions on the regular opers in such a way that the associated diagram commutes.

(10/8/2020) Today I learned the idea behind the determinant bundle on the affine Grassmannian (roughly, the derived global sections of the associated line bundle tensored the inverse of the standard one, to make this independent of the actual piece you look at in the colimit sort of?) and that this forms the structure of a *factorization algebra*, and some rough heuristics for why a factorization algebra, defined as isomorphisms for every ring and every set of geometrically disjoint points for that ring, can be recovered from the data of things happening on the square of the space satisfying some relations.

(10/9/2020) Today I learned an idea behind holonomic $\mathcal{D}$-modules on a smooth space. Specifically, you can consider a cyclic $\mathcal{D}$ module for a given function f, $\mathcal{D}/\mathcal{D}f$ and take its solution functor, i.e. maps of $\mathcal{D}$ modules from that space into functions. The idea behind holonomic *functions* is that this solution space is at worst finite dimensional.

(10/11/2020) Today I learned a functor in the other direction to the obvious map $\mathcal{D}(N\backslash G/N)_n \to \text{End}_{\mathcal{H}_\psi}(\mathcal{D}(T))$, which is given by the fact that $\Gamma(\mathcal{D}_T)$ is its own opposite, and so we can (at least abstractly) write $\text{End}_{\mathcal{H}_\psi}(\mathcal{D}(T)) \simeq \mathcal{D}(T)\otimes_{\mathcal{H}_\psi}\mathcal{D}(T))$, from which we can apply the induced averaging functor.

(10/12/2020) Today I learned that one can work with the space $\mathfrak{t} \times_{\mathfrak{t}//W^{aff}} \mathfrak{t}$ a lot more easily than the associated quotients by Λ's because we literally have that that space is an ind-scheme, given by a certain union of graphs! I also learned one cannot have good lattices in families. I also learned that my endomorphism statement above is false, because we are using the wrong monoidal structure.

(10/13/2020) Today I learned a trick which will help prove the above stuff. Specifically, to show that the functor $\text{IndCoh}(\mathfrak{t} \times_{\mathfrak{t}//W^{aff}} \mathfrak{t}) \to \text{End}_{\text{QCoh}(\mathfrak{t}//W^{aff})}(\text{QCoh}(\mathfrak{t}))$ is fully faithful on compact object, it suffices to show it on coherent sheaves on the subsheaf of various collections of lines, because compact objects of ind-coherent sheaves on ind-schemes are supported on schemes.

(10/14/2020) Today I learned how to construct a functor $\text{IndCoh}(\mathfrak{t}\times_{\mathfrak{t}//W^{aff}}\mathfrak{t}) \to \mathcal{D}(N\backslash G/N)^{T\times T,w}$. Specifically, one can use the adjunction of a $G \times G$ scheme of $\text{Av}_*^{-\psi\times\psi}$ being left adjoint to Av_*^G (where G acts by the diagonal action) which allows one to map the whole of $\text{IndCoh}(\mathfrak{t} \times \mathfrak{t})$ to $\mathcal{D}(N\backslash G/N)^{T\times T,w}$.

(10/15/2020) Today I learned at least a rough sketch about how to show the equivalence of the above. Specifically, one can argue that the functors to *IndCoh*$(\mathfrak{t} \times \mathfrak{t})$ are fully faithful on the compact objects, and then to show essential surjectivity on the appropriate part, you can simply argue that by $\mathfrak{t}\times\mathfrak{t}$ equivaraince, both functors already map into where you want them to map into, and then you can just show that they hit the lines, which ideally can be seem geometrically.

(10/16/2020) Today I learned the algebro-geometric version of the fact that maps of connected Lie groups are determined at the level of their Lie algebra. Specifically, this is expressed as an isomorphism of formal groups in characteristic zero of the Lie algebra formally completed at 0, which acquires a formal group structure via the Baker-Campbell-Hausdorff formula (which is not abelian!) and identifies with the formal group structure that can be put on the formal completion of your group at 1.

(10/17/2020) Today I learned one canonical way to describe the Springer sheaf as an ad-equivariant sheaf. Specifically, one can describe the canonical functors between the G via adjoint action invariant $\mathcal{D}$ modules on G and the $(N \times N)\delta_T$ invariant sheaves on G via the 'obvious' averaging functors and the counit of this associated adjunction is given by averaging with the Springer sheaf.

(10/19/2020) Today I learned of the notion of Bernstein asympototic maps for a spherical variety. The idea is this–roughly, certain spherical varieties have compactifications where the functions are G or $G(K)$ equivariant and are detectable by their compactification near infinity, and then you can use G equivariance to determine them near 0.

(10/20/2020) Today I learned a potential way to get at the Ind-coherent sheaves on the product. Specifically, one can use the $G \times G$ equivariance of the averaging while applied to $\mathcal{D}(G)$ to get a functor, and this lifts to a functor from the tensor product of the three desired categories.

(10/21/2020) Today I learned a definition of a compactly generated category to be 'proper'–specifically, one requires that the Exts between compact objects are finite dimensional. I also learned the idea that one can define the stack of Langlands parameters (with certain technical assumptions) and is representable by unions of affine schemes.

(10/22/2020) Today I learned a loose outline of how to possibly prove that the averaging comonad is actually a regular W representation. Specifically, the W action I'm considering is an action of $\mathrm{Sym}(\mathfrak{t})$ modules, and so the two actions commute. So it suffices to find some object of this doubled average that behaves like the unit object of the group ring. I'm not sure if this is possible.

(10/23/2020) Today I learned ('finally') the notion of strongly linked and how it applies to category $\mathcal{O}$. Specifically, there's an easier question to ask than exact multiplicities of simples in Vermas–instead, you can ask when $[M_\lambda : L_\mu] \neq 0$. This is given by the notion of two weights being strongly linked, meaning that the smaller weight can be gotten from the larger weight from a simple reflection s which makes them a nonpositive distance away (or they are the same), and extend this notion transitively. I also learned that Category $\mathcal{O}$ is only acted on by generalized central character (as in Humphreys).

(10/24/2020) Today I finally wrote down the two projective resolutions for the two simple objects of the zero block of category $\mathcal{O}$. Specifically, the resolution $\Delta_0 \hookrightarrow P_{-2\rho}$ gives a projective resolution of the simple at -2ρ and the three step complex $\Delta_0 \hookrightarrow P_{-2\rho} \to \Delta_0$ gives a projective resolution of the simple L_0. This in particular shows that the category is of finite length.

(10/26/2020) Today I learned one way to show that the two comonad structures agree on the two functors. Specifically, you can just note that the literal functors are the same on the Whittaker Hecke category and $\mathcal{D}(T)^W$ and $\mathcal{D}(T)$ (i.e. just by writing the appropriate commutative diagrams) and noting all of the comonad structure is determined by where δ is sent.

(10/27/2020) Today I learned a no go idea I had on the fact that in the monodromic category for $\mathcal{D}(G/B)^N$ that the endomorphisms of the IC sheaf were given by $k[\epsilon_2]$ related to the fact that there's a canonical morphism of the IC sheaf in degree 2 which is given by the symplectic Fourier transform. Specifically, the problem is that the first comes from the delta sheaf at the origin, but without requiring B equivariance, the endomorphism of the IC sheaf is given in degree 3 (by the de Rham cohomology of SL_2).

(10/28/2020) Today I learned a main tenant of Springer theory–one can take the preimage of any nilpotent element of the Springer resolution and get a certain variety for which the top cohomology group gives rise to a W action, and this gives a bijection between conjugacy classes and nilpotent orbits (at least for type A). I also learned that the Springer fiber of the nonzero, non-regular nilpotent element of $\mathfrak{sl}_3$ is given by gluing two spheres at a point.

(10/29/2020) Today I learned that the Schubert stratification of the flag variety explicitly gives a basis for the cohomology of the space. For SL_3, this says that the top and bottom cohomologies have one dimension and the middle two have two dimensions. I also corrected some error in my mind–if I'm looking for a 'universal generator' of the middle nilpotent orbit for SL_3 the object should have two dimensions of Homs, but one dimension of endomorphisms in degree 2 (by Poincare duality, at least, if it holds).

(10/30/2020) Today I learned a really fun way to prove that the projective dimension of the zero block of category $\mathcal{O}_0$ is twice the longest element in the Weyl group. Specifically, you can show by induction that the projective dimension of the Vermas is given by the length itself by starting at the Verma indexed by zero, which is itself projective. Then, you end up at -2ρ, and can go back up for the simple objects!

(10/31/2020) Today I learned a fact I probably should have known about before, but didn't, but, you know, whatever. Anyway, the fact is that subobjects of projective objects in a module category need not be projective–witness the simple subobject of the Verma at zero!

September 2020

(9/1/2020) Today I learned a fun theorem of Yakov Varshavsky which says that in a certain topoology (which roughly can be defined by declaring that the sheaves are with respect to 'formalized blow ups'), one can recover sheaves equivariant with respect to loops on G by the quotients with respect to all of the maximal parahoric subgroups. The idea is an inductive proof/ind-ing your way up the Bruhat decomposition, where you have two cases for each maximal w in the extended affine Weyl group–either it is minimal with respect to its parahoric or it is not.

(9/2/2020) Today I learned a critical mistake of my original argument that $Av_!Av_*$ is a vector bundle. Specifically, I assumed that Av_* preserves compact objects. But sadly, that's not the case, and in fact, if it preserves compact objects, then it admits a left adjoint, which is currently not expected (although not impossible).

(9/3/2020) Today I learned a theorem which classifies all of the simple holonomic $\mathcal{D}$ modules on a smooth variety X. Specifically, all such examples come as the $!*$ of a closed embedding of a smooth(?) variety of an integrable connection, which specifically for a coherent $\mathcal{D}$ module means that it is coherent as an $\mathcal{O}$ module as well.

(9/4/2020) Today I learned a potential solution to help with the Weyl groups of type B-D. Specifically, one can consider the half integral weights via $\mathbb{G}_m^2$'s action. The thing I actually learned today was the quotient $\mathbb{G}_m/\mathbb{G}_m^2$ is $B\mathbb{Z}/2\mathbb{Z}$!

(9/5/2020) Today I (finally!) learned the general decomposition of the root systems of types B-D. In type C, it is specifically given by the coordinate axes L_i where the positive roots are given by the vectors $L_i + L_j$ for i possibly equal to j. This allows one to see that the Weyl group contains a normal subgroup of reflections of order 2^{rank} because they are just the reflections about the coordinate axes!

(9/6/2020) Today I learned an equivalent formulation (and a sketch of the proof of!) the Riemann hypothesis. Sadly, it was only the Riemann hypothesis for curves for finite fields. Specifically, you can define a certain power series via the property that its log is the power series whose n^{th} term counts the number of $\mathbb{F}_{q^n}$ points. Then you can show for curves that, by Euler characteristic arguments, the expected defect number from the number of points N_n is given by $1 + q^n$, and the error being bounded by a certain constant, namely $2g\sqrt{q^n}$, is equivalent to the Riemann hypothesis.

(9/7/2020) Today I finally learned what the Gromov-Witten invariant is on the stable moduli space of a certain number of curves. The idea is that you pull back the fundamental classes of each curve on the Moduli space, and then you integrate along the virtual fundamental class to obtain a number!

(9/8/2020) Today I learned a definition of a real form, which I had not just sat down and asked for a long time (it's just a choice of the real group which complexifies to the implicitly complex thing you're talking about). I also learned that on a classifying space of a finite group, the $\mathcal{D}$ modules agree with the quasicoherent sheaves (because for finite groups, strong actions are equivalent to weak actions!)

(9/10/2020) Today I learned a fact which may be helpful which says that every closed subroot system contained in some parabolic subroot system. This can be proven, for example, on type C_n algorithmically–you write a sequence of maximal subroot systems going to yours, and if it's the same rank, you keep the same simple roots (but relabel them), and if you lost a rank, you delete the labeled simple root.

(9/11/2020) Today I learned that I was semi-wrong about a path to prove that the functor was fully faithful. Specifically, I mixed up the directions–both pushforward and quotient have left adjoints, but the fixed point functor is not conservative that's the one associated to $BH \to *$ (for H a finite group).

(9/12/2020) Today I learned a potential cheating way to prove the functor is fully faithful, continuing from yesterday. Specifically, taking the same invariants as yesterday, if the domain of the functor is $*/A$ for a finite abelian group A, then the fact that our functor is $*/A$ linear allows us to check fully faithfulness just on the trivial representation, since all other irreps are invertible and the regular rep is a direct sum of these one dimensional irreps. I'm honestly not sure whether this works.

(9/14/2020) Today I learned some basic facts about the space $\mathfrak{t}//W^{aff}$. Specifically, I learned that the canonical map from $\mathfrak{t}//W$ is not a surjection, but yet there is a bijection of points given by the exponential map, at least for $T = \mathbb{G}_m$.

(9/15/2020) Today I learned one way to define the big projective object of the zero block of category $\mathcal{O}$. Specifically, you can convolve the delta sheaf with ψ with the $Av_!$ convolution and prove it's self Verdier dual and can compute Homs out of it explicitly being concentrated in degree 0.

(9/17/2020) Today I learned a fun theorem of Chan and Oi which says that one can determine representations of the $\mathbb{F}_q$ of a reductive group by an associated character which is easy enough to compute. I also learned a potential way to go about the half integral case of the functor–specifically, so long as $Av_!^{\psi}$ is $t-$ exact, the functor on Vect preserves injective objects.

(9/18/2020) Today I hopefully learned a way to glean the multiplicities off of the image of $Av_*^N(k)$ for a given central character. Specifically, one can use the specific central character and the fact that the categories split into pieces associated to the coset to argue that the minimal simple objects correspond to objects which are supported in exactly one Bruhat cell. One can use this and the fact that Av_*^N restricted to the torus is the identity to argue that the multiplicities are one of the minimal simple objects and zero otherwise.

(9/19/2020) Today I learned another fun way to show that the image of the averaging functor is the big injective. Using BGG reciprocity, it suffices to see what the maps from Vermas are, and then one can specifically compute what the averaging does on the ψ averaging to argue that the appropriate maps from Vermas vanish (and possibly that the other maps have multiplicity one, but I haven't checked).

(9/20/2020) Today I learned some more fun stuff about the theory of IndCoh. Specifically, I realized that there is a fully faithful embedding of QCoh into IndCoh (for schemes) because all coherent complexes are perfect, but this embedding only commutes with colimits that stay in the category of perfect complexes, so to speak. In particular, the functor of QCoh into IndCoh sends noncompact objects to compact ones and doesn't commute with colimits.

(9/22/2020) Today I learned an obvious quick shortcut to computing the essential image of Av_*^N. Specifically, since that functor is t-exact, we know that whatever the essential image is must lie in the bounded by below category. However, without W equivariance, the boudned by below category is generated by the big projective.

(9/23/2020) Today I learned a reasonable conjecture I can have on the essential image of Av_*^N.

Specifically, I can divide everything up on each central character λ (for any field valued λ!) and ask that the associated integral Weyl group W_λ has the associated B_λ averaging be an entirely trivial W_λ representation.

(9/24/2020) Today I learned how to actually construct a functor $\text{IndCoh}(\mathfrak{t}//\Lambda \times_{\mathfrak{t}//W^{aff}} \mathfrak{t}//\Lambda) \to \mathcal{D}(N\backslash G/N)_n$. Explicitly, such a functor exists on a quasicoherent level by biaveraging, using the fact that if you have a quasiaffine morphism then the quasicoherent sheaf of a cartesian product is the tensor product of the quasicoherent sheaves. Then, you can reduce to checking on each $\lambda \in \mathfrak{t}/\Lambda$, where the explicit description comes from earlier work. I also learned that I do not know if this functor is monoidal.

(9/25/2020) Today I learned a result of James Tao and Roman Travkin which says that you can recover the full Hecke category for the loop group (meaning with bi-Iwahori equivaraints) as a colimit in the category of monoidal stable categories. I also learned that it's not the case that $\mathbb{G}_m(K)/\mathbb{G}_m(\mathcal{O}) = \mathbb{Z}$, but you actually need to formally complete the $\mathbb{G}_m(\mathcal{O})$.

(9/26/2020) Today I learned an intuitive picture for how the Nilhecke algebra works! Specifically, one can define for $\mathbb{A}^1$ the *Demazure operator* which takes a polynomial $f(x)$ to $\frac{f(x)-f(-x)}{x}$ which is still a polynomial and you can define the subring of endomorphisms of the ring of polynomials which has the polynomial ring itself and these operators. I also learned that these are a decategorification of Soergel bimodules.

(9/27/2020) Today I learned that the map from $\mathfrak{b} \to \mathfrak{g}$ on the regular locus is more likely to have a locus that behaves as a W cover rather than behaves like a W^{aff} cover. This is because one of the things that causes the integral translation behavior comes from the Gauge action of loops on the torus, and therefore 'remains' in the regular locus. I also learned that it's not the case that the regular singular local systems can be checked up to $G(\mathcal{O})$'s Gauge action, for this fails even for $\mathbb{G}_m$!

(9/28/2020) Today I learned that it's not the case that we have a canonical quotient functor from IndCoh of the product above to quasicoherent sheaves on it, because this theory only works for schemes. I also learned a potential sketch for showing that the characteristic polynomial on regular singular local systems is given by projecting onto the -1 coordinate and then taking the usual characteristic polynomial (with more descent to make it map to $\mathfrak{t}//W^{aff}$) because $G(\mathcal{O})$ preserves these connections, so we can appeal to the Bruhat decomposition on the flag variety–but maps from affine schemes need not map into a single cell or in the closure (eg for nonreduced reasons). Finally, I also learned a fun sketch of the fact that there is an action of the affine Hecke category on the category $\text{Rep}(G)_0$ over a field of characteristic p, which is given by a certain monoidal functor from the affine Hecke category to the completed Harish-Chandra category.

(9/29/2020) Today I learned that you can recover the category of $\mathcal{D}$ modules in characteristic p as a colimit of inverted Frobenius maps of quasicoherent sheaves! I also learned (finally!) of an explicit example of an object in the formal completion of positive loops on $\mathbb{G}_m$ which is not in positive loops itself– specifically $\epsilon t^{-1} + 1$.

(9/30/2020) Today I learned the existence of a so called Moy-Prasad filtration on a category indexed by the loop group of G. I also learned that through this, you can generalize Sam's proof of the affine Beilinson-Bernstein localizaiton theorem for general G.

August 2020

(8/1/2020) Today I learned how to translate between the fact that (one incarnation) of the Langlands correspondence for GL_1 gives the class field theory correspondence, at least for $\mathbb{Q}$! Specifically, one can state that this Langlands correspondence gives a representation of the connected components of the ideles, which, for $\mathbb{Q}$, is the product of all of the $\mathbb{Z}_p^\times$'s. In particular, a representation of an abelian Galois extension then corresponds to a Dirichlet character, and you can view Dirichlet characters as (probably) living in $\mathbb{G}_m(\mathbb{Q})\backslash G_m(\mathbb{A})$.

(8/3/2020) Today I learned a fun perspective from Rok about derived algebraic geometry. Specifically, DAG is often times simply a context which is a better language to make problems come out more cleanly when viewed homotopically. While one can take intersection theory or Riemann-Roch as examples, another problem that wasn't already solved without the language of derived algebraic geometry was an equivalence between certain moduli problems and certain deformations of a given Lie algebra.

(8/4/2020) Today I learned a recasting of class field theory, which says that for a given global field K (or at least an algebraic number field, but I think any global field) you can identify the Galois group of the maximal abelian extension of K with the group of connected components of the invertible adeles modulo the scalars of the ground field. In more human terms, this at least says that for any character of the group of an abelian extension we can find a matching Hecke character so that the associated L-functions agree.

(8/5/2020) Today I learned the computation that the symplectic Fourier transform takes the constant sheaf on $\mathbb{A}^2\backslash 0$ and sends it to itself with a shift by 3, it seems. This may match with the fact that the epsilon in $k[\epsilon_3]/(\epsilon_3^2)$ vanishes.

(8/6/2020) Today I learned (solidified?) a consequence of the filtration of the big projective of category $\mathcal{O}$. Specifically, you can actually realize the endomorphism as $\mathcal{P}_{-2\rho} \to \mathcal{P}_{-2\rho}/\nabla_0 \cong L_{-2\rho} \hookrightarrow \mathcal{P}_{-2\rho}$ which in particular has kernel and image $L_{-2\rho}$ and creates the desired complex which makes it easy to compute cohomology. Using the fact that i^*j_* of this complex is invariant under shift, you can make the problem of computing $i^*j_*(k)$ go away for the zero block of category $\mathcal{O}$ with fancy colimits.

(8/7/2020) Today I learned that the computation I did two days ago was off by a shift, which makes everything more plausible because now this matches with the endomorphism being degree -2. I also learned that Drinfeld also proved the Langlands correspondence for function fields, but also constructed the notion of Hecke eigensheaves for unramified Langlands.

(8/8/2020) Today I learned the notion of an automorphic representation for $G(\mathbb{A})$. Specifically, you can define the functions $G(\mathbb{A})/K^\times$ which are smooth with respect to a compact open subgroup, and define the irreducible subreps of this as the automorphic representations.

(8/9/2020) Today I learned a way to actually conceptualize these pseudo-Levis that, while obvious, I hadn't realized before. Specifically, the difference between pseudo-Levis and levis is that Levis are merely the reductive quotient of a parabolic subgroup.

(8/10/2020) Today I learned that the two nontrivial parabolic subgroups of SL_3 are not actually conjugate, only their Levi factors are.

(8/12/2020) Today I learned that on the glued category constructed by Polishchuk, one can fix a simple reflection and recover any object supported only in the categories associated to the lower half by only knowledge of how the various pieces indexed by parabolic cosets which are subsets of this lower half.

(8/13/2020) Today I learned that there is a problem in the proof that the W functors are exact. Specifically, while the usual symplectic Fourier transform is exact, the inclusion is not t-exact. On the other hand, I learned that a theorem of Positselski which says that if you restrict your vector bundle to an affine open subset for which the vector bundle splits.

(8/14/2020) Today I learned a solution to my problem yesterday. Specifically, you can use the Positselski theorem to argue that even though the composite functors F_w are not exact for (at least most) $w \in W$, you can use the fact that the δ sheaf lives in an affine open subset to argue that $F_w(\delta)$ still remains in the heart of the category. Specifically, you can then apply the same reduction to one categorical things as if the functors were exact.

(8/15/2020) Today I learned a fun fact which implies that the Riemann ζ function divides the zeta function of any number field! Specifically, you can show with some fun conections that you can always write the ζ function of a given algebraic number field as the L function associated to the representation of its Galois group, and furthermore you can show that the L-function takes direct sums of representations to the products of L functions. Identifying the usual Riemann ζ function with the L function associated to the trivial representation, we obtain our claim.

(8/17/2020) Today I learned a fact that implies nondegeneracy in my category $\mathcal{D}(G/N)^{N,T^{\perp}_{\Phi^c}}$ because if the simple reflection you excluded maps your closed subroot system out of itself then you can map your associated SL_2 averaging via left-right stuff to the averaging for an SL_2 associated to a coroot where the associated torus average vanishes.

(8/18/2020) Today I learned that a strategy I came up with yesterday to prove nondegeneracy of the associated quotient was incorrect, because doing the left W action could potentially change which roots average to zero. I also learned that ρ is not integral for just the long roots, i.e. is not an integral linear combination of the long roots, for type $B_2 = C_2$.

(8/19/2020) Today I learned that the averaging functor above is vacuously fully faithful on the strange maximal subroot systems. Specifically, if you are integral with respect to a set of weights which span the torus, and are non integral for some other roots, vanishes, because if T acts trivially on Vect then there can't be any non monodromic objects!

(8/21/2020) Today I learned that I have been technically getting the real Mellin transform that people have been talking about wrong this whole time, slightly. Specifically, the one people seem most interested in is that $\mathcal{D}(T) \simeq \text{QCoh}(\mathfrak{t}^*//\Lambda)$, which makes sense, because given a map $\text{Sym}(\mathfrak{t}) \to k$ gives rise to a k-point of $\mathfrak{t}^*$ so you can imagine $\text{QCoh}(\mathfrak{t}^*)$ acting on $\mathcal{D}(G/N)^{T,w}$.

(8/22/2020) Today I learned a discrepency further given by above in the two books I have. One is the description of the roots associated to a character in $\mathfrak{t}^*$ and one is a description to the coroots, and only as a subset of the coroots do these definitions form a closed subroot system.

(8/24/2020) Today I learned some intuition for what's happening with the representations of quantum groups with divided powers. Specifically, just as the category of representations of a group is entirely determined by the data of its monoidal functor, which is in turn determined by the data of the Hopf algebra structure on functions, the category of quantum representations is determined by a similar structure on the functions, and the failure of the braided monoidal structure to be symmetric is given by dynamical R-matrices.

(8/25/2020) Today I learned some ideas of Yakov Varshavsky which says that any category with an action of the extended Weyl group can be recovered from a compatible family of actions from the actions of the various Weyl groups of parahorics. Furthermore, one can determine the actions of equivaraiant sheaves with respect to the action of loop group from the quotients of the various parahorics.

(8/26/2020) Today I learned the full main theorem of Pavel Safranov and Artem Kalymkov which says that you can recover the generic category $\mathcal{O}$ from just the generic Harish-Chandra category for the torus on the generic subcategory.

(8/27/2020) Today I learned a maybe more fully rigorous statement of the cobordism hypothesis. Specifically, this is a theorem which says that given a certain tensor category C, one can evaluate it on a (framed) point and this object will be fully dualizable, and furthermore, this evaluation yields a homotopy equivalence of topological field theories as such and the space of fully dualizable objects in the original category.

(8/28/2020) Today I learned that a method of attack to construct a filtration on the kernel of my functor fails. Specifically, the filtration on the kernel of $Av_!Av_*$ is actually not given by a bunch of identity functors, even though at the central character zero it is.

(8/29/2020) Today I learned that $V = Av_! Av_*(Z\mathfrak{g})$ is a vector bundle! Specifically, I learned that if you take these two composite functors, since both maps are t-exact (the latter for $Av_! = Av_*$ reasons) the functor $Av_* Av^*$ is given by the derived tensor product over $Z\mathfrak{g}$ with v. Therefore since these two composite functors are exact, tensoring with V is also exact, so V is flat, which on affine space implies V is projective.

(8/31/2020) Today I learned a potential hangup in the idea of working with $Z\mathfrak{g}$. Specifically, when working with the highest weight $-\rho$, the functor becomes a map from Vect to Vect and you can test conjectures here.

July 2020

(7/1/2020) Today I learned another approach that will make it easier to show that my functor is fully faithful. Specifically, I can only show that the T-equivariant restriction functor to the appropriate pseudo-Levi subgroup is W equivariant only on the right (as opposed to on both sides) and I can do the same argument that I had using the right W equivariance again.

(7/2/2020) Today I learned a slick way to show that we have a W action on the category $\mathcal{D}(G/N)_n$. Specifically, I learned that Polishchuk constructed a W quasiaction on the category $\mathcal{D}(G/N)^{\heartsuit}$ and constructed the connecting morphisms whose only obstruction to being an isomorphism is that the square identity endomorphisms are not actually isomorphisms. However, on the nondegenerate category, these maps are (quasi)isomorphisms, so after dealing with the $t-$structure things, you get a W action.

(7/3/2020) Today I learned the full statement of a generalizaiton of the Beilinson-Bernstein theorem applied to the loop group. Specifically, the naive guess for this conjecture would be to say that $\mathcal{D}(Gr_G) \xrightarrow{\sim} (\mathfrak{g}-\text{Mod})_\chi$ where χ is some oper. However, the idea behind local geometric Langlands is that there's a symmetry of $Rep(G^{vee})$ acting on $\mathcal{D}(Gr_G)$ which you also have to quotient out to get the correct statement of the theorem.

(7/4/2020) Today I learned an idea to show that $\mathcal{D}(T \rtimes W)$ acts on a category. Specifically, when one shows that the nonsemidirect product $\mathcal{D}(T \times W)$ acts on a category like $\mathcal{D}(G/N)$ it suffices to give objects in $\mathcal{D}(N\backslash G/N)^T$ where the T equivariance is given diagonally. Instead of the T equivariance being diagonal, for the semidirect product I hope that it suffices to simply show it for each w by twisting the T action by the orbit of $T(1, w)$.

(7/5/2020) Today I learned that there is a map from algebraic K-theory which maps to Hochschild homology which factors through an invariant called negative cyclic homology, and that this negative cyclic homology can often detect the rational rank of the algebraic K-theory.

(7/6/2020) Today I learned what the derived Geometric Satake theorem says and a heuristic for why it is true. Specifically, the derived Geometric Satake theorem says that you can identify $\mathcal{D}(G(\mathcal{O})\backslash G(K)/G(\mathcal{O}))$ with the (ind-completed) derived category of $G^\vee$ equivariant coherent sheaves on $* \times_{G^\vee} *$. The idea is to use the fact that the Whittaker sheaves on the affine Grassmannian is $Rep(G^\vee$ and identifying $\mathcal{D}(G(\mathcal{O})\backslash G(K)/G(\mathcal{O}))$ as the tensored product of the Whittaker sheaves over a monoidal category, so the strategy becomes identifying that monoidal category.

(7/8/2020) Today I learned that I may have my direction of non-paritally integrable incorrect when I say that W acts on $\mathcal{D}(G/N)_{npi}$. Specifically, fixing a simple root s for G, the associated symplectic fourier transform is a G, SL_2 functor where SL_2 acts on the right. Therefore, it seems more likely that the non partially integrable condition happens with respect to the (N equivariance of) the right action.

(7/9/2020) Today I learned a sort of way to solve the above problem and a modification. Specifically, you can compute the canonical map of the symplectic Fourier transform on G/N

doubled as a $j_*j^!$ of the 'usual' one. Therefore, to show that this map is an isomorphism on the kernel, it suffices to compute the kernel and show it is monodromic... I'm at least pretty sure.

(7/10/2020) Today I learned the condition above more explicitly. Specifically, you can instead of asking for non-paritally integral/nondegenerate subcategroies, which requires that you quotient out by all nonminimal P for a choice of Borel B, you can instead quotient out further by all of those SL_2 monodromic objects for each simple roots. Inclusion of a smaller subgroup (associated to a closed subroot system) is functorial with respect ot this property, for if you have a simple G root, it is either a simple subgroup root or in the center.

(7/11/2020) Today I learned how to possibly generalize the notion of Koszul duality to the category $\mathcal{D}(G/N)^N$ for an arbitrary reductive group G, or at least a toy model suggesting that such a thing might hold. This is because the category for $G = \mathrm{SL}_2$ is built out of three pieces–a copy of Vect, and a quotient category with a canonical W action such that the W invariants can be built up from two pieces which are explicitly describable in their Soergel bimodule combinatorics.

(7/12/2020) Today I learned a few other ways to classify Spherical varieties. Specifically, a pretty good one is to argue that the action of the Borel on functions vanish. I also learned of one potential way to show that the remaining functor is W equivariant because each w functor takes δ_1 to the sheaf $\overline{K(w)}$.

(7/14/2020) Today I learned a heuristic which, roughly speaking, says that if $K(w)$ as above is restricted/quotiented to the full subcategory for which the associated $\mathbb{G}_m$ averagings vanish for the associated coroots, then in the specific construction of the sheaf $K(w)$ for G, the only coroots that don't contribute to zero in the exponential averaging is the coroots in the associated M subgroup constructed earlier.

(7/15/2020) Today I learned the way that Polishchuk constructs the quasi W-action given a limited skeleton of data. Specifically, Polishchuk shows how to, for each w, construct the composition of w and the functor associated to a simple reflection s. With this, hopefully it can be generalized to show that the restirction functor on the appropriate subcategory can be given W_M equivariant structure.

(7/16/2020) Today I learned one way to finally pin down what it means to identify two W actions for a finite group W. Specifically, each action can be identified as a monoidal functor, and the notion of a natural isomorphism of lax monoidal functors is defined as a condition for one one categories (namely, compatibility with the monoidal structures!)

(7/17/2020) Today I learned that for a given quantum group representation (well, at least type I representations) for q not a root of unity, one can define a certain operator called an R matrix, and through the composition with the R matrix and the usual swap operator, one can put a braided monoidal structure on the category of representations.

(7/18/2020) Today I learned some sort of heuristic for how the tensor product of a finite group over a category with a subgroup action behaves. Specifically, one simply needs to define functors for each coset rep and given a canonical way to compose them. I'm not actually sure about this, but I believe I can do it for Weyl group elements at least.

(7/20/2020) Today I learned a quick way to prove Bezout's theorem for $\mathbb{P}^n$. The idea is to recognize that in the Chow ring, because $\mathbb{P}^n$ is stratified by affine spaces, you can show that the classes of the closures of these affine spaces generate the Chow ring (because, roughly speaking, if you take a point you can collapse to that point) and therefore you can show that $\mathbb{P}^n$ has a single generator as a ring. Defining the degree as the coefficient of interest on this generator, you can obtain that the degree is multiplicative once you get down to the point grading of the ring.

(7/21/2020) Today I learned some general frameworks on how to do intersection theory problems, and the notion of a r-dimensional *linear system* of degree d hypersurfaces in $\mathbb{P}^n$. This definition is specifically cut out by $r+1$ degree d homogeneous polynomials which vary linearly based on $r+1$

coordinates in $\mathbb{P}^r$. This is equivalently a degree r hypersurface in $\mathbb{P}^N$ where N is $n+d$ choose n.

(7/22/2020) Today I learned how to do computations in the Chow group of projective lines in the plane (i.e. Gr(2,4)). Specifically, by working with general flags one can compute the general computation structure of the Chow ring with an affine stratification.

(7/23/2020) Today I learned more general overview of intersection theory, which was given by verifying that there are 32 tangent lines on four general cubic surface. Specifically, the solution to this problem is given as follows. In the specific Chow ring, the class of lines which are tangent to a general cubic surface is three dimensional (one direction of the line, two on the surface) so in terms of the Chow ring, it is some multiple of σ_1. You can use the relations of the Chow ring to show that this multiple is given by the intersection of a this class with a hyperplane with a specified point, and show that this gives precisely two lines, so then computing that $(2\sigma_1)^4$ has degree 32 gives the proof.

(7/24/2020) Today I learned a nice way to help understand the map $\mathfrak{t}/\Lambda \to \mathfrak{t}//W^{aff}$. Specifically, one can take a point $\lambda \in \mathfrak{t}//\Lambda$ and consider the etale neighborhood about that point. Then the point is that when the category is formally completed, you can isomorphically work with the formal completion of the usual map $\mathfrak{t} \to \mathfrak{t}//W$.

(7/25/2020) Today I learned a possible distinction in showing that the inclusion of a pseudo-Levi subgroup induces an equivariant functor with respect to the Weyl group. Specifically, there are two ways that one can define the switching between the various Bruhat cells–one by mere multiplication of the Weyl group elements and the other by doing the associated (combination of) symplectic Fourier transformations.

(7/27/2020) Today I learned a way to reduce a bunch of cases while checking if two W actions are the same for a Weyl group W. Specifically, you can turn a theorem of Polishchuk's on its head, so to speak. Specifically if you have two W actions on a certain (1,1) category, to show they are the same, you need only check that they agree on the braid group and that the isomorphisms $ss \to 1$ agree. This is because if you already know they are group actions, you can show that associativity forces other morphisms to already be given by this data.

(7/28/2020) Today I learned a mild technical difference between Soergel bimodules and Soergel modules. Specifically, Soergel bimodules are meant to classify the category of bi-B monodromic sheaves on G, whereas the Soergel modules are meant to informally kill one of these and require it to be actually equivaraiant. Roughly speaking this is the R module k while closed under tensoring with any Soergel bimodule.

(7/29/2020) Today I learned a strategy which I'm pretty sure guarantees that the functor is fully faithful, but doesn't as obviously give me W_M equivaraince of the inclusion or the restriction functor. Specifically, fix a reflection across a root hyperplane–one can show that you can conjugate to any other N' containing the torus and show that your map takes your root to a simple root to show that the inclusion element is equivariant, and so the reflection acts as a symplectic Fourier transform. You can use this to move any sheaf down to the trivial coset and show the same extensions are the same.

(7/30/2020) Today I learned a different kind of 'stratification' of the affine Grassmannian. Specifically, one can take a coset labeled by a coweight and right multiply it by $N(K)$, and one can use this interaction between this and the associated N^- cells to stratify the space. I also finally learned about the existence of a *Hecke eigensheaf* for a local system, and why the local system itself should be an eigenvalue for the curve!

(7/31/2020) Today I learned another feature of the proof of the Geometric Satake. Specifically, I learned a way to deal with the monoidal condition on the affine Grassmannian. Specifically, we use the Beauville-Lazlo theorem above which allows you to identify the curves at various points, and you move the points around using a sort of chiral-algebra-esque giant (probably Beilinson-Drinfeld)

Grassmannian.

June 2020

(6/1/2020) Today I learned that earlier when considering the complement of the T_ϕ equivariant $\mathcal{D}$ modules on $N\backslash G/N$. Specifically, for each simple root s for which the associated coroot averages to zero, we also have that the Bruhat cell NwB cannot have an extension by the cell $NswB$. I also learned the general framework of the proof of (one of) Serre's GAGA theorems–specifically that for any coherent sheaf on a smooth projective algebraic variety, the global sections and cohomologies agree with that of the analytification of that coherent sheaf. This is because you can check it for $\mathbb{P}^n, \mathcal{O}$, and then for twisting sheaves, and then diagram chasing to find it for a general sheaf.

(6/2/2020) Today I learned a fun way to define a modular form of weight k. Specifically, one can define it as a certain analytic map on the set of lattice inside of $\mathbb{R}^2$. which transforms by $-k$ upon scaling by k. You can use this fact to get back to the upper half plane definition by taking your first lattice basis coordinate to be $(1,0)$.

(6/3/2020) Today I learned that the full subcategory of $\mathcal{D}(G/N)^N$ for $G = SO_5$ for which is nonintegral for the complement of the SO_4 subroot system is entirely determined by two copies of the embedded SO_4 via the symplectic Fourier transform. The missing insight was that for the symplectic Fourier transform associated to a simple root s will lower *any* sheaf totally supported on the $s-$ big part of the cell. For instance, any sheaf supported on $\mathcal{D}(N\backslash B^c/N)$ for $G = SL_2$ will have support only inside of $\mathcal{D}(N\backslash B/N)$.

(6/4/2020) Today I learned a small flaw in the plan to make an algorithm to prove that all connections with respect to certain integral conditions are explained through connections determined by the Weyl subgroup. Specifically, I learned that I need to consider connections of open subsets by nontrivial extensions of a closed subset, not just the singular Bruhat cells in and of themselves!

(6/5/2020) Today I learned more into the algorithm above. Specifically, one can apply that algorithm and prove that the decomposition of the category of N-equivariant $\mathcal{D}$ modules on SO_5 with an associated subroot system is determined by the associated psuedo-Levi subgroup.

(6/6/2020) Today I learned a way to chop off a bunch of connections in the Bruhat decomposition when you are attempting to determine which connections are and aren't valid when a complement of a T averaging is nonzero. Specifically, for all but two cases (E_8 and G_2) you can divide the roots into two connected components–those which have the affine simple root and those that don't, and you can run the parabolic analysis on the roots that don't contain the affine simple root.

(6/7/2020) Today I learned another (sub)-case for which the functor is fully faithful. Specifically, when the maximal closed subroot system associated to the simple root is determined by removing the 'first' usual root in the affine Dynkin diagram for type C_n, you can break your Weyl group into $2n$ pieces: namely $P, Ps_1, Ps_1s_2, ..., Ps_1...s_{n-1}, Ps_1...s_n = s_0Ps_1...s_{n-1}, ..., s_0P$ and use the symplectic Fourier transform in that line to show that the only pieces that can talk to each other are $Ps_1...s_j$ and $s_0Ps_1...s_j$, where P is the parabolic given by $\{2, ..., n\}$.

(6/8/2020) Today I learned one characterization of a reductive Lie algebra–specifically it's a Lie algebra which is a direct sum of a semisimple Lie algebra and an abelian one. Because of this, you can state and verify theorems regarding any reductive group (like Soergel's endomorphismensatz) often times by just checking them for semisimple and abelian components. I also learned that you can check a functor of G-category's fully faithfulness or essential surjectivity on the $G-$monodromic part by taking the functor $(-)^G$ on both sides, which by construction sends no nonzero $G-$category to zero and thus is conservative. (6/9/2020) Today I learned how to show that my functor is fully faithful provided that we have a mild functoriality of the W action. Specifically, if we can show that the B-average zero subcategories of the inclusion $\mathcal{D}(C_t/N_t)^{N_t} \hookrightarrow \mathcal{D}(G/N)^N$ admit $W \times W$ equivariant structures, then the functor is fully faithful. Roughly speaking, you can make a sequence to move any open Bruhat cell to a parabolic using the right action and then use the left to move that open set down to 1, proving the structure of the nonintegral part of $\mathcal{D}(G/N)^N$.

(6/10/2020) Today I learned that the Fourier-Laplace transform is exact, and that to actually define it and show that it determines an equivalence for complex constructible sheaves only if you include the $\mathbb{G}_m$ equivaraince.

(6/11/2020) Today I learned that the functor of forgetting N invariants $\mathcal{D}(G/N) \to \mathcal{D}(G)$ is t-exact, but doesn't preserve truncations. I also learned that the Hom's of the smallest object in a block of category $\mathcal{O}$ usually do not agree with their quotient in the nondegenerate category. This is because the former has no nontrivial extensions, while the latter does.

(6/13/2020) Today I learned a way to reduce general infinity categorical computations of a continuous functor $A - Mod \to \mathcal{C}$ for some stable infinity category $\mathcal{C}$. Specifically, if one can put a $t-$structure on $\mathcal{C}$ and, noting that a continuous functor is determined by where it sends A, show that this object lives in the heart of the $t-$structure, we can use the fact that the heart of a $t-$structure is a one category to compute everything in the derived (1-)category of an abelian category.

(6/14/2020) Today I learned a proof that if you give the two subcategories of $\mathbb{G}_m$ given by the $\mathbb{G}_m$-equivariant and $\mathbb{G}_m$ monodromic structure the $t-$structures which come from being a subcategory of $\mathcal{D}(\mathbb{G}_m)$ then the quotient functor $\mathcal{D}(\mathbb{G}_m) \to \mathcal{D}(\mathbb{G}_m)^0$ is $t-$exact, I think. The reason being is by construction the right adjoint to this functor given by inclusion is by construction $t-$exact, and therefore by general nonsense the right adjoint must be right $t-$exact and now we must only check left $t-$exactness. But on the other hand, $\underline{Hom}(\underline{k}, \mathcal{F})$ is concentrated only in positive degrees for the constant sheaf, so the long exact sequence gives us that if $\mathcal{F}$ is in the ≥ 0 part of the $t-$structure, so too is $Q(\mathcal{F})$. Granted, I'm not 100 percent sure on this proof and just thought of it. So maybe today I learned a proof idea?

(6/15/2020) Today I learned what a Riemannian manifold is! Specifically, a Riemannian manifold is a manifold with a smoothly varying inner product on the space. Furthermore, given such a Riemannian manfold, there is a canonical connection on it which is compatible with the metric and which is 'torsion free', meaning that the obvious commutator you can write is given by the Lie bracket of the vector fields. This is called the Levi-Civita connection.

(6/16/2020) Today I learned the following theorem. Fix a quantum parameter $q : \Lambda \to \mathcal{C}$. Then one manifestation of the fundamental local equivalence says that there is a factorization algebra Ω_q^{Lus} whose factorization modules can be identified with $q-$twisted Whittaker sheaves on the affine Grassmannian.

(6/17/2020) Today I learned another piece of the fully faithful-ness of the averaging puzzle. Specifically, on the B monodromic objects, this work was done by Beilinson-Ginzburg where they explicitly identify the 'geometric' translation functors and therefore show that the canonical W action (hopefully) is the W action on $\chi^{-1}(0)$.

(6/18/2020) Today I learned some implicit assumption that I had been making all along was false! Specifically, it is believed (although no one has written this down in this language anywhere) that $\mathcal{D}(G/N)^B_{nondeg} \simeq \text{IndCoh}(\chi^{-1}(0))$ where $\chi : \mathfrak{t} \to \mathfrak{t}//W$ and I further believe that the action of W on this category and the obvious action on $\chi^{-1}(0)$ makes this a W equivalence. However, $\chi^{-1}(0)/W \neq *$ because $\chi^{-1}(0)$ is not W!

(6/19/2020) Today I learned a lemma of Sam Raskin's about the $t-$structure on quotient categories. Specifically, if one is given a fully faithful functor of DG Categories which admits a right adjoint (for example, the functor $i_{*,dR}$ for a closed embedding i) then there is a unique t structure on the quotient category (i.e. the kernel of the left adjoint of the functor) for which the functor from the category to the quotient is $t-$exact. The example of this is the $t-$structure on the category $\mathcal{D}(\mathbb{A}^2\backslash 0)$, which can be realized as the quotient of $\mathcal{D}(\mathbb{A}^2)$ by the objects supported at the origin.

(6/20/2020) Today I learned a fun way to remember the adeles versus the ideles. The adeles start with a, and they associated to the additive group, whereas the ideles are the units because the word invertible starts with the letter i. How cool! I also learned that the required bonus data for two groups T, W acting on a (1,1) category $\mathcal{C}$ to lift to an action of the semidirect product $T \rtimes W$. Specifically, all that is required (I am 99 percent sure) is just the commutation data. But I definitely learned that if my proof is to go through, I need the $T \rtimes W$ action on $\mathcal{C}^N_{nondeg}$, not just the W action!

(6/22/2020) Today I learned a sort of general conceptual framework for how to define things with the name 'Hall algebra' in them. Specifically, given some abelian category $\mathcal{A}$ (really, of global dimension 1 which I at least heuristically learned means that there are no higher exts than 1) and any functor F which takes in objects of $\mathcal{A}$, we can view the stack of short exact sequences as associated to a correspondence from pairs of objects to an extension, and ergo for each functor F on this stack we can associate an algebra structure on F. When F is the Borel-Moore cohomology functor, this gives the cohomological Hall algebra.

(6/23/2020) Today I learned about the notion of Schur-Weyl duality. I suspect that there is a more general formulation of this, but the most basic example of it occurs as follows–we have commuting actions of the group S_q and GL_n acting on the space $k^n \otimes ... \otimes k^n$ (q-times) and furthermore you can decompose this as a direct sum of tensor products of S_q-reps and GL_n reps, one for each Young diagram with at most q rows (which you can remember by decomposing $k^n \otimes k^n$ into the $\wedge$ and the Sym direct sum).

(6/24/2020) Today I learned (solidifed?) the idea behind Harish-Chandra's idea about representations of a real Lie group $G_{\mathbb{R}}$. Specifically, the idea is that a $G_{\mathbb{R}}$ representation is determined by the data of its Lie algebra representation and its maximal compact, and if they agree, you get your representation of your real Lie group.

(6/25/2020) Today I learned an extension of Koszul duality. Specifically, one version of Koszul duality involves the equivalence of mixed sheaves on the derived category of B-equivariant ($\mathcal{D}$-modules/constructible sheaves) on the flag variety, and in fact in characteristic p this is the only kind of equivalence that holds. The way to make this work in an ungraded context is to ask that the instead we work with the category of *motives*.

(6/26/2020) Today I learned a result about the structure of reductive groups over fields. Specifically, for any reductive group there is a central isogeny of the group from the product of a semisimple group and a torus. More in English, this means that there's a short exact sequence of a reductive group G of the form $1 \to \mu \to S \times T \to G \to 1$, where S is a semisimple algebraic group, G is a torus, and μ is finite central closed subgroup. This gives a reduction step in proving that Av_*^N is fully faithful because the N averaging only affects the S factor in $S \times T$!

(6/28/2020) Today I learned the ideal essential image of the averaging functor on the Whittaker invariants and a rough sketch of how to prove it. Specifically, the essential image is cut out by noting that the coroot averaging gives you $\mathbb{Z}/2\mathbb{Z}$ representation and we can simply require that that action is trivial. We can show this on the B-monodromic objects of the category by a Beilinson-Bernstein-esque argument and then reduce via induction (on $|W|$) to get the general case, assuming that $M/N_M \overset{G}{\hookrightarrow} /N$ induces a W equivariant map on $\mathcal{D}$ modules.

(6/30/2020) Today I learned a mistake in my thoughts that I was making earlier. Specifically, I had hoped that a $T \rtimes W$ action could be obtained by lifting a monoidal functor $QCoh(W) \to \mathcal{D}(N\backslash G/N)_n$ to the T_{diag} invariants. However, this would give rise to a *commuting* action of T and W, which is not what I want.

May 2020

(5/1/2020) Today I learned a generalization of the endomorphsismensatz, which states when the endomorophisms of projective covers in category $\mathcal{O}$ are commutative. Specifically, the endomorphisms of a projective cover is commutative if and only if the multiplicity of the largest Verma in a standard filtration is one if and only if there is a surjection from the center of the universal enveloping algebra.

(5/2/2020) Today I learned the notion of a *projective functor*, which is a functor of $\mathcal{O}$ or O_λ that is a direct summand of tensoring with some finite dimensional module. I also learned that translation functors, at least by ρ-dominant weights, take projectives to projectives (by functoriality), which is enough to make a conjecture as to the decategorified statement that the fully faithfulness of Av_*^N is determined by the ability to reduce to the integral case.

(5/4/2020) Today I learned more into the proof of why $H^{4g-6}(\mathcal{M}_g)$ grows exponentially. Specifically, you can use the fact that this cohomology surjects onto a piece of the Lie algebra associated to the graph complex, which is a free algebra. Then by the PBW theorem, you can take the associated *Poincare series* of the two graded vector spaces (defined to be $\sum_n dimgr(V)t^n$) and use clever rearrangements to show they agree and one side grows more easily exponentially.

(5/5/2020) Today I learned some fun facts about the K_0 of the category of varieties. Specifically, I learned that the exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$ shows that torsion is trivial in the K_0 of the category of abelian groups. I also learned a relatively trivial fact that given a closed root system of, say $\mathfrak{sl}_3$, one cannot always necessarily chose a closed complementary subroot system.

(5/6/2020) Today I learned some cool facts in algebraic topology and related things. First, I learned another conceptual reason that Poincare duality holds. Namely, given any triangulation of a manifold, one can construct a dual cell complex which, as a manifold, is homeomorphic to the original manifold. This fact in essence gives Poincare duality. I also learned that in computing the Euler characteristic of $\mathcal{M}$, one can use the fiber bundle $oblv : \mathcal{M}_{g,1} \to \mathcal{M}$ and use the fact that the (orbifold) Euler characteristic is multiplicative on fiber bundles.

(5/7/2020) Today I learned one place that I haven't checked that I can get information from. Specifically, I learned that I haven't well enough explored the right T averaging on $\mathcal{D}(G/N)_n^N$. I also learned that $T_{1,2}$ is not a normal subgroup of $T \rtimes W$ where T is the torus for SL_3.

(5/8/2020) Fix an arbitrary algebraic group G, and let R be its unipotent radical. Today I learned in characteristic zero the existence of a *Levi subgroup*, which is a splitting of the short exact sequence $1 \to R \to G \to G/R \to 1$. I also learned that such a group need not exist for characteristic p!

(5/9/2020) Today I learned that there is an isomorphism given by multiplication which identifies any basic affine space with any other basic affine space and this isomorphism is allegedly W equivariant when it induces a map $\mathcal{D}(G/N)_n \cong \mathcal{D}(G/N')_n$.

(5/10/2020) Today I learned a fact about the invariants vs. the coinvariants of $\mathcal{L}N$ where G acts on a category. Specifically, it was conjectured that the invariants of the opposite Cartan $\mathcal{L}N$ was the $\mathcal{L}N$ coinvariants of the category, and this is allegedly false. However, it is true when $\mathcal{C}$ is the category of $\mathcal{D}$ modules on the affine Grassmannian.

(5/12/2020) Today I learned that, given any parabolic subroot system $i \in I$ of a root system, one can separate the Weyl group W as $W = W_I W^I$ where the W_I is the group generated by the I

and the W^I are objects for whose lengths are raised by multiplication by every simple root indexed by I.

(5/13/2020) Today I learned that one can use the decomposition above to argue that, given a parabolic subgroup of the Weyl group, that every Bruhat cell can be fourier transformed through valid fourier transforms (meaning that the associated left transposition multiplication is not in W_I to be in W_I itself.

(5/15/2020) Today I learned an alternative characterization of the classification of closed subroot systems of a root system. Specifically, one can characterize any closed subroot system as the centralizer of a semisimple element of the semisimple algebraic group G, and that the centralizer's connected component of the identity (also known as a *psuedolevi subgroup*) will be a reductive group, and the quotient of the Weyl group modulo the (canonically embedded) Weyl group of this reductive group will also be the associated subgroup modulo its connected component.

(5/16/2020) Today I learned a theorem which states that the centralizer of any semisimple element of a simply connected algebraic group is connected. I also learned that this is not true for groups of general type (eg, PGL_2), but in general there are finitely many components of the space, in bijection with a certain subquotient of the Weyl group associated to the semisimple element.

(5/17/2020) Today I learned a way to generalize the basic affine space for a reductive Lie group G. Specifically, one can parametrize all such choices without canonically making a choice, similar to that of the flag variety, as the set $\{(\mathfrak{b}, x) : x \in \text{simple}(b)\}$ where here simple(b) means the product of the vector space of roots of simple weights, where you remove zero at each root.

(5/18/2020) Today I learned a fun theorem of Sam's which says that you can view the Whittaker invariants of the category of $\mathfrak{g}_\kappa$ modules as the category of modules over the W algebra $\mathcal{W}_\kappa$, and this is in part done by proving that there is a canonical sequence of invariants of DGCategories $Whit^{\leq n}$ (the *adolescent Whittaker construction*) which realizes the Whittaker subcategory.

(5/19/2020) Today I learned a heuristic (finally!) for the fact that if the spectrum of a (bounded?) linear operator is $\geq \epsilon$ for some positive ϵ then it's invertible. The idea behind this is that if the operator were diagonalizable, then the worst it can do is multiply by $\frac{1}{\epsilon}$.

(5/21/2020) Today I learned the structure of the center of the universal enveloping algebra of the Lie algebra of a reductive group over a field of characteristic p. Specifically, there is a part that analogues the theory of characteristic zero, i.e. a center which is isomorphic to $k[\mathfrak{h}^*]^{W,\cdot}$ and a part which stems from the fact that p^{th} powers of differential operators commute with lots of things (eg the center of $p-$differential operators is large). Specifically, this is the embedding of the Sym of the frobenius twist. With this two pieces of info, one can define central reductions of the center, which I also learned are finite dimensional algebras.

(5/23/2020) Today I learned the definition of a Soergel bimodule. Specifically, one can define for any simple reflection the graded module $C \otimes_{C^s} C(1)$ as a graded C bimodule, and similarly do iterated constructions for any $w \in W$ using a reduced word decomposition of the w. It turns out that the category closed under grading shifts, direct sums, direct summands, and tensors (and maybe something else I'm forgetting?) actually has Grothendieck group the same of the Hecke algebra, and that every Bott-Samuelson module described above has a unique 'largest' indecomposible submodule which corresponds to the Kazhdan-Lusztig basis of the Hecke algebra.

(5/25/2020) Today I learned two seemingly important facts. One is in representation theory, and says that on the level of modules over the coinvariant algebra, tensoring with an associated Soergel bimodule indexed by a $w \in W$ is given by $Hom(\mathcal{P}, \theta_w(-))$ where $\mathcal{P}$ is the big projective. I also learned how to define the analytification functor–one can specifically define the category of all locally ringed spaces which have an open cover whose open subsets admit an immersion into $\mathbb{C}^n$, and show that the inclusion functor into all locally ringed spaces admits a partially right adjoint defined on all complex analytic varieties, which is known as the analytification functor!

(5/26/2020) Today I learned the general proof that the block of a category $\mathcal{O}$ for a given semisimple Lie algebra only depends on the class of the associated antidominant weight $\lambda \in \mathfrak{t}$ in $\mathfrak{t}/\Lambda$ and the actual $W, \cdot$ stabilizer of the associated weight! The idea is this–this category is obviously generated by the direct sum of the projective covers associated to each weight in the $W_[\lambda]$, and so by abstract nonsense this block can be identified with the category of right modules over the endomorphism algebra of this generator. The next step is to use Soergel's theory that the functor $Hom(P_\lambda, -)$ is fully faithful on projective objects, that each projective object can be given as the successive 'new' simple objects of the wall crossing functors, i.e. they are obtained by a successive wall crossing functors of an associated Verma. This can also be constructed on the level of Soergel bimodules as well, which in particular are just defined only in terms of the associated $W_{[\lambda]}$ and the associated stabilizer!

(5/27/2020) Today I learned that one can define a certain isomorphism of G representations $\oplus_{\lambda \in \Lambda} V_\lambda^\vee \otimes V_\lambda \to Fun(G)$ given by the function $\phi, v \mapsto (g \mapsto \phi(gv))$ and this is an isomorphism of algebras and induces a filtration on functions indexed by the weights.

(5/28/2020) Today I learned that a naive strategy I had hoped might work in proving that the averaging functor is fully faithful (and more accurately understanding the W action on $\mathcal{D}(G/N)^N_n$ fails. Specifically, I had hoped that I might intersect two Bruhat cells to deal with the case of having an embedded root system which doesn't correspond to (up to conjugacy) a parabolic, but the intersection of two big cells will have the same dimension as the big cell itself, so this can't work... at least I'm pretty sure.

(5/29/2020) Today I learned ways to define, for a space X with an action of $\mathbb{G}_m$, the *attractor* and the *repellor* loci. Specifically, one can define the attractor loci as the G_m maps from $\mathbb{A}^1$ to the space, and the repellor loci as the inverse action's attractor loci, and the fixed point loci as the fiber product of both of these.

(5/30/2020) Today I learned a contradiction hiding in my brain. Specifically, I believe that for any given simple root, we can embed C_t/N_t for some semisimple t into G/N as a closed embedding (or at least take the map $C_t \to G$ and quotient out by the associated unipotent radicals) and this seems to suggest that in the symplectic Fourier transform associated to a random C_t we will always be on the 'small' part of the vector bundle with respect to the Bruhat cell, where on the other hand the embedding of $SO_4 \hookrightarrow SO_5$ seems to suggest that there are things that hit the biggest cell. I also learned that an alternative way to state the Bruhat decomposition is to argue that W parametrizes the G orbits of $G/B \times G/B$.

April 2020

(4/1/2020) Today I learned a general way to prove that the general averaging functor is essentially surjective on those components which $B-$ average to zero. Specifically, it suffices to show this on N invariants and afterwards the left category becomes that subcategory of $\mathcal{D}(T)$, and one need only show that the subcategory pf $\mathcal{D}(G/N)^N$ which B averages to zero splits up into W many pieces (which don't map to each other!)

(4/3/2020) Today I learned some kind of a toy model which is a non-affine version of the Av_*^N functor. Specifically, with the quotient map $\mathfrak{t}/W \to \mathfrak{t}//W$, one can pull back and identify the essential image as those sheaves which, with their W action, can be recovered by their fixed points. I also learned that I made an earlier mistake in my connections and that I computed the deRham cohomology incorrectly for the restriction of $Av_!^\psi Av_*^N(\delta_\psi)$ to the big biN^- cell.

(4/4/2020) Today I learned a toy example of my project (on the Mellin dual side, anyway). Specifically, letting $q : \mathfrak{t}/W \to \mathfrak{t}//W$ denote the quotient map, one can show that because $\mathfrak{t}//W$

is affine you can characterize the essential image of $q^* : Sym(\mathfrak{t})^W - Mod \to (Sym(\mathfrak{t}) - Mod)^W$ is characterized precisely by those modules whose restriction to the singular point has a trivial representation.

(4/6/2020) Today I learned that you can pick off all but the big cell terms when taking the W fixed points associated to $\mathcal{F} = Av_! Av_*(\delta)$. Specifically, you can push forward the sheaf $B(\delta_{w_0 s})$ and average it and restrict it to the torus associated to some other root, and find that the sheaf $\psi(\alpha + \alpha^{-1})$ appears and the equivariance is similarly controlled by the averaging.

(4/7/2020) Today I learned two possible ways going forward to classify the essential image of the $N-$averaging functor on the Whittaker subcategory. One of them is that we can either obtain that $\mathcal{D}(G/N)^N$ is generated by those $B-$monodromic objects and those which integrate to zero, in which case we can classify this by arguing that the $B-$monodromic parts of the Whittaker Hecke category has only one object and so does the subcategory of $\mathcal{D}(N\backslash G/N)^W$ which is hit, and then do the same game for the averaging to zero part of $\mathcal{D}(N\backslash G/N)$ by arguing that the zero-B-averaging part decomposes as a W cat into a direct sum of $|W|$ many copies of $\mathcal{D}(T)$.

(4/8/2020) Today I learned a way to define a generalization of an E_n algebra for any manifold M and a conceptual way to view operads–namely, categories which are allowed to take multiple domains at the same time. For your manifold M you can take the *colored operad* associated to M, whose objects are abstractly isomorphic to $dim(M)$ dimensional disks and can be defined as the homotopy product of embeddings of the disjoint unions of disks with remembering how you got there from moving the disks around. When $M = \mathbb{R}^n$ this recovers the notion of E_n algebra once you take the associated colimit to this colored operad.

(4/9/2020) Today I learned definitively that there is no actual B equivariant biWhittaker function on SL_2 besides zero. This is because of a calculation. I also learned one way to view the factorization homology associated to a manifold M. Specifically, one can (left Kan) extend the category of framed disks into the category of all framed disks on a manifold and define the factorization cohomology associated to a manifold as the colimit over this functor.

(4/10/2020) Today I learned another restriction on the essential image of the $N-$averaging functor on the Whittaker subcategory. Specifically, on the category $\mathcal{C} := \mathcal{D}(G/B)_{nondeg}$ you can show a that each object $\mathcal{F} \in \mathcal{C}$ in the essential image has a canonical extension $\mathcal{F} \to \mathcal{F}[1]$, and there is a G-closed condition with $colim_n \mathcal{F}[1] = 0$.

(4/11/2020) Today I learned that the functor on nondegenerate G categories given by $\mathcal{C}^{G,w} \otimes_{Z\mathfrak{g}-Mod} Sym(\mathfrak{t}) - Mod \to \mathcal{C}^{N,(T,w)}$ is W equivariant, where the W action is given by the Gelfand-Graev action on the right hand side and by the usual action on the left. I also learned that a conservative functor of DGCats preserves the properties of a functor being essentially surjective and fully faithful, because any functor can be factored into the composite of an essentially surjective and fully faithful functor.

(4/12/2020) Today I learned a fun way to potentially alternately prove that the averaging functor is fully faithful. Specifically, since the G-functor is Morita equivalent to considering the category over the Harish-Chandra category, we can take the $G, weak$ equivariance of our desired functor to see that it is equivalent to show that the compact generator of $(\mathfrak{g} - Mod)^{N^-,\psi}N$-averages to something whose endomorphisms are (conjecturally) $Sym(\mathfrak{t})$ as a W-vector space. Then you can take the W fixed points (corresponding to W-equivariance) and then via Harish-Chandra would obtain $Z\mathfrak{g}$ as the endomorphisms, thus giving fully faithfulness!

(4/13/2020) Today I learned a full way to possibly show that the averaging functor is fully faithful. Specifically, we can also take the conservative functor of $N, (T, w)$ invariants and then using standard manipulations get that our averaging functor then becomes a functor $\mathfrak{t} - Mod \to \mathcal{D}(G/N)_n^{N,(T,w),W}$. It remains, then, to show that the object that the ('Whittakered') $Sym(\mathfrak{t})$ maps to something whose endomorphisms admit a W action and whose fixed points are $Sym(\mathfrak{t})$. One possibility for the endomorphisms of this object is, as a W ring, $Sym(\mathfrak{t}) \rtimes W$.

(4/15/2020) Today I learned some homotopical facts about the classifying space of a discrete group G. Specifically, you can (as I knew before) construct BG as the quotient of a contractible space with a free G action, but you can construct this action for a discrete G as a CW complex whose n cells are tuples of points of G. I also learned that BG has no higher homotopy groups and its π_1 is G itself.

(4/16/2020) Today I learned a fact that I can prove, ground up, without appealing to any other fully faithfulness arguments or anything else. Consider the functor of $Av_*^N : \mathcal{D}(G)^{N^-,\psi} \to \mathcal{D}(G/N)^W$, and let $\mathcal{D}$ be the full subcategory of $\mathcal{D}(G)^{N^-,\psi}$ which maps to things whose right T averaging vanishes. Then this functor is an equivalence onto its image, and the image is the full subcategory of $\mathcal{D}(G/N)^W$ whose right T average vanishes. I also learned that the constant sheaf on $\mathcal{D}(T)$ under the Mellin transform goes to some object of $QCoh(\mathfrak{t}/\Lambda)$ which forgets to the module $\oplus_{\lambda\in\mathbb{Z}}\delta_\lambda$.

(4/17/2020) Today I learned that the universal Verma module has self exts, so you don't actually have a derived equivalence of $Sym(\mathfrak{t})$ and $End_{\mathfrak{g}}(U\mathfrak{g} \otimes_{U\mathfrak{n}} k)$–just on the level of the zeroth homology. I also learned that these extensions are classified by those $x \in U\mathfrak{g}$ for which $ex \in U(\mathfrak{g})e$, the condition which matches the condition of localizing the ideal.

(4/18/2020) Today I learned so many things! Specifically, I learned that it's possible to define a W action on $Ext^1_{\mathcal{O}}(M_{w_o}\lambda, \lambda)$ such that that vector space identifies as a W vector space with $Sym(\mathfrak{t})$. On the other hand, I also learned there are no extensions of a Verma by itself in category $\mathcal{O}$, essentially because you can always lift highest weight vectors if $\mathfrak{t}$ acts diagonalizably. On the other other hand, I learned there are extensions of Vermas by itself that do not Lie in category $\mathcal{O}$ because you can $U\mathfrak{g} \otimes_{U\mathfrak{b}} -$ the short exact sequence of $\mathfrak{b}$ reps $0 \to k \to k^2 \to k \to 0$ which allows $\mathfrak{t}$ to act on both $k's$ by your favorite scalar while the $1, 2$ coordinate can be any other number.

(4/20/2020) Today I learned a potential proof strategy to show that $(\mathfrak{g}-Mod)^N_{nondeg} \equiv \mathfrak{t}-Mod$. Specifically I learned that we can take the general form of the Beilinson-Bernstein functor $\mathcal{D}^{G,w} \to \mathcal{D}^{N,(T,w)}$ and substitute the category $\mathcal{D} := \mathcal{D}(G/N)_{nondeg}$, and show that the corresponding fully faithful functor is actually W equivariant where the W action on the left hand side only acts on the $Sym(\mathfrak{t}) - Mod$ factor. We then can take our alleged fully faithful functor $\mathcal{C}^{N^-,\psi} \to \mathcal{C}^{N,W}$ for the nondegenerate $\mathcal{C} = \mathcal{D}(G/N)^{T,w}$ and classify the essential image. By showing an equivalence which would hypothetically move the W action from one side to another and showing that the two essential images match, we'd get the desired equivalence.

(4/21/2020) Today I learned an interpretation of $\mathfrak{t}//W^{aff}$ in the setting where $\mathfrak{t} = Lie(T)$ for T the torus of SL_3. Specifically, one can draw the picture of a point and six lines coming out equi-angularly and the dot in the center corresponds to the integral point, the non dot lines correspond to having a $\mathbb{Z}/2/Z$ stabilizer, and the other points correspond to the generic section. I also learned that the $G(\mathcal{O})$ equivariant perverse sheaves on the affine Grassmannian which are constructible with respect to any stratification are constructible with respect to the Bruhat stratification.

(4/22/2020) Today I learned a vague claim that any $\otimes$-excisive functor is given by factorization homology on some manifold, and this is used to prove non-abelian Poincare duality. I also learned that there are functors for each simple root α on category $\mathcal{O}$ which is defined by pull pushing on the map $\mathcal{D}(G/B) \to \mathcal{D}(G/P_\alpha)$, and that these functors commute with the *wall crossing functors*, which are endofunctors of the translation functors going to a μ from, eg, 0 and back. I also learned a quick way to show that these wall crossing functors are self adjoint (which stems from the fact that the adjoint of the translation functor $T_{0\to\mu}$ is $T_{\mu\to 0}$.

(4/23/2020) Today I learned (finally!) an interpretation of the Mellin transform of the $\mathbb{G}_m$

averaging being zero for $\mathbb{G}_m$ associated to a coroot of the torus. Specifically, each coroot gives rise to an object in $\mathfrak{t}^*$ and the averaging to zero should (conjecturally) correspond to being integral on the weight associated to this coroot (since you can naturally view coroots as living inside $\mathfrak{t}^*$). I also learned that there are three types of quantum groups (the small Quantum group, Lusztig's quantum group, and a Kac dC quantum group) and that a quantum group is in particular a q deformation of the Hopf algebra associated to a group.

(4/24/2020) Today I learned a general heuristic which says that a factorization $\mathcal{E}_n$ structure corresponds to an $\mathcal{E}_{n+2}$ structure, and that it is through this which the strongest version of the Geometric Satake is proven. Specifically, one can show that the derived category of bi-equivariant sheaves on the affine Grassmannian is actually symmetric monoidal. I also learned that the object $\mathcal{M}_{univ}$ is a compact generator of the category $(\mathfrak{g} - Mod)^N$ since the Ind functor is a left adjoint and, by unipotence of N, the trivial representation generates $(\mathfrak{n} - Mod)^N$.

(4/25/2020) Today I learned a useful lemma. Specifically for SL_3 (although this can be generalized) we have that on the full subcategory of $\mathcal{D}(T)$ for which averaging with respect to the subgroup $\mathbb{G}_m$ associated to the coroot of (2,3), then either all other lattice points are nonzero averaging or we are in the subcategory of objects for which the full T averaging is zero. I think this will be useful someday.

(4/27/2020) Today I learned some ideas that go into proving that the wheel graph is a nonzero element in the graph complex. Specifically, one can use the universal coefficient theorem to reduce this to showing this for the reals or the complex numbers, and one can create an integral on a configuration space of points for these specific $g + 1$ vertex graphs with $2g$ edges. One can then integrate this and compute that it is nonzero on the wheel graph and satisfies the conditions we want to actually be a cycle in the graph complex.

(4/28/2020) Today I learned a potential interpretation which unifies being integral with respect to all roots for all groups. Specifically, I learned that (at least for the SL_3 case) for a simple root s, if the T averaging associated to every other transposition is zero, then the restriction to the associated embedded $SL_2 \times T'$ for a complementary torus entirely determines the whole space.

(4/29/2020) Today I learned a theorem of Weyl's which states that any finite dimensional representation of a compact Lie group must be semisimple. I also learned that this generalizes what's happening in finite groups, where the hypothesis of compactness is used so that the notion of summation in finite groups can be replaced with the notion of integration.

(4/30/2020) Today I learned that for any class $\lambda \in \mathfrak{t}/\Lambda$, the associated subroot system Φ_λ as above is a closed subroot system, and so in turn determines an embedding of semisimple Lie algebras. I also learned (for a fact today!) that the averaging functor on $\mathcal{D}(G/B)$ takes the Whittaker sheaf on N^- to the big projective object.

March 2020

(3/1/2020) Today I learned an alternative characterization of quasiaffine schemes–specifically, I learned the notion of quasiaffine morphism, which is simply just a morphism of schemes for which the preimage of any affine open set is a quasiaffine variety! This makes it clear that the vector bundle of $G/[P_s, P_s]$ associated to a simple root s is quasiaffine, because it maps down to a quasiaffine space via an affine map. I also learned that it need not be the case that, even though a left adjoint is exact, the right adjoint must be. Specifically, the morphism $\mathbb{A}^2\backslash 0 \to *$ is flat because it factors as a composite of two flat morphisms, and therefore the pullback is exact. However, the pushforward is not exact!

(3/2/2020) Today I learned a sketch of the proof of Riemann-Roch without Serre duality.

Namely, you can show it explicitly for effective divisiors by induction, with your base case being the curve. I also learned a theorem of Drinfeld, which says that for any two dimensional representation of the absolute Galois group of the local field, you can find the associated trace and determinant of the Frobenius and construct an associated unramified automorphic form whose Hecke operators have eigenvalues of the associated trace and determinant.

(3/3/2020) Today I learned an outline of the proof of the above theorem of Drinfeld. Specifically, given any (absolutely) irreducible representation of the etale fundamental group of a curve, you can take the associated trace at each Frobenius and the determinant of the rep (by geometric class field theory) gives rise to a map $Pic(X) \to \mathbb{Q}_l$. In turn, you can look at the Whittaker functions on $G(\mathbb{A})$ which are automorphic, and we can use the fact that these sorts of reps are the same as "almost automorphic" unramified cuspidal functions and you can use a specific criterion to show that these functions actually descend to an honest automorphic unramified cuspidal function.

(3/4/2020) Today I learned a fun little fact about the space $\mathfrak{b}/B$. Explicitly, you can pull back the map $\mathfrak{b}/B \to \mathfrak{g}/G$ via the inclusion of the regular semisimple elements of $\mathfrak{g}$, which in particular will yield a W torsor!

(3/5/2020) Today I (re?)learned the definition of a Poisson bracket on an algebra. Specifically, this is a bracket on some algebra A making the algebra into a Lie algebra for which all $f \in A$, 'bracketing with f' acts as a derivation. I also learned that given a symplectic manifold, you can create a Poisson bracket structure on global functions as follows. Explicitly, given a symplectic manifold, one can create an assignment of $f \mapsto \xi_f$, where ξ_f is a vector field given by $\omega(df, -)$.

(3/8/2020) Today I learned a possibly easy consequence of the fact that you can view the category $H^0\Gamma(\mathcal{D}_{G/N}) - Mod$ as glued together from $|W|$ many copies of the category $\mathcal{D}(G/N)$. Specifically, there is a category whose modules come from the untwisted copy of G/N, and for each simple reflection you can define the full subcategory closed under that simple reflection subgroup (of order 2). I expect that this will coincide with the associated vector bundle, and easily!

(3/9/2020) Today I learned some motivation for using Deligne-Mumford stacks. Specifically, the definition of a normal divisor requires that the ambient space be smooth. However, the moduli space of genus g curves with n marked points is only smooth as a Deligne-Mumford stack; the coarse moduli space is not smooth. I also learned of a generalization of a delta complex, known as a *symmetric delta complex*, which allows for delta complexes without a canonical choice of coordinates. This allows one to construct, for example, the 'half interval' as a geometric realization of a symmetric delta complex which cannot be realized as the geometric realization of a usual delta complex.

(3/10/2020) Today I learned that any irreducible representation of any Lie algebra $\mathfrak{g}$ in characteristic p is automatically finite dimensional, and in fact, it is bounded by $p^{dim(\mathfrak{g})}$. I also learned the existence of something called the *Metaplectic group*, a double cover of the symplectic group which comes equipped with a unique irreducible unitary representation.

(3/11/2020) Today I learned an interpretation which makes the Whittaker character and the associated Whittaker sheaves more canonical. Namely, given any G-bundle $\mathcal{P}$ on a scheme, one can use this bundle to identify G bundles with $Aut_G(\mathcal{P})$ bundles such that the trivial bundle associates to $\mathcal{P}$. With this interpretation, given a curve X one can consider the *pure inner form* of G, $\tilde{G}$. This has an associated $\tilde{N}$ which identifies with one forms on the curve and so this has an associated character $\tilde{N} \to \mathbb{G}_a$ given by the residue map!

(3/12/2020) Today I learned a general proof strategy to knocking off my functor being fully faithful (so long as it exists). Specifically, I can find some sheaf $\mathbb{G}$ in H_ψ supported away from N^- for which $Hom(\mathcal{F}, \mathbb{G}) \neq 0$ while the associated fixed points $Hom(\mathcal{F}, \mathbb{G})^W$ are zero, then you've shown that $F^W = \delta_{N^-,\psi}$. This is, of course, assuming the existence of the functor!

(3/13/2020) Today I learned a heuristic idea for a proof of why the tempered/cuspidal Geometric

Langlands conjecture is true. Specifically, for any cuspidal automorphic form for $G = GL_2$, we can forget the $G(F)$ (where F is the function field of the associated curve) down to $N(F)$ equivariance and then, using a lemma which says that $N(\mathbb{A})/N(F)$ is compact, we can Whittaker average to obtain an object in $Fun(N(\mathbb{A})_\psi \backslash G(\mathbb{A})/G(\mathcal{O})$, a a miracle says that this Whittaker averaging procedure yields an isomorphism between that sheaf and $Funct(H(F)\backslash G(\mathbb{A})/G(\mathcal{O})$. The heuristic of the proof is that for any action, the N part either acts trivially or by another character, and these the multiplicative part of $B, \mathbb{G}_m$, permutes.

(3/14/2020) Today I learned a way to prove that there can be biWhittaker sheaves on the big cell. Specifically, because $w_0N^- = Nw_0$, we obtain that (on k points) there is an injective map $N^-Tw_0N^- \to G$ given by multiplication for all G. Specifically, picking your favorite point g in that Bruhat stratification, we obtain that by base change, the restriction of the pushforward of the 'bi-averaged' delta sheaf at g restricts to a one dimensional vector space at g itself.

(3/15/2020) Today I learned some basics as to how to encode things in quantum mechanics. Specifically, now that things like position are viewed as probabilities, the space of positions is a Hilbert space (eg, like $L^2(\mathbb{R}^3)$ and has a certain time evolution attached to it, which can be encoded as a Hamiltonian. Further still, you can realize things like momenta as certain invariants attached to symmetries that you expect the thing associated to your Hamiltonian to hold, and you can obtain the associated Lie algebra map as an invariant quantity.

(3/16/2020) Today I learned the basic examples of local systems on the punctured disk for various reductive groups G. Specifically, if G is a torus, one can realize the quotient $\mathfrak{g}((t))/G(K)$ as a (product of) quotients such that when you just quotient by $G(\mathcal{O})$ (or more accurately, its formal completion inside $G(K)$), you will get a product of spaces of the form $Kdt/\mathcal{O}dt$, where the remaining/residual $G(K)/G(\mathcal{O})_0^{hat}$ action acts by translations and only acts on the t^{-1} factor.

(3/17/2020) Today I learned more examples of local systems on the punctured disk for various groups. Specifically, I learned that you can construct a map from $\mathbb{A}^\infty \to LocSys_B$ which does not factor through any finite $\mathbb{A}^n$, and therefore shows that $LocSys_B$ is not locally of finite type. This same proof also shows that the space $\mathfrak{g}[[t]]\frac{dt}{t}/G(\mathcal{O})$ is not locally of finite type either, since the map to $LocSys_B$ factors through this subspace.

(3/18/2020) Today I learned an un-fun but necessary fact about showing that the nondegenerate $G-$subcategory of $\mathcal{D}(G/N)$ obtains a W action. Specificailly, for each simple reflection s, one may embed G/N as the complement of the zero section of a certain vector bundle associated to s. One might hope to show that the nondegenerate subcategory of $\mathcal{D}(G/N)$ is the complement to the $P-$monodromic subcategories of $\mathcal{D}(V_s)$. However, there are certain sheaves in the zero section of the vector bundle which are not $P-$monodromic, and you must show that the symplectic Fourier transform maps anything in the zero section to something monodromic.

(3/19/2020) Today I learned the best outline of the proof that the characteristic polynomial exists so far, concerning surjectivity of the restriction map $Sym(\mathfrak{g}^*) \to Sym(\mathfrak{t}^*)$. Namely, one can pull back the W equivariant functions on T up to global functions Grothendieck-Springer resolution. Then you can argue that the map $\mu : \tilde{\mathfrak{g}} \to \mathfrak{g}$ is induces a map on the regular elements $\tilde{\mathfrak{g}}^{reg}/W \to \mathfrak{g}^{reg}$ is $O-$connected and so you have constructed a way to get a global function on $\mathfrak{g}^{reg}$, which are (by codimension reasons) global functions on $\mathfrak{g}$.

(3/20/2020) Today I learned a strategy/conjecture which may lead to showing that the averaging functor is fully faithful for general G. Specifically, one can decompose G into its N^-, N^- Bruhat cells and conjecture that for any simple reflection $s \in W$ associated to a simple root, that $Hom_{\mathcal{D}(T)}(\delta_1, Av_*(\mathcal{G}))$ has W fixed points if and only if W is supported on the $N^-w_0sN^-$ cell. Using this, I learned that $Av_*(B(\delta_{(1,2,3)}))$ can be restricted to the torus associated to the root $(2,3)$ and we obtain the same equivariant sheaf as in the SL_2 case.

(3/21/2020) Today I learned the restriction to the sheaf $Av_*^\psi Av_*(\delta)$ to the Bruhat cell associated to the reflections (1,3)(1,2) is zero on the $T(1,3)(1,2)$ part except for on the diagonal where the lower right T coordinate is 1.

(3/23/2020) Today I learned that for all nondegenerate characters $\psi : N^- \to \mathbb{G}_a$ and any category acted on by G, say $\mathcal{C}$, the associated Whittaker invariants are isomorphic, and canonically isomorphic if G is adjoint. This is because G acts on the space of all characters transitively, and if G is an adjoint group, this action is simply transitive, furthermore. One interpretation of this (which I also learned today) is that the character is equivalently a choice of Chevalley generators.

(3/24/2020) Today I learned where the Biwhittaker of the delta sheaf at a representative of the longest element of the Weyl group group restricts to only a certain codimension 1 subset of the torus, and furthermore on that torus there is (mostly) a 2:1 cover of the points hitting it.

(3/25/2020) Today I learned a kind of fun way that integers appear in $\mathcal{D}$-modules. Specifically, consider the connection $e^{t+\frac{c}{t}}$ for some nonzero $c \in \mathbb{C}$. Then this connection yields a collection of "half integers" which vanish if and only if $c \in \mathbb{Z}$.

(3/26/2020) Today I learned what it means for a flat morphism of integral finite type schemes to be smooth of relative dimension n. Specifically, such a morphism is smooth if and only if the sheaf of relative differentials is a locally free sheaf of rank n. I also learned about a specific non-example that came up in the fully faithfulness thing–namely, the scheme $k[x,y,z]/(xy = z)$ is $k-$smooth, but the morphism to $Spec(k[z])$ is not because dz vanishes at the point (0,0,0).

(3/27/2020) Today I learned the specific example of how to show that the nondegenerate category of $\mathcal{D}(G/N)$ is closed under the W action for $G = SL_3$. Specifically, I learned that for each fixed root, the parabolic at each fiber will either be (modulo a constant sheaf) a delta sheaf on the punctured line or the punctured plane.

(3/29/2020) Today I learned an outline of showing that the nondegenerate subcategory of $\mathcal{D}(G/N)$ acquires a W_s action. Specifically, you can simply show that the full G subcategory of $\mathcal{D}(V_s)$ (where V_s is the vector bundle associated to the simple root s) which contains the entire zero section of the vector bundle and the higher P-monodromic objects of the embedded $\mathcal{D}(G/N)$ category is closed under the W action. To do this, you can simply argue this for the minimal parabolics. For the minimal parabolic associated to s you can almost write it out explicitly and argue that the G-subcategory is W stable, and for the other minimal parabolics you mostly use the fact that the constant sheaf on $\mathbb{A}^1$ is W stable.

(3/30/2020) Today I learned the general idea behind why the category of $H^0\Gamma(\mathcal{D}_{G/N}) - Mod$ acquires a strong $G \times W$ action. Specifically, given an algebra A with a strong G action and a finite group W, the strong G action upgrades to a strong $G \times W$ action with the 'same' data if the associated lifting of the map $\mathfrak{g} \to Der(A)$ to inner derivations (i.e. the $G-$map $\mathfrak{g} \to A$ of Lie algebras) upgrades to a $G \times W$ equivariant map. In other words, the original Harish-Chandra data maps into W equivariant objects of A.

(3/31/2020) Today I learned the specific example of the Linkage principle, which says that there is an extension of two simple objects in SL_2 if and only if their associated highest weight integer is in the same W^{aff} orbit with the $p-$dialated dot action, i.e. the group of automorphisms of the integers generated by reflecting about -1 and translating by p. This in particular says that the highest weight representations of weight p and $p-2$ are linked for we have a short exact sequence $L_p \to \nabla_p \to L_{p-2}$.

February 2020

(2/1/2020) Today I learned that for all quasiaffine varieties X, the derived global sections functor is conservative, and furthermore I learned a specific example for a (non-quasiaffine) variety X, namely

$\mathbb{P}^1$, which derivedly sends a sheaf, namely $\mathcal{O}(-1)$, to zero. I also learned an explicit computation for $Av_*Av_!(\delta)$.

(2/3/2020) Today I learned the statement of the Fundamental Local Equivalence in (quantum) local geometric Langlands, which says that the categories $Whit(\mathcal{D}_\kappa(Gr_G)) \cong (\mathcal{D}_{\check{\kappa}}(Gr_{\check{G}})^{G(K),w}$, which is an abstraction of the local geometric Langlands conjecture, which says that for positive or irrational $\kappa \in \mathcal{C}$, we have an equivalence of infinity, 2 categories given by $G(K) - Mod_\kappa$ and $\check{G}(K) - Mod_{\check{\kappa}}$.

(2/4/2020) Today I learned the actual explicit computation in verifying the symplectic Fourier transform on the delta sheaf on G/N for $G = SL_2$. Specifically, I learned that this Fourier transform takes δ to the pushforward of ψ in the line $x = 0$, which, up to a constant function, goes back to the delta sheaf when you reapply this symplectic Fourier transform!

(2/5/2020) Today I learned a way to actually compute (using ideas from 2/2) that the sheaf cohomology of a quasicoherent sheaf on a quasiaffine variety X actually vanishes. Specifically, you can write X as the union of $Spec(A_f)$'s, and then when you're doing the tensor product given by testing whether the open embedding $X \to Spec(A)$ is fully faithful, you can write out $\Gamma(\mathcal{O}_X)$ as a the cokernel of things that are of the form A_f's for the various f's. All the A_f are flat A-modules though, so the tensor product becomes localization and gives fully faithfulness for at least commutative f.

(2/6/2020) Today I learned a sort of general framework which allows one to view the Fundamental Local Equivalence as a quantum version of the Geometric Satake equivalence. Specifically, the "usual" equivalence of $Rep(\check{G}) \cong \mathcal{D}(G(\mathcal{O})\backslash G(K)/G(\mathcal{O}))$ is not an equivalence at the derived level, and so it needs to be replaced with the derived equivalence $Rep(\check{G}) \cong \mathcal{D}(G(K)/G(\mathcal{O}))^{N(K),\psi}$. With this, and the knowledge that if κ is an integral level, $Rep(\check{G}) = (\hat{\check{g}}_\kappa - Mod)^{G(\mathcal{O})}$, one can view the fundamental local equivalence as this κ quantum deformation.

(2/7/2020) Today I learned a neat little way to construct a twisted $\mathcal{D}$ module on some space or group X. Specifically, given a punctured line bundle $\mathring{\mathcal{L}} \to X$, you can view this as a $\mathbb{G}_m$-torsor over X. Specifically, given any $\lambda \in \mathbb{C}$ there are special sheaves to be viewed as $z^\lambda \in \mathcal{D}(\mathbb{G}_m)$, you can set $\mathcal{D}_{\lambda,\mathring{\mathcal{L}}}(X) := \mathcal{D}(\mathring{L})^{\mathbb{G}_m,\lambda}$.

(2/8/2020) Today I learned a sort of motivation for spectral sequences. Namely, in my actual research life I have come up with a graded complex with a filtration–specifically, the tensor product $\Gamma(\mathcal{D}_{G/N}) \otimes_{H^0\Gamma(\mathcal{D}_{G/N})} \Gamma(\mathcal{D}_{G/N})$ and am trying to compute the homology groups of this complex. You can think that if you have a filtered map of complexes that for the i^{th} stage of the filtration that that map (or homology of the complex if the differential preserves grading) might send your i^{th} associated graded piece, $(i-1)^{th}$ piece, ..., all the way down to the 0^{th} graded piece. This explains the page turning of spectral sequences, so to speak.

(2/10/2020) Today I finally learned the proper motivation behind Riemann-Roch theorem! Namely, given a d fixed distinct points x_i on a Riemann surface (for example, $\mathbb{C}$), you can ask what the dimension of the space of functions is which have at worst one pole at those points (which you can encode as a divisor). You can encode this information as "genuine" global functions (i.e. taking away the constant functions) and obtain the exact sequence $0 \to \mathbb{C} \to L(D) \xrightarrow{Res_{x_i}} \mathbb{C}^d$ (which you can view as this space of functions satisfying this) and you can ask for the constraints on the residue to recover the dimension of $L(D)$, i.e. identify the dimension of the images. Then you look for the conditions that for all $\omega \in \Omega^1(X)$, we need for any $f \in L(D), \sum Res_{x_i}(f\omega) = 0$ by some general complex analysis. You then mod out by the constraints which are trivial, which are cut out by one forms that totally vanish at all the points. If you denote this correction factor via $dim(\Omega_D(X))$, it turns out that this is in fact all of the constraints, and so this gives the basic Riemann Roch theorem, which says that $l(D) = 1 + d - dim(\Omega^1(X)) + dim(\Omega_D(X))$, and so then you obtain the first portion of the Riemann-Roch theorem, which says that $l(D) \geq 1 + d - g$, for $g := dim(\Omega^1(X)$ the genus. I also learned that all closed embeddings are projective, and that (at least over an algebraically closed field) the obstruction for a projective morphism to be a closed embedding is injectivity on points and injectivity on the maps on tangent spaces (and that the Frobenius shows that the last condition, i.e. that the map is *unrammified*, is needed).

(2/11/2020) Today I learned that the obvious diagram that I might hope commutes in order to reduce my functor to the SL_2 case does a general reductive G does not in fact commute. The fact that inclusion doesn't commute with averaging can be shown by checking where the delta sheaf for the identity point goes, because for one direction of the commuting the sheaf will have support only on the $N \cap SL_2$ part.

(2/12/2020) Today I learned a few things from analyzing my averaging functor on the N invariants of a category. Specifically, I learned a recollement setup on $N\backslash G/N$ given by $\mathbb{G}_m$ (viewed as $N\backslash B^c/N$) as an open substack of $N\backslash G/N$, with complimentary closed substack $N\backslash B/N$, which, since the action of N on B/N is trivial, is the stack $\mathbb{G}_m \times B\mathbb{G}_a$.

(2/13/2020) Today I learned a few new approaches in showing that my functor has essential image characterized by having T averaging giving a trivial W representation. Specifically, using the fact that the N invariants of a $DGCat$ with a G action G generates the category itself, you can take N invariants to get a functor $\mathcal{D}(T) \to \mathcal{D}(N\backslash G/N)^W_{nondeg}$, which you can quickly argue T invariants. The problem then becomes showing the appropriate category is free.

(2/14/2020) Today I learned (solidified?) a way to classify maps to $\mathbb{P}^1$. Specifically, you can declare that any map to $\mathbb{P}^1$ is a line bundle plus two sections which collectively don't vanish anywhere. The map then corresponds to '$p \mapsto [s_0(p), s_1(p)]$', and you can use this to show that any curve minus a point q is affine. This is because Riemann-Roch essentially says that for any fixed q that for $j \gg 0$ that the line bundle $\mathcal{O}(jq)$ is ample and thus accordingly you get a map into $\mathbb{P}^N$ for some large N but you can use this nonvanishing section to then construct an affine map from your curve to $\mathbb{P}^1$ whose preimage at infinity is q!

(2/15/2020) Today I learned how to reduce an argument involving sheaves on the stack $N\backslash G/N$ to an argument involving just an open and closed subset of the line. Specifically, due to the fact that the pullback functor yields an equivalence $\mathcal{D}(*/N) \xrightarrow{\sim} Vect$, you can reduce a question about extensions along the closed subset $*/N$ to questions about the actual point without its N automorphisms. I also learned a fun lemma which says that if you have a sheaf $\mathcal{F}$ and an open and closed setup for which $Hom(j_*\mathcal{F}, i_*\mathbb{G}) = 0$ for all $\mathbb{G}$, then there cannot be any extensions of $\mathcal{F}$. This is simply because you can take any sheaf on your space and write its recollement sequence and then apply $Hom(j_*\mathcal{F}, -)$ to it. Using the fully faithfulness of j_* then yields the result!

(2/17/2020) Today I learned that in the hypothetical direct sum decomposition $\mathcal{D}(N\backslash G/N)^0 \cong \mathcal{D}(N\backslash B/N)^0 \oplus \mathcal{D}(N\backslash B^c/N)^0$, the associated map to $\mathcal{D}(N\backslash B^c/N)^0$ cannot just be given by (!-)restriction to the open embedding. This is because on the line, the exponential sheaf has sheaf cohomology zero and, yet, however, the restriction to the punctured line has nonzero deRham cohomology, which can be shown by applying the deRham pushforward to the recollement sequence, which says that the dR cohomology will be the (!-)restriction to the closed shifted by 1.

(2/18/2020) Today I learned the notion of the Tate conjecture, which is a generalization of the BSD conjecture, which says that for any smooth proper variety X over a finite field, the etale cohomology is generated by smooth proper subvarieties. In particular, I learned this conjecture reduces to a claim that an isogeny of elliptic curves over a finite field is equivalent to an isogeny of their *Tate modules*, i.e. the limit of their l torsion for l a fixed prime not p.

(2/19/2020) Today I learned the notion of a *Kuranishi family*, which is, roughly speaking, a local moduli space. I also learned the fact that states that a curve admits a Kuranishi family of deformations if and only if it is stable. I also learned (solidified?) the existence of a Hilbert polynomial, and that a morphism is flat if and only if along the fibers the Hilbert polynomial is constant.

(2/20/2020) Today I learned a slick way to define twisted $\mathcal{D}$ modules on the flag variety for SL_2 (or really, any reductive group). Specifically, note that you have a central extension (or, really, T torsor) given by $G/N \to G/B$, and on $\mathcal{D}(T)$ there exist a $\mathcal{D}$ module t^λ for any $\lambda \in \mathfrak{t}^*$. With this, you can define λ twisted $\mathcal{D}$ modules as $\mathcal{D}_\lambda(G/B) := \mathcal{D}(G/N)^{T,z^\lambda}$

(2/21/2020) Today I learned one way to possibly show that on $\mathcal{D}(N\backslash G/N)^0$ (i.e. those sheaves which have no T-average), that all sheaves extend cleanly *on the quotient category*, i.e. if $\mathcal{F} \in \mathcal{D}(N\backslash B^c/N)$ and $\mathbb{G} \in \mathcal{D}(N\backslash B/N)$ have no de Rham cohomology, then $Hom(j_*\mathcal{F}, i_*\mathbb{G}) = 0$. This is because the W action swaps those things supported on the big cell and those things supported on the small cell, so this then becomes a $Hom(i_*\mathbb{G}', j_*\mathcal{F}')$, which is zero!

(2/22/2020) Today I learned a way to describe the ring of functions on the basic affine space. Specifically, one can describe the ring as a direct sum of irreducible representations associated to each dominant weight (so, for example, the 1st weight for SL_2/N is $kx_1 + kx_{-1}$ as a vector space) with multiplication given by projecting onto the tensor product which contains the sum of the associated weights as a subspace. I also learned that this is finitely generated by the weight one elements!

(2/23/2020) Today I learned the explicit construction for the simple root $\alpha := L_1 - L_2$ of $\mathfrak{sl}_3$ which exhibits a symplectic rank two vector bundle over the space $\mathbb{A}^3\backslash 0 = SL_3/Q_s$. Specifically, one can take SL_3 and mod out by everything in N except for that the slot where the simple root would go must be zero. This in turn has an action of SL_2 (embedded as a root subgroup) and then one can define the associated vector bundle as $G/U_s \times^{SL_2} k^2$. Using the map $SL_3 \to G/U_s \times^{SL_2} k^2$ obtained by sending $g \to gU, e_1$, this induces an open embedding $SL_3/N \to G/U_s \times^{SL_2} k^2$.

(2/24/2020) Today I learned a kind of cool way to piece together how the Weyl group action on the affine closure of G/N breaks into pieces. Specifically, for each simple reflection s in the Weyl group (or for each simple root s) you can construct the vector bundle V_s as above. This vector bundle contains G/N as a codimension two subset and therefore the ring of functions are the same, but it is also quasiaffine because it admits an affine map to a quasiaffine scheme $G/[P_s, P_s]$ and therefore it admits an open embedding to $\overline{G/N}$!

(2/24/2020) Today I learned how to explicitly identify the quotient of SL_3 by a parabolic and its associated commutator subgroup. Specifically, you can view SL_3 as acting on either k^3 or its projectivization and determine that the stabilizer of the vector e_3 is given by the associated parabolic subgroup (up to transpose). I also learned that the commutator of this group is the group with zeroes in the third row except for the bottom right corner, which has a one.

(2/25/2020) Today I learned some kind of physical notion which gives rise to the statement that all observers see light traveling at the same speed. Specifically, one can consider the worldline of a particle moving in a lightlike manner in all points and assume it meets (really, is very close to) an observer moving in a timelike direction. You, the timelike observer, can then, after rescaling the light's "objective" time to your time, and then project onto your spacetime coordinates and compute that you will always measure the velocity to be c!

(2/28/2020) Today I learned a neat little way to compute the Hochschild Homology of the space BG. Specifically, one way you can compute the Hochschild homology of any space is to compute the categorical Hochschild homology of its associated category of quasicoherent sheaves on it. But then one can note that quasicoherent sheaves on BG are equivalent to representations on G and so it suffices to compute Hochschild homology of the category of representations!

January 2020

(1/1/2020) Today I learned (worked out?) the specific details behind the proof of the following mostly tautological lemma. Specifically, given two groups H, K and a map of groups $K \to Aut_{Gp}(H)$, then we can form the semidirect product $H \rtimes K$. Tautlogically, an action of this semidirect product is given by an action of H and an action of K which satisfy the compatibility condition of $k * (h * x) = (k * h) * (k * x)$. Using this, I can upgrade the argument to show that the functor of Av_*^N is monoidal when lifted to $\mathcal{D}(T)^W = End_{\mathcal{D}(T \rtimes W)}(\mathcal{D}(T))$.

(1/3/2020) Today I learned a specific example of a sheaf which has a nonzero stalk with respect to the ! fiber but not with respect to the * fiber. Specifically, considering the inclusion $j : \mathbb{A}^1 \setminus 0 \hookrightarrow \mathbb{A}^1$, we can ! pushforward the constant sheaf. By base change, the * fiber is nonzero. However, noting that the Verdier dual of the constant sheaf is the constant sheaf up to shift, we obtain that $i^! j_!(k) = Di^* DD j_* D(k) = Di^* j_*(k[?]) \neq 0$, so $i^! j_!(k) \neq 0$.

(1/4/2020) Today I learned that the quotient functors by an affine algebraic group is exact, because the quotient map is an affine map and pushing forward by an affine map is exact. A consequence of this is that, for example, the functor of pushing forward via $* \to */W$ is exact, and pushing forward via $G \to G/N$ (so weak averaging is exact). However, strong averaging is not, even if $n = \mathbb{A}^1$.

(1/5/2020) Today I learned about the *categorical sign representation*. Specifically, one can consider two actions of the standard Hecke category $\mathcal{H} := \mathcal{D}(B\backslash G/B)$ on the category $Vect$–one given by the action of $\mathcal{H}$ on $\mathcal{D}(G/B)^{N^-,\psi}$, and the other on the most singular block of the action of $\mathcal{H}$ on $\mathfrak{g} - Mod^B$. It is a theorem that these two actions agree as $\mathcal{H}$ modules, and hopefully I can work through the argument that shows that they agree as $\mathcal{H}_\psi \cong QCoh(\mathfrak{t}//W^{aff})$ modules as well.

(1/6/2020) Today I learned some fun orienting facts about some of the groups involved in the local Langlands program. In particular, while the Iwahori subgroup is explicitly (by definition) contained in $G(\mathcal{O})$, neither are contained in the group $B(K)$ because $\begin{pmatrix} 1-t & 1 \\ 1+t+t^2+... & 1 \end{pmatrix}$ is contained in $G(\mathcal{O})$ and not in $B(K)$, and $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ is in $B(K)$ but not in $G(\mathcal{O})$ (and thus not in the Iwahori I). Of course, by construction, we have $B(\mathcal{O}) \subseteq G(\mathcal{O})$ and $I \subseteq G(\mathcal{O})$ and thus a surjection $G(K)/I \to Gr_G = G(K)/G(\mathcal{O})$.

(1/7/2020) Today I learned a conjectural thing inside of the twisted Hecke category which could potentially hit the constant sheaf inside $\mathcal{D}(T)$ when applied to with the Av_*^N functor. Specifically, assuming that the identification $\mathcal{H}_\psi \cong QCoh(\mathfrak{t}//W^{aff})$ is monoidal, the delta sheaf at zero is associated (when localized or whatever) to the integral part of the category. Therefore it makes sense that when you average it (conjecturally, the same as pulling back to the stack quotient) it should go to something that is B-equivariant. Conveniently, there is only one thing that is B equivariant, the constant sheaf!

(1/8/2019) Today I learned a heuristic for why the equivalence of categories $\mathcal{D}(N(K), \psi\backslash G(K)/I) \cong \hat{\check{\mathfrak{g}}} - Mod^{\check{I}}$ is true. Specifically, roughly viewing the Iwahori as the analogue of the Borel, you can check that the left hand side has orbits labeled by $W^{aff}/W = \check{\Lambda}$, and meanwhile, the right hand side is analogous to $\mathfrak{g} - Mod^B$, whose objects are labeled by Vermas with dominant integral weights, i.e. Λ.

(1/10/2020) Today I learned the notion of a graded category. Specifically, one can define a graded category as a category with an action of the monoidal category $Rep(\mathbb{G}_m)$, or, similarly to vector spaces, you can define it as a category with a $\mathbb{G}_m$ action. These categories are the graded lifts, and the forgetful functor is given by the map which "degrades" vector spaces or forgets the action of the monoidal category $\mathbb{G}_m$.

(1/12/2019) Today I learned the definition of a map of Artin stacks being flat. Specifically, it is defined to be local on the target (for any map of affine schemes into the stack) and then a map is flat if for some (equivalently, for any) smooth covering of the domain the map is flat.

(1/13/2020) Today I learned that the functor of induction $w_{Ind} : \mathcal{D}(T)^W \to \mathcal{D}(G)^G$ is exact. I also learned the general outline of David and Sam's paper about the Ngô functor. Specifically, they show that for any groupoid mapping on a stack X, you can show that there is an $\mathbb{E}_2$ map from the equivariant aheaves on the stack to the center of modules for the associated Hecke category. Running this machine through the geometric Satake correspondence yields the (cohomologically sheared) Ngô map.

(1/14/2020) Today I learned that the formal completion of a Noetherian ring A at any ideal is a flat A module. I also, with this information, learned a potential atlas for a flat covering of $\mathfrak{t}//W^{aff}$– specifically, the covering of the "open" subset of nonintegral points and the formal completion of the closed point.

(1/15/2020) Today I learned the notion of 1-affineness of a prestack $\mathcal{Y}$. Specifically, for a prestack $\mathcal{Y}$, one may define the notion of a category over the prestack as a $QCoh(\mathcal{Y})$-module category or a compatible system for every A-point for every commutative algebra A. There are accordingly localization and global sections functors, and a prestack is one affine if these are equivalent categories. I also learned any Artin stack is 1-affine, which is a theorem of Gaitsgory.

(1/16/2020) Today I learned about the notion of 2IndCoh. Specifically, for a prestack X, one can define the notion of $2QCoh(X) := QCoh(X) - Mod$ as a two category. However, due to the notion of Betti Geometric Langlands (which I also learned today, a conjecture which identifies topological sheaves on Bun_G with a certain kind of coherent sheaf of local systems of the Langlands dual group) modeling after the usual geometric Langlands, it requires a different category of Ind coherent sheaves. This lends itself to the notion of taking Ind-Coherent sheaves on IndCoh(X) for X smooth.

(1/17/2020) Today I learned some ideas that go into the Ngô map and the group scheme of regular centralizers. Specifically, I learned that for any regular element $x \in \mathfrak{g}$, you can identify the centralizer of x in G canonically with its associated *Kostant slice*, a certain section of the characteristic polynomial (think "all 1's on the off diagonal). Because of this isomorphism and for codimension reasons, one can give a map from the regular centralizers to all centralizers (really, the "inertia stack").

(1/18/2020) Today I learned an interpretation of the Ngo map on a torus T. Specifically, the map is not the pushforward from $T \to T \times BT$, but rather a map only guaranteed by the fact that any symmetric monoidal category gets a map to its Drinfeld center.

(1/19/2020) Today I learned (okay, solidified?) one way to classify the essential image of the averaging functor. Specifically, one can show that the left adjoint, when restricted to the full subcategory generated by those objects, is fully faithful. Essential surjectivity then follows from the equivalence of fully faithfulness with the associated unit of the adjunction being an isomorphism (the latter giving the essential surjectivity).

(1/20/2020) Today I learned a strong hint as to why the category of adjoint equivariant sheaves on G should be the center of the category of $\mathcal{D}$ modules on G. Specifically, if $h \in G$ is an ad-equivariant point, that's just another way to say it's in the center of the group. This idea provides the map $\mathcal{D}(G)^G \to Z(\mathcal{D}(G))$.

(1/21/2020) Today I learned how to actually make rigid (in a categorical sense) the Loc functor, which for an action G on X yields a map $U\mathfrak{g} \to \Gamma(X, D_X)$. Specifically, one has a map $X/G_1^{\wedge} \to X_{dR}$ (because the associated action of $G_1^{\wedge}$ on X_{dR} is trivial, so we can take the map $X/G_1^{\wedge} \to X_{dR}/G_1^{\wedge} \cong X_{dR} \times */G_1^{\wedge} \to X_{dR}$. Push pull (and the fact that $IndCoh(BG_1^{\wedge}) = \mathfrak{g} - Mod$ gives us the map.

(1/22/2020) Today I learned one interpretation of the geometric Satake theorem. Specifically, one can (noting the Morita equivalence of $\mathcal{D}(G)$ and the Harish-Chandra category) consider the cohomologically sheared version of the Harish Chandra version, $\mathcal{HC}_{\hbar}$. This geometric Satake theorem says that this category for $\check{G}$ is equivalent, as a monoidal category, to $\mathcal{D}(G(\mathcal{O})\backslash G(K)/G(\mathcal{O}))^{\mathbb{G}_m}$, where the action of $\mathbb{G}_m$ is given by loop rotation.

(1/24/2020) Today I learned a trick which allows you to say that tensoring with certain categories commutes with Cartesian products. Specifically, if the category is dualizable, then we can write tensoring as Homs out of the dual category, which commutes with limits. You can then dualize the category back over.

(1/25/2020) Today I learned a neat little trick to classify some maps from $\mathcal{D}(G)_{nondeg}$. It's not hard to actually compute maps from this semi-explicitly (it's a colimit), but the real "trick" here is that if $\mathcal{C}$ is any nondegenerate category itself, all of the "later terms" in the limit actually cancel and you can identify with the quotient functor the functor $Hom_G(\mathcal{D}(G)_{nondeg}, \mathcal{C}) \xrightarrow{-\circ Q} Hom_G(\mathcal{D}(G), \mathcal{C})$.

(1/26/2020) Today I learned a further reduction in the computation of the essential image of the long twisted averaging functor. Specifically, one can show that because the property of being "B-monodromic" (really, being in the essential image of $oblv^T$) is a G equivariant property, that it suffices to show (if true) that the map on the usual category $Av_*Av_!(\delta_{1N}) \to \delta 1N$ has kernel which is B equivariant. I also learned (sort of again?) what an oper is, at least on the punctured disk. Specifically, you can define an oper on the punctured disk as the objects $(f + \mathfrak{b}((t)))/N(K)$.

(1/27/2020) Today I learned a neat little computation which compares N^-B and BN^-. Specifically, you can identify the former as those matrices in SL_2 whose 11 coordinate is nonzero, and you can identify the latter as those matrices whose 22 coordinate is zero. I also learned that my finite group analogy approach for essential surjectivity of the functor is bunk, because it in essence can't take into account that the de Rham cohomology is zero. Alternatively, one can note that the $Av_!$ functor (as a functor of W categories) really does take into account the W equivariance of the objects.

(1/28/2020) Today I learned of a potential hangup in using a universal case argument to classify the essential image of the averaging functor. Specifically, it is difficult to $\mathcal{D}(G/N)_n$ as a W category to satisfy a universal property l. I also learned that, given a map of connective cdga's, there exists a unique map $R \to H^0R$ and this map is fully faithful it and only if the map is a ring isomorphism.

(1/30/2020) Today I learned that, given connective (derived) rings A, B, C there is a spectral sequence starting on the second page with $Tor_p^{\pi_*C}(\pi_*A, \pi_*B)_q$ which converges to $\pi_{p+q}(A \otimes_C B)$. The rings π_* are to be interpreted as classical graded rings, which in turn the Tor then becomes a graded ring.

(1/31/2020) Today I learned a theorem of Sam's which is the correction needed to get the Beilinson-Bernstein localization theorem to work at critical level in the local story. Specifically, one can recognize that the original global sections functor is a map of $Rep(\check{G})$ modules, and that the delta sheaf goes to the vacuum representation, which does not have endomorphisms k. You tensor this category over $Op_{\check{G}}$ and you get the associated functor that's conjectured to be an equivalence!

December 2019

(12/1/2019) Today I learned an explicit construction which yields an equivalence between rank n vector bundles on some scheme X and GL_n bundles over them. Specifically, given a GL_n torsor $\mathcal{P} \to X$, we in particular obtain a (right, let's say) GL_n action. Thus, if we denote the standard representation by k^n, we can form the total space of a vector bundle $\nu : \mathcal{P} \times^{GL_n} k^n \to X$. The local triviality of $\mathcal{P} \to X$ says that locally ν looks like $X \times k^n \to X$!

(12/3/2019) Today I learned a heuristic which explains Scholze's Primitive Comparison Theorem (one of the key steps in proving the comparison theorem in p-adic Hodge theory). Specifically, if V is a vector space over an algebraically closed field k of characteristic p, then we can recover the vector space from its Frobenius fixed points (as an $\mathbb{F}_p$ vector space) by tensoring up with k. This holds due to the fact that the Frobenius twisted conjugation action $g \cdot^{\sigma} h := Fr(g)hg^{-1}$ when $h = 1$ yields an etale map (by the differential criterion, since differentials make the Frobenius properties and so roughly speaking this map has all the differential properties of the right hand action).

(12/4/2019) Today I learned that I can explicitly compute equivariance given by the action of an algebraic group, at least when a finite group acts on $Spec(A)$ for some ring A. Explicitly, equivariant structure on M at s in the group is most literally given by an isomorphism $M \to M \otimes_A A$. This explains why "identity" can't usually be used as equivariant structure!

(12/5/2019) Today I learned that the spaces G/B and G/N are smooth, clearing up a misconception that I had in the past (it's the affine closure of G/N which isn't necessarily smooth!) This is because you can locally write them as the product of two affine spaces. I also learned some more specifics in the statement of homological mirror symmetry. Specifically, we need to π-twist the Fukaya category in order to obtain the notion of distinguished triangle and the closure under direct sums.

(12/6/2019) Today I learned how to define (one of two of the) categories of D modules on a scheme, possibly of infinite type. Explicitly, you reduce the problems to affixes via the Yoneda lemma, and for affine schemes, you write your ring as the colimits of the finitely generated sub rings and then for either the push forward or the pulled back modules you simply write the appropriate limit.

(12/9/2019) Today I learned a slightly different construction of an equivariant sheaf on the torus. Specifically, one can include the points λ and its inverse on the torus (for $G = SL_2$) and define the W action on the two points to make the closed embedding W equivariant. The W action on the left hand side creates a W action on the direct sum of two copies of $Vect$ for which the action is nontrivial, and allows one to construct different W equivariances on $\mathcal{D}$ modules on the torus.

(12/10/2019) Today I learned a theorem of Sean Keel which allows one to compute the Chow Ring of the moduli space of genus 0 curves with n marked points. Explicitly, Keel writes out a set of generators indexed by the set of subsets of $\{1, ..., n\}$, and they are given by symmetry $\theta_I = \theta_{I^c}$, a certain sum relation and their product is nonzero only when one index set is contained in the other its complement.

(12/11/2019) Today I learned a recent result of X. Wang which says that the Gelfand-Graev W action on the affinization of the cotangent variety of the basic affine space $(T^*(G/N))_{aff}$ can also be realized as a certain type of quiver variety (known as a Nakajima quiver variety) which comes with a certain W action.

(12/12/2019) Today I learned the étale fundamental group of $\mathbb{P}^1$ minus three points is the profinite completion of the free group on two generators, so in particular, any representation of any finite group is contained in a representation of this group. I also learned an interpretation of the Harish-Chandra classification of a real Lie group–specifically, the picture to have in mind for the representation of a real Lie group $G(\mathbb{R})$ by imagining the action of an actual compact subgroup K, as well as the infinitesimal action of the Lie algebra $\mathfrak{g}$.

(12/13/2019) Today I learned (at least a strong belief in the fact that) the recollement procedure works for categories over some monoidal $QCoh(X)$ for some scheme X. Specifically, I mean that, given a functor of $QCoh(X)$ module categories $\mathcal{C} \to \mathcal{D}$, provided that the functor is fully faithful we can argue essential surjectivity by showing that the restriction to an open (i.e. the monoidal functor induced by an open $U \subseteq X$) and to a closed are both essentially surjective. This is because, assuming you can write any object in $\mathcal{D}$ as a distinguished triangle from an object in the closed part and the open part, you can reach those via the (assumed) essential surjectivity on each of the parts, and then use the fully faithfulness (a statement about Hom *spaces*!) to recover the extension class of the actual object.

(12/15/2019) Today I learned (or at least clarified) a specific test case in classifying the essential image of the Whittaker restricted N averaging functor. Specifically, I learned that the hangup to testing this at zero lies at showing that, as a category over $QCoh(\mathfrak{t}//W^{aff})$, that localizing the category $D(T)^W$ at zero yields a category with one object.

(12/16/2019) Today I learned the existence of a map in Spectra called the *Tate diagonal* map (associated to a prime p), i.e. a natural transformation in $Spectra = Mod_S$ which sends $M \to (M^{\otimes p})^{tC_p}$, where the lower case t is the Tate construction, invariants modulo coinvariants. This is specifically a construction which (as a no go theorem by Nikolaus-Scholze says) cannot exist for the derived category for any ordinary ring. This map is also used to construct the Frobenius, by composing this Tate diagonal with the multiplication map. I also learned a sort of heuristic for why the Tate valued Frobenius of an $\mathbb{E}_\infty$ ring A should take values in A^{tC_p}–specifically, because for a usual ring, if you expand out $(x+y)^p$ to make a ring map, you can first check that it is totally invariant under the C_p action and secondly that, in order to make this a ring map, roughly speaking we have to kill the parts that come from a norm.

(12/17/2019) Today I learned a very strong heuristic (via the finite group analogy) for why the functor Av_*^N on the twisted Hecke category is monoidal. Specifically, you can restrict the convolution on $G \times^{N^-} G$ to parts for which the first coordinate does or doesn't stem from the big cell of $G = SL_2$. In the case where the coordinate does come, you can use the fact that the N action of B/N is trivial to show that that part of the convolution agrees with that of the T convolution. You can then show that the other part of the averaging can be made to N-average a bi-N^- invariant sheaf and evaluate it off the big cell, which by the BBM theorem must evaluate to zero.

(12/18/2019) Today I learned another alternative way to show that the averaging functor $Av_*^T : \mathcal{D}(G/N) \to \mathcal{D}(G/B)$ is W equivariant. Explicitly, this can be done by showing that the adjoint is W equivariant, which in turn can be computed by identifying the kernel with the associated quotient of the sheaf $\delta_{B/N}$. I also learned the full statement of the Geometric Langlands conjecture, which corrects the naive one. Specifically, the error in the conjecture of $\mathcal{D}(Bun_G) \xrightarrow{\sim} QCoh(LocSys_{\check{G}}$ is in that the compatibility with induction, the automorphic (i.e. $\mathcal{D}(Bun_G)$ side) parabolic induction functor preserves compacts, while the Galois side's parabolic induction functor doesn't. This is rectified by noting that the compact objects of $\mathcal{D}(Bun_G)$, under this functor, don't map to just perfect complexes but instead map to the coherent sheaves with *nilpotent singular support*, and the refined Geometric Langlands conjecture now states that the functor above identifies $\mathcal{D}(Bun_G)$ with $Ind(Coh_{\mathcal{N}}(LocSys_{\check{G}})$.

(12/19/2019) Today I learned a finite group analogy argument for why the delta sheaf on $B/N \subseteq SL_2/N$ may actually have nondegenerate "piece" be W invariant. Specifically, after noting that the fga allows for constant functions to be modded out, I computed that the generator of W takes $\delta_{B/N}$ to a constant plus the delta sheaf itself. Unfortunately, I also learned that the quick way to try and realize the actual non-degenerate piece of this delta sheaf cannot be realized by $oblv \circ Av_*^G(\delta_{B/N}) \to \delta_{B/N}$ because this map does not H^*_{dR} to an isomorphism (the left hand side turns out to be $H^*_{dR}(k \oplus k[1])$.

(12/20/2019) Today I learned an alternative description of a W equivariant object in a G category $\mathcal{C}$. Specifically, you can view this as a functor $\mathcal{D}(G) \to \mathcal{C}$ in the category of categories with a $G \times W$ action.

(12/22/2019) Today I learned a flaw in my thinking from a few days ago. Specifically, I had originally hoped that I could view the functor of Av_T^* as a W equivariant functor by equipping the object $Q(\delta_{B/N})$, the kernel of the forgetful functor, with a W equivariant structure. However, that was slightly flawed, at least, because what matters is not only the kernel, but the kernel with its B equivariance (whereas before the Whittaker invariants of a category could be viewed as a subcategory).

(12/23/2019) Today I learned that for finite groups there (at least one categorically) is only one action of a finite group W on the category $Vect$. This is because via comonoidal things you can inject (in the one categorical sense) all W actions into objects of $QCoh(W)$, and you can use the group-ness to argue that the stalk of the associated object at each point must be an invertible object in $Vect$ and so it must also therefore be one dimensional, and you can use finiteness to get that this determines the associated object is the constant sheaf.

(12/24/2019) Today I learned how to actually get W equivariant structure on the sheaf $Q(\delta_{B/N})$. Specifically, I learned that the above logic is covered because the W and G actions commute (i.e. 12/22/2019 is wrong!) and that you can get equivariance on that sheaf by realizing it as the same as $Q(\delta_{\mathbb{A}^1 \times \{0\}})$, a sheaf which is canonically W equivariant.

(12/25/2019) Today I learned another way to prove that the spherical Hecke algebra is isomorphic to the nil Daha (I.e. the ring for $QCoh(\mathfrak{t}//W^{aff})$). Specifically, both of these can be realized as loop and $\check{G}[[t]]$ equivariant cohomology on the affine Grassmannian.

(12/26/2019) Today I learned a way to show that the averaging functor on $\mathcal{C} = Whit(\mathcal{D}(G))$, i.e. the averaging functor $\mathcal{H}_\psi \to D(T)$ (after restriction) is monoidal. Specifically, you can view the Hecke category as the G linear endomorphism of $\mathcal{C}$, and we have a functor of 'tensoring with the identity' on $\mathcal{D}(G/N)$ which is a monoidal functor $End_G(\mathcal{C}) \to End_T(\mathcal{C}^N)$, mapping into the T functors because $\mathcal{D}(G/N)$ has commuting G and T actions.

(12/27/2019) Today I learned a bit more about what a lax monoidal functor actually is. Specifically, the point is that a lax monoidal functor of monoidal categories $\mathcal{C} \to \mathcal{D}$ is a functor where the appropriate diagrams commute up to natural transformation. Specifically, we have a map for any $\mathcal{F}, \mathbb{G} \in \mathcal{C}$ given by $F(\mathcal{F}) \otimes F(\mathbb{G}) \to F(\mathcal{F} \otimes \mathbb{G})$, where you can remember the direction of the arrow by remembering that lax monoidal functors preserve the property of having a 'multiplication.'.

(12/28/2019) Today I learned a heuristic for why the averaging functor hits precisely those objects in $\mathcal{D}(T)^W$ whose objects have trivial de Rham cohomology as a W representation. Specifically, I came up with some kind of a base change formula for modules over monoidal categories and, assuming my diagrams are correct, this predicts precisely what the functor are in the localized picture.

(12/29/2019) Today I learned a picture that I hope to improve further and expand and see where it might take me in the context of $QCoh(\mathfrak{t}//W^{aff})$. Specifically, I learned that, even though more naturally (for me at least) the category $\mathfrak{g} - Mod_0$ is viewed as $\mathfrak{g} - Mod \otimes_{Z\mathfrak{g}-Mod} Vect$, you can apparently also (at least abelian categorically) view it as the full subcategory of $\mathfrak{g}$ modules by which the center acts by zero.

(12/31/2019) Today I learned a way to define the monoidal structure on the category $\mathcal{D}(T)^W$. Specifically, one can realize $\mathcal{D}(T)$ as an algebra object in the category $\mathcal{D}(W) - Mod$ because the multiplication map $T \times T \to T$ is W equivariant, i.e. acting by any element in W is a group homomorphism. Therefore, we can discuss the category $\mathcal{C} := \mathcal{D}(T) - Mod(\mathcal{D}(W) - Mod)$, which in particular has the property that endomorphisms of any object is monoidal. Because $End_{\mathcal{C}}(\mathcal{D}(T)) \simeq \mathcal{D}(T)^W$, we obtain a monoidal structure on $\mathcal{D}(T)^W$.

November 2019

. (11/1/2019) Today I learned some facts about the basic affine space G/N. Explicitly, I learned that the ring of functions on G/N, say A, can be written as a direct sum of all of the finite dimensional irreducible representations, with multiplication stemming from the fact that there is a projection map $V(\lambda) \otimes V(\mu) \to V(\lambda + \mu)$. This is generated by the representations associated to the fundamental weights, which gives the description below for SL_3/N.

(11/2/2019) Today I learned how a class $0 \to A \to B \to C \to D$ being trivial in $Ext^2(D, A)$ implies for the extension. Specifically, this means that, if $I = B/A \hookrightarrow C$ is the inclusion, this implies that there is an (obvious) surjection $B \to I$, and the claim that this class is trivial implies that there is some $E \to C$ which "extends" this surjection.

(11/3/2019) Today I learned a consequence of $i^*\mathcal{F} = i^!\mathcal{F}$. Specifically, we can compute $i^!_N(\mathcal{F})$ and realize it as an extension of the δ sheaf at zero by $j_!j^!(\mathcal{F})$.

(11/5/2019) Today I learned the actual specific definition of a toric variety–namely, it's a normal reduced separated irreducible variety with a torus inside of it, equipped with the data of a torus action which extends the action of the torus inside of it on itself. I learned that, with this definition, toric varieties (and the maps preserving the torus actions between them) is equivalent as a category to maps between fans, as they've been defined. I also learned that a toric variety is proper if and only if the associated fan is *complete*, i.e. the union is the entire vector space.

(11/6/2019) Today I learned one way to explicitly construct a monoidal functor. Explicitly, given a morphism of monoidal categories $\mathcal{C} \to \mathcal{D}$ and a module category $\mathcal{M}$ for $/D$, we can explicitly construct the functor $End_{\mathcal{D}}(\mathcal{M}) \to End_{\mathcal{D}}(\mathcal{M})$. I also learned that, giving an R bialgebroid structure on a space S is equivalent to the data of a monoidal structure on $S-$Mod such that the canonical functor $S - Mod \to R - BiMod$ is monoidal.

(11/7/2019) Today I learned that a functor very closely related (and I suspect is the same) to the averaging functor $\mathcal{C}^{N^-,\psi} \to \mathcal{C}^N$ for the category $\mathcal{C} = \mathcal{D}(G)^{N^-,\psi}$ is fully faithful. This can be obtained by explicitly identifying $\mathcal{C}^{N^-,\psi}$ with modules for the nilHecke algebra of the affine Weyl group, and then taking the subring of elements whose modules give $D(T)^W$ as your functor.

(11/8/2019) Today I learned that for any $\mathcal{F} \in \mathcal{D}(X)$ there exists a map $i^*\mathcal{F} \to i^!(\mathcal{F})[2codim(Z)]$ for $i : Z \hookrightarrow X$ which measures the failure of $\mathcal{F}$ being lisse. In particular, this map is an isomorphism if $\mathcal{F}$ is lisse.

(11/11/2019) Today I learned the restriction of a potential counit map $\mathcal{F} \to \delta$ given by $m_{*,dR}$-ing the biWhittakerness of the map $\kappa \to \delta$ is zero. In particular, I learned that if the functor of Ben-Zvi and Gunningham from the nilHecke modules to $D(T)^W$ is the same functor as the Gaitsgory averaging functor, then the counit cannot be the above map.

(11/12/2019) Today I learned the statement of *Kashiwara's conjecture*, now a theorem of Gaitsgory and Drinfeld, which says that the pushforward of a simple D-module along a proper map breaks up as a direct sum of simple objects. This follows from a conjecture of de Jong (which was proven by Drinfeld), and the conjecture of de Jong was proven by Gaitsgory later. This in particular implies that the sheaf $\mathcal{F}$ splits as a direct sum of the two simples (and in particular, invalidates last night's learned because that map is actually zero, since $m_*(\psi \boxtimes \psi) = \psi[-2]$.

(11/13/2019) Today I learned what a W algebra is and why I care about it. Specifically, in the context of an affine Lie algebra with a twist κ, one can create the associated Virasoro algebra $\mathbb{V}^\kappa := Ind^{\mathfrak{g}_\kappa}_{\mathfrak{g}[[t]]}$ and apply the *semi infinite cohomology* functor to it, a functor which attempts to mix the properties of the Lie algebra cohomology of $\mathfrak{n}[[t]]$ and the homology of $\mathfrak{n}((t))/\mathfrak{n}[[t]]$. One reason that the W algebra is important is that one can identify the completion of the moduli space of G local systems on a curve X at the subset of opers as the 0^{th} Chiral cohomology of some limiting version of the W_κ, which is a commutative algebra $W(\mathfrak{g})$. I also learned that this semi infinite cohomology functor extends to the setting of W_κ modules and provides a duality between (renormalized) categories $W_\kappa - Mod^\vee \cong W_{2\kappa-\kappa_{crit}}$.

(11/14/2019) Today I explicitly showed that, given a W equivariant map $X \to Y$, you can explicitly transfer the given W equivariant structure on a sheaf on X to a sheaf on the associated pushforward. This allows us to specifically say that we equip our functor Av_* with canonical W equivariance, but we still made a choice in the sense that we had two options of the W equivariance to equip it with.

(11/15/2019) Today I learned the actual definition of the dual object in the category $\mathcal{O}$. Specifically, one should expect that the unit is maps to the unit of the monoidal structure, eg, the linear dual should go to k and not $U\mathfrak{g}$. But furthermore, one would like the dual of an object in category $\mathcal{O}$ to remain in category $\mathcal{O}$. Explcitly, this involves changing the obvious action of $\xi\phi(x) := \phi(\xi x)$ to the *Cartan involution* $\tau : \mathfrak{g} \to \mathfrak{g}$, which, for $\mathfrak{sl}_n$ is the transpose map.

(11/17/2019) Today I learned that there is a filtration of the projective object P_e in category $\mathcal{O}$ for $G = SL_2$. Specifically, it is a three term filtration whose subquotients at the first and last stages are $L_e = M^{-2}$. In particular, this allows us to explicitly construct the square zero endomorphism of P_e.

(11/18/2019) Today I learned that, given certain pairs of 'log Calabi-Yau''s with an snc divisor D, one can construct canonical coordinates on them as analogous to the construction that, while $\mathbb{A}^1$ does not have canonical coordinates, you can write a canonical basis of global functions on the open subset of the complement of the vanishing of two lines as the invertible elements of global functions on the space. Gross-Hacking-Keel used *θ-functions* to generalize this in their 2011 paper to surfaces with certain divisors, the philosophy being that the divisor determines these canonical coordinates.

(11/19/2019) Today I learned that the pushforward of the map $G/N \to G/B$ reasonably induces, upon restriction to the identity point inside G/B, gives de Rham cohomology (by base change). This provides plausible reason to believe that the pushforward map, as a map of categories with a G action, could restrict via taking Whittaker invariants, to the de Rham cohomology functor $\mathcal{D}(T) \to Vect$ as W categories. In particular, using this would show that the kernel of the functor $\mathcal{D}(G)^{N^-,\psi} \to \mathcal{D}(Flag)^W$ maps entirely into the trivial component.

(11/20/2019) Today I learned a theorem of Gurbir Dhillon which says that, for any κ, some Beilinson-Bernstein Theorem holds for the κ twisted category $\mathcal{D}_\kappa(N(K)_\psi\backslash G(K)/I)$. Specifically, the theorem states that this category is a full subcategory of modules over the algebra $\mathcal{W}_\kappa$.

(11/21/2019) Today I learned a mistake in the original proof I had in my head about ample line bundles and very ample line bundles. Specifically, very ample line bundles *do not* have the property that every coherent sheaf tensors with it to have global sections (for example, even for N large, $O(N) \otimes O(-N-1)$ on $\mathbb{P}^1$ does not have global sections. The actual property that an ample line bundle $\mathcal{L}$ has is that once we fix a coherent sheaf $\mathcal{F}$, we can find some high enough power N such that the sheaf $\{ \otimes \mathcal{L}^{\otimes N}$ has global sections for $N\backslash gg0$.

(11/22/2019) Today I learned a specific way to compute the full W equivariance of the kernel of the averaging functor explicitly. Explicitly, this involves writing out the two diagrams you get from the fact that $* \to T$ is a W map and the action being given by pushforward by $a : T \times W \to T$. I also learned that the twisted convolution for N can be computed by just smearing the usual sheaves together, because the forgetful functor is fully faithful.

(11/23/2019) Today I learned an annoying little discrepency in the Borel-Weil theorem. Specifically, the input is most naturally an *antidominant* coweight, which you then view as a representation of B and then use the torsor structure on $G \to G/B$ to get the *dual* of the highest weight representation associated to the opposite of the coweight. In particular, some annoying minus sign must occur.

(11/25/2019) Today I learned that the definition often used of a *dominant weight* $\lambda \in \mathfrak{t}^*$ is not the one I would naturally think. Specifically, one says that $\lambda \in \mathfrak{t}^*$ is antidominant if and only if $w\lambda - \lambda$ cannot be expressed as a nonzero positive coroot. Equivalently, we can ask the condition that for all coroots $\check{\alpha}, \lambda(\check{\alpha}) \notin \mathbb{Z}^{<0}$. In particular, for example, $\frac{-1}{2}$ is a dominant weight. One reason that people take this as a definition is that the Beilinson-Bernstein theorem holds for all dominant weights (when interpreted derivedly, otherwise we'd need regular dominant weights).

(11/26/2019) Today I learned a theorem of M. Brion which gives explicit generators and relations for the equivariant Chow ring of a scheme with the action of a torus. I also learned that it implies the *localization theorem*, which says that up to inverting elements of $Hom(\Gamma, \mathbb{C})$, we have an isomorphism from the equivariant Chow group of the fixed points of the T action (with the trivial action!) to the equivariant Chow group of the full space.

(11/27/2019) Today I explicitly worked through how to construct the moment map $T^*X \to \mathfrak{h}$ for any algebraic group H which acts on a space X. Explicitly also, I computed explicitly the fact then, when using the Killing form and identifying $T^*G \cong G \times \mathfrak{g}^*$ with *right invariant* vector fields (because they are coming from the left action of G, which commutes with the right action of G), we can compute the explicit preimage of zero under the moment map $G \times \mathfrak{g} \to \mathfrak{n}^* = \mathfrak{g}^*/\mathfrak{b}^*$ as given by the universal resolution $\tilde{\mathfrak{g}}$.

(11/28/2019) Today I learned the statement of a few cool facts. Specifically, I learned that there is an elliptic curve whose solutions (or, more naturally, p minus the number of solutions) over $\mathbb{F}_p$ is given by the appropriate coefficient on the series $q(1-q)^2(1-q^11)^2(1-q^2)^2(1-q^22)^2...$ computed as a power series. I also learned the statement of a theorem of Kazhdan and Lusztig, which says that if you take the $\mathfrak{g}((t))$ representations at level $\kappa < 0$ which are integrable with respect to $G(\mathcal{O})$, you obtain the category $Rep_q(G)$.

(11/29/2019) Today I learned that my naive definition of the symplectic Fourier transform at least doesn't work in the finite group analogy, because it doesn't preserve the delta sheaf at 1 for the torus. However, I learned that I could correct this by taking the opposite of the usual symplectic form.

(11/30/2019) Today I learned one perspective on viewing the equivariance we equipped on the kernel of the averaging functor on the Whittaker category. Specifically , we can view the isomorphism as being given by the fact that the support of the sheaf is on 1 (inside the torus) and so we can also compute (only!) that the associated vector space at one is one dimensional, and then we can equip the equivariance of being a one dimensional vector space.

October 2019

(10/1/2019) Today I learned an interpretation of Kan extensions along a map $u : A \to B$ in terms of the six functor formalism. Specifically, you can view pullback by u as u^*, and basically ask for the Kan extension of a map $f : A \to C$, and this is the right adjoint u_* in the sense that $Fun(g, u_*f) = Fun(u^*g, u^*u_*f)/f$. I also learned that you can view the right adjoint of a functor $F : \mathcal{C} \to \mathcal{D}$ as the right Kan extension of the identity map by F, since this by definition is a map $L : \mathcal{D} \to \mathcal{C}$ and a natural transformation $LF \to id$. This seems to be giving the counit. Granted, I am not 100 percent sure what the unit is. Oh but there's a little more, it's that the right Kan extension exists *and is preserved by* F. But I still learned some connection between right Kan extensions of functors and left adjoints.

(10/2/2019) Today I learned a method to produce/define a twisted cotangent bundle $T^*_\psi(G/N)$. Specifically, one can construct the *moment map* of the right action of N on G, i.e. the associated map $T^*(G) \to \mathfrak{n}^*$. Taking the preimage of zero or ψ has an N actoin, and quotienting by the former gives $T^*(G/N)$ and taking the quotient of the latter gives the twisted cotangent bundle.

(10/3/2019) Today I learned a broad strokes reason that the theory of deformations and moduli spaces are so related. Specifically, when constructing a moduli space, one would like it to be compact. In the complex analytic interpretation of compactness, sequential compactness can be viewed as requiring that anything we deform to is also allowed in the moduli space. Because we can deform, eg, smooth curves to nodal curves, we allow nodal curves as well.

(10/4/2019) Today I learned a few ways to define the category of Spectra. Specifically, you can define it as the iterated limit of pointed spaces of the loops functor, which forces it to be invertible. From this comes the sphere spectrum which is given by the stabilization of the iterated suspensions of the sphere. I also learned that you need to add a bit more limits to invert σ instead.

(10/5/2019) Today I learned about the notion of *strength* of a polynomial, which is the minimal number of ways to write it as a sum of product of lower degree terms. With this notion, the *Ananyan-Hochster* principle says that if the collective strength (ie strength of any k linear combination) is sufficiently large then the set behaves like independent variables.

(10/7/2019) Today I learned that for an analytic perfect ring, the condition of the Frobenius being a homeomorphism comes for free. This can be thought of as a consequence of the Banach open mapping theorem. I also learned that the valuative criterion for proper ness only applies for finite type morphisms by the definition of properness.

(10/8/2019) Today I learned that a method of attack of understanding $\mathcal{D}$ modules by mapping tangent vectors $Spec(D) = Spec(k[\epsilon]/(\epsilon^2))$ into the space. Specifically, if the map from $Spec(D)$ is ever given by a closed embedding, the pullback then is more or less determined by the underlying pullback of schemes, and the additional data is given by completing some etale basis of derivations. But since $Spec(D)$ has dimension zero, the data is entirely determined by the sheaf pullback, so in particular, the behavior of the exponential sheaf cannot be detected by the tangent vectors $Spec(D)$ mapping into them.

(10/9/2019) Today I learned some basics of intersection theory and the difference between two types of divisors on a scheme. The first notion is a notion that is much more usable on a pure dimension n scheme, specifically, the notion of *Weil divisors*. Specifically, this is the formal sum of classes of codimension one closed irreducible subvarieties. But there is a more general notion of divisor, known as a *Cartier divisor*. This comes with the data of an open cover and a rational function on each open subset of the open cover such that the quotient is a unit in the ring of functions on that open subsets.

(10/10/2019) Today I learned a new class of examples of $\mathcal{D}$ modules on a line, which are just modifications of the exponential $\mathcal{D}$ module where now, if $p(t) \in k[t]$, you view the $\mathcal{D}$ module as having connection given by deriving $q(t)e^{p(t)}$. I also found some polynomails for which the first deRham cohomology for these $\mathcal{D}$ modules don't vanish, specifically for $p(t) = \frac{1}{t}$.

(10/11/2019) Today I learned how the notion of cluster varieties connect to the notion of local systems (or more accurately, decorated local systems). Specifically, I learned that there is a way to decorate G local systems on a surface with boundary or marked points, and a way to associate a seed to them and construct cluster varieties. I also learned that one way to see the local Whittaker functor is not an equivalence is because it sends $Vect$ (a category with a $G(K)$ action) to zero.

(10/13/2019) Today I learned a fun fact which says that, given any functor L of abelian categories which admits an exact right adjoint R, L preserves projective objects. This is essentially a formal property.

(10/14/2019) Today I learned a bunch of stuff about the *big projective* object in the BGG category $\mathcal{O}$ and projective objects in general in $\mathcal{O}$. Specifically, I learned about the statement of *BGG reciprocity*, which says that for any two weights $\lambda, \mu \in \mathfrak{t}^*$, we have that $(P(\lambda) : M(\mu)) = [M(\mu) : L(\lambda)]$, where the right hand side is a remark about how projective objects in $\mathcal{O}$ associated to a weight admit a *standard filtration* whose subquotients are Vermas and the multiplicity of each Verma is unique.

(10/15/2019) Today I learned some tenants of intersection theory. Specifically, I learned that for a regular embedding $i : X \to Y$, the *normal cone*, defined as the $Spec_Y(\oplus(I^d/I^{d+1}))$, which admits a closed embedding into the *normal bundle* $Spec_Y(Sym(I/I^2))$. When i is a regular embedding, this map is an isomorphism. I also learned that more about the Chern class–specifically, on a variety X, the Chern class takes in a line bundle and returns an automorphism of the Chow ring $A_*(X)$. For a line bundle L, one can define specifically the first Chern class as the intersection product of any divisor D for which $\mathcal{O}(D) = L$. I also learned that you can define the Lurie tensor product on presentable or stable presentable categories with left adjoints as morphisms. For stable presentable categories, I learned that the unit is the category is Spectra (or, equivalently, for any presentable category $\mathcal{C}, \mathcal{C} \otimes Sp \cong St(C)$. Because Spectra is the unit of a symmetric monoidal category, we obtain an essentially unique multiplication map $Sp \otimes Sp \otimes ... \otimes Sp \to Sp$.

(10/16/2019) Today I learned an alternate way to construct a potential splitting map $\mathcal{F} \to \delta_{N^-,\psi}$. Specfically, this can be realized by pushing forward by $m : N^- \times N \times N^- \to SL_2$ the map given by $\psi \boxtimes \underline{k} \boxtimes \psi \to \boxtimes \delta_0 \boxtimes \psi$.

(10/17/2019) Today I learned an explicit way that, using the definition of W equivariance as giving an object $\mathcal{F}$ together with an identification of it with $F_s(\mathcal{F})$ for all $s \in W$, at least for finite W, gives rise to a W action on the vector space of maps between any two equivariant objects. Namely, you simply apply F_s to your Hom space and use equivariance to go back to the original space, so to speak.

(10/18/2019) Today I learned a theorem of Borel which says that there is a canonical surjection of rings $\mathbb{Q}[x_i] \to H^*(G/B, \mathbb{Q})$ whose kernel is the symmetric polynomials. i also learned that there are a collection of primes, all of which divide the order of the Weyl group of the associated group (but not every prime is such), which are the obstruction to that map being an isomorphism when phrased in $\mathbb{Z}$. Specifically also, given two commuting objects in G of order those primes, the orders being these primes are the only obstruction to these elements all being inside a maximal torus.

(10/19/2019) Today I learned a further extension of the above which relates to the cohomology ring of the Grassmannian $Gr(m, n)$, the $m-$hyperplanes in $\mathbb{A}^n$. Specifically, for each partition of m (i.e. a nonincreasing sequence $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_m \geq 0$, we can create its associated Young diagram and for each Young diagram, we obtain the *Schubert variety* X^λ in the Grassmanninan, and the cohomology rings are generated by the length m or less partitions of numbers less than m. Furthermore, I learned a specific hands on example which sheds some light on the fact that $Vect^G = Rep(G)$. Specifically, for a finite group G at least, one can note that you can view equivariant sheaves having identifications in $QCoh(G)$, which by the finiteness assumption is a direct sum of finitely many points. Therefore for this identification to play nicely, you need to check where it goes under the three arrows of maps in the simplicial diagram of $\mathcal{C}^W$.

(10/20/2019) Today I learned one way to see that the map $Av_* : Vect \to Rep(W)$, for a finite group W, takes k to the group ring $k[W]$. Specifically, one can compute by adjunction that homs into $k[W]$ compute the dual representation, and that the dual representation has underlying vector space the dual of the original vector space. I also learned some classification that equivariant cohomology and K theory are both generalizations of cohomology. I also learned that the first singular Schubert variety is given by the single box inside of $Gr(2, 4)$.

(10/22/2019) Today I learned a few results in homotopy theory which relate to the fully faithfulness of the cochairs functor. Specifically, it's a theorem that simply connected finite type rational spaces (ie spaces localized if the $\mathbb{Q}-$homotopy agrees) embed fully faithfully via chain complexes, and you can describe the essential image as what it should be.

(10/23/2019) Today I learned the definition and a cool theorem involving the *spectrum* of a linear operator A of Banach spaces, defined to be $\{\lambda \in \mathbb{C} : (A - \lambda I) \text{ is not invertible}\}$ (which is larger than the notion of eigenvalues because the failure of surjectivity is not the failure of injectivity). Specifically, for any holonomic function $f : \mathbb{C} \to \mathbb{C}$ (or more specifically, any f which is holonomic at points in the spectrum of A!) you can talk about the linear operator $f(A)$.

(10/24/2019) Today I learned an explicit construction of the Miura bimodule $\mathbb{M}$. Specifically, you can literally write it as $H^0\Gamma(\mathcal{D}_T) \otimes_{U\mathfrak{t}} \mathfrak{n}\backslash H^0\Gamma(D_G)/\overline{\mathfrak{n}}^{\psi}$. The proof of the existence of a map between global differential operators on G/N and the twisted equivariance is given by explicitly constructing an algebra map by showing that global differential operators acts on $\mathbb{M}^{Z\mathfrak{g}}$ (i.e. the Hochschild homology) and that $\mathbb{M}^{Z\mathfrak{g}}$ is as a module the other ring in question and the associated map is a ring hom.

(10/25/2019) Today I learned (solidified?) the notion of a *level* associated to an algebraic group G. By the letter of the law, it's just a certain central extension of G by some abelian group A, and for a simple group and $A = \mathbb{G}_m$, these extensions are classified by scalar multiples of the Killing form. I also learned that because we can do this construction with the affine Grassmannian, yielding a G equivariant line bundle on the Grassmannian.

(10/27/2019) Today I learned a neat little trick which says that, assuming that all of the functors are defined, you can get a second excision sequence $j_!j^*\mathcal{F} \to \mathcal{F} \to i_!i^*\mathcal{F}/$. I also learned that, because the constant sheaf $k \in \mathcal{D}(T)$ is canonically W equivariant, the map $Av_!^{\psi}$ takes the constant sheaf to a sheaf with a trivial W action in a category acted on trivially, and thus the map $H_{N^-,\psi} \to D(G/N)^{N^-,\psi} \to Rep(W)$ where the second arrow is de Rham cohomology maps into the trivial representations.

(10/28/2019) Today I learned a way to show that my sheaf splits, assuming that we have our alternate excision sequence. Specifically, using both excision sequences and the fact that $\psi = i^*(\mathcal{F})$, we can explicitly show that the composite $\delta_{\psi} \to \mathcal{F} \to \delta_{\psi}$ is nonzero by arguing that, if it were not, we could explicitly compute that $i^!j_!j^!(\mathcal{F}) \cong \psi$, and this would violate that our weird distinguished triangle was in fact a distinguished triangle.

(10/29/2019) Today I learned a result of Bezrukavnikov, Braverman, and Positselskii which gives the conservativity of the collection of functors from the heart of $D(G/U)$ mapping into $|W|$ many copies of $D(G/U)$ given by twisting the localization functor by the Gelfand-Graev W action on the ring of differential operators on the base affine space. I also learned a result about the Fourier transform which that a function is smooth if and only if its Fourier transform is rapidly decaying.

(10/30/2019) Today I learned about the notion of *decorated flag varieties*, i.e. the thing for which G/U parametrizes. Through this, I learned that one can explicitly write the map $SL_3/U \hookrightarrow Spec(\mathbb{C}[x_1, x_2, x_3, y_1, y_2, y_3]/(\sum_i x_i y_i))$ via taking the first three coordinates of the line and using the decoration to pick out the second three coordinates.

(10/31/2019) Today I learned a few heuristics on what the notion of rigid analytic geometry is supposed to study. In particular, I learned that the disk $D_{\leq 1} \subseteq A^{1,an}_{\mathbb{Q}_p}$ is open but not closed, and furthermore the analytic line is not compact because it can be covered by the disks of radius r for $r > 0$. I also learned the tc the notion of field valued points are given by topological fields with continuous valuations.

September 2019

. (9/1/2019) Today I learned a quick way to show that any N, ψ equivariant sheaf on G or on any scheme X has no de Rham cohomology. Specifically, this is because the terminal map $X \to *$ is necessarily N equivariant, and so the push forward maps to $(Vect)^{N,\psi} = 0$. I also learned the proof of the *Poincare lemma*, which says that the sheaf complex of differential forms is isomorphic to the constant complex. Specifically, for $\mathbb{R}^n$, constructing the form $\iota_t \sigma^* w$ and integrating "from zero to one" gives a form whose exterior derivative yields the form w if the original form is closed.

(9/2/2019) Today I learned an alternate construction of the functors associated to the action of W on (at least the one category) $D(G/N)_n$. Specifically, for each element $w \in W$ you can construct a sheaf on the diagonal $G/N \times G/N$ and pull back, convolve, and push forward.

(9/3/2019) Today I learned the computation which says that if k is a separably closed field with a prime l invertible in k, then $H^1_{et}(\mathbb{A}^1, \mathbb{Z}/l\mathbb{Z}) = 0$. This is proven by showing directly that there are no l torsors of $\mathbb{A}^1$ by using Riemann Hurwicz. I also learned an idea of mirror symmetry, which specifically using the example of a quintic in $\mathbb{P}^4$ says that the moduli space of deformation equivalent curves is "mirror" to the Picard group.

(9/4/2019) Today I learned a quick proof of the fact that the Whittaker differential operators $\Gamma(D^{N,\psi,N,\psi})$ is isomorphic to $\mathbb{H}^{sph}$, i.e. the spherical subalgebra of the affine nil Hecke algebra. This proof comes from realizing both sides as ${}^LG[[t]] \rtimes \mathbb{G}_m$ equivariant cohomology on the affine Grassmannian. Furthermore, I learned a cool proof that, given a continuous, fully faithful right adjoint to a category $A - Mod$ with left adjoint Q, then the resulting category is $Q(A) - Mod$.

(9/5/2019) Today I learned a pathology that occurs in the recollement setup. Namely, one can have a functor of two categories and (the data of) commuting restrictions to open and closed associated categories such that both restrictions are equivalences, but the original functor is not. This is given by the fact that $D(\mathbb{A}^1 \backslash 0) \times Vect$ isn't $D(\mathbb{A}^1)$.

(9/6/2019) Today I learned an interpretation of an $\mathcal{E}_2$/braided monoidal stucture on a 1-category $\mathcal{C}$. Specifically, you can interpret the monoidal structure as a map of the configurations on a finite subset of points in $\mathbb{R}^2$ giving rise to some tensor product, and any isotopy of this configuration spaces given an isomorphism. In this way, by viewing each point as locally being $G/^{ad}G$, and you can push pull to a circle around ∞ on the Riemann sphere, giving an interesting monoidal structure on the category $D(G/^{ad}G)$.

(9/8/2019) Today I learned an intuitive explanation as to why the constant sheaf should be O_X with a shift, at least for smooth X. Specifically, using the adjunctions argument, one can argue for smooth k, defining the constant sheaf as $p^*(k)$ (where p^* is the left adjoint to the de Rham global sections), one can argue that this should be the sheaf which is associated to "mapping somewhere where the de Rham differential vanishes," which, via the isomorphism $\mathcal{D}_{\mathbb{A}^1} \cong \mathcal{D}/\mathcal{D}\partial$, realizes this.

(9/9/2019) Today I learned an important technical distinction that, even though there are no maps from the $D_{\mathbb{A}_1}$ modules $\delta \to k[t]$, there is a nontrivial extension of δ by $k[t]$. This is because one computes the extensions by the space $Hom(\delta, k[t][1])$, which by protectively resolving δ we obtain that this Hom space is not zero.

(9/11/2019) Today I learned the definition of a *Poisson algebra*, which is a commutative algebra equipped with an anti symmetric bracket , which is a Lie algebra with respect to the bracket and such that the bracket satisfies the Leibniz rule. I also learned that a symplectic manifold canonically comes with a Poisson algebra structure.

(9/12/2019) Today I learned the notion of the *Weil-Deligne group*, a variant of the Galois group of a local field K. Specifically, by taking the residue field map, we obtain a map $Gal(K) \to Gal(k) \cong \hat{Z}$ and take the preimage of the geniune integers to obtain the Weil group. You can then characterize the Weil-Deligne group by noting that the tamely ramified part of the Galois extension is given by some N which commutes with the Frobenius via $F^{-1}NF = qN$ (where q is the order of the ground field) and define representations of the Weil-Deligne group to work like this. +

(9/13/2019) Today I learned that the idea behind cluster varieties and cluster algebras. Specifically, given a lattice, then every choice of basis gives a coordinate of the torus, and you can take certain piecewise linear maps associated to each change of basis and glue along them to form a new scheme.

(9/15/2019) Today I learned that if you take a Mobius band and you cut exactly on the inner circle, you end up with a cylinder (after untwisting the bends). On the other hand, if you instead cut a third of the way from one edge to the other, you will actually cut out a connected component, since you're cutting out a cylinder.

(9/16/2019) Today I learned the full statement of a theorem of Ginzburg and Kazhdan, which says that, under the usual setup, $H^0(\Gamma(\mathcal{D}_{G/N})) \cong (\Gamma(\mathcal{D}_T) \otimes_{Z\mathfrak{g}} \Gamma(\mathcal{D}_{G/_\psi N}))^{Z\mathfrak{g}}$, where the right hand side denotes the nonderived fixed point functor. I also learned that the general fact that the functor $\mathcal{C}^{G,w} \otimes_{Z\mathfrak{g}-Mod} (Sym(\mathfrak{t})) - Mod \to (\mathcal{C}^N)^{T,w}$ being fully faithful implies (even though the proof I know is a consequence of) the full Beilinson-Bernstein theorem, which comes from the substitution $\mathcal{C} = D(G)$.

(9/17/2019) Today I learned that there is a notion to view stacks that includes homotopy coherence via the notion of *groupoids* over a base scheme S. Specifically, defining $\mathcal{S}$ as the category of schemes over S, one can define a category fibered over $\mathcal{S}$ as a category $\mathcal{C}$ equipped with a functor $\mathcal{C} \to \mathcal{S}$, and to be a groupoid (or also called fibered in groupoid) means that for any object over X and any map $X' \to X$, you can, roughly speaking, take fiber products over X. I also learned a proof which says that the functor $Ring \to Set$ $A \to \{\text{curves over } Spec(A)\}$ isn't representable by a scheme. Specifically, this is because if it were representable, we could feed in the identity map and obtain a *universal family* of curves. Then if this were a map of schemes, we could use the fact that there is a nontrivial *isotrivial* family of curves, i.e. a family of curves which is trivial after finite base change to show that this can't happen.

(9/18/2019) Today I learned that the ring $\Gamma(D_T)$ and the ring of twisted global differential operators are flat as $Z\mathfrak{g}$ modules. In particular, this says that the isomorphism relating differential operators on G/N to the twisted tensor product can only have cohomology coming from taking HH with respect to the center.

(9/19/2019) Today I learned that the ring $\mathbb{Z}_p[\zeta^{\frac{1}{p^\infty}}]$ p-adically completed is an integral perfectoid ring. I also learned that you can construct the standard global sections functor $\Gamma(\mathcal{F})$ on $D(X)$ can be realized as $Hom_{D(X)}(D_X, \mathcal{F})$. This is because of the fact that $D_X = Ind(O_X)$.

(9/20/2019) Today a way to construct the *scattering* functions of a affine log Calabi-Yau toric variety with a cluster structure. Specifically, given any open subset of a Calabi-Yau toric variety, the volume form restricted will be the volume form and so the tropical points (defined so that they're divisorial valuations on the field of fractions) will be the same. Thus the map on the tropical points will give rise to a rational map on tori.

(9/21/2019) Today I learned that, if $\mathcal{F} := Av_! Av_*(\delta_{N^-,\psi})$ is the sheaf associated to the composite BBM functor, that $Hom_{\mathcal{H}_\psi}(\mathcal{F}, -) \cong H^*_{dR}(i^!_N(-))$. Specifically, this means that if the sheaf $\mathcal{F}$ splits as a direct sum of its closed and open factors, then this says that for any object in $\mathcal{H}_\psi$, you can restrict it to the identity and that will be a direct summand of its de Rham cohomology of the restriction to N.

(9/22/2019) Today I learned that there is a nonzero map $\psi \boxtimes k \to \mathcal{F}$ in the BBM adjnction setup, which is the actual map that comes from the fact that $Av_!^{N^-,\psi}$ is defined on $\mathcal{C}^N$. I also learned that our original map $Av_! Av_*(\delta_{N^-,\psi}) \to \delta_{N^-,\psi}$ comes from the fact that we have a map $Av_*(\delta_{N^-,\psi}) \to \delta_{N^-,\psi}$ given by integrating the constant N factor.

(9/23/2019) Today I learned the connection between the actual setup of the functor $\mathcal{C}^{N^-,\psi} \to (\mathcal{C}^N)^W$ for a $\mathcal{C}$ that $G = SL_2$ acts on. Namely, it's a theorem that the adjunction above can be interpreted to mean that $\mathcal{C}^{N^-,\psi} \to \mathcal{C} \to \mathcal{C}^N$ has a right adjoint. (Okay, I sort of already knew that, but I sorted out some details today.) I also learned of the notion of the *derived completion at t* of some A-module M. Specifically, there is a notion of the homotopy limit of a sequence of maps

$... \to M_2 \to M_1$ and the derived completion is the homotopy limit of $hcoker(t^n : M \to M)$.

(9/24/2019) Today I learned two different statements of the *HKR Theorem*, which, roughly speaking, says that the Hochschild homology of a smooth $k-$ ring or a ring in characteristic zero is given by differential operators. I also learned that the Harish-Chandra isomorphism implies that the center $Z(\mathfrak{sl}_2)$ is a polynomial algebra generated by the Casimir element, a fact that I have quoted a lot but never actually thought through the proof of.

(9/26/2019) Today I learned that the dual theorems about Hochschild Homology are true for Hochschild cohomology. In particular, I explicitly computed that if M is an A bimodule, then the first Hochschild cohomology group precisely classifies derivations modulo brackets.

(9/27/2019) Today I learned that $U\mathfrak{g}$ is a flat $Z\mathfrak{g}$ module. This is because $U\mathfrak{g}$ is a flat $U\mathfrak{b}$, module, which in turn is a flat $Sym(\mathfrak{t})$ module which by Harish Chandra is a flat $Z\mathfrak{g}$ module.

(9/28/2019) Today I learned a few of the terms in basic game theory and how it relates to a game called *Nim* and a game called *one rook chess*. Specifically, there are certain starting positions in Nim (with two piles, they are classified by the piles being equal) such that the second player can always win, with perfect play. You can show certain games are equivalant and that they even form a group!

(9/30/2019) Today I learned some motivation for the stack $\overline{M_g}$. Specifically, this stack is defined to be the moduli space of curves with at most nodal singularities, but also has a condition which guarantees separateness (which basically amounts to controlling the blow up of $\mathbb{P}^1$ at points. I also learned a fun lemma which says that for a $t \in A$ adically complete ring, a (derived) t adically complete module is acyclic if and only if its quotient is.

August 2019

. (8/1/2019) Today I learned a result of David Ben-Zvi and Sam Gunningham which says that the functor $QCoh(\mathfrak{t}//W^{aff}) \to D(G)^G$ (where G acts on itself via the adjoint action) is a braided monoidal, i.e. $\mathcal{E}_2$, functor. I also learned the construction of the affine nil Hecke algebra, i.e. the ring $\mathcal{H}^{sph}$ for which modules yield $QCoh(\mathfrak{t}//W^{aff})$. Specifically, one can take the group ring of the affine Weyl group (over the weight lattice, i.e. $W^{aff} := \Gamma \rtimes W$), and make a tensor-like product on $Sym(\mathfrak{t})$, or one can take a certain *spherical subalgebra* of the affine Hecke algebra $\mathcal{H}(\mathfrak{t}, W^{aff})$.

(8/2/2019) Today I learned the construction of the Hochschild homology of a dualizable category. Namely, you can assign a certain (complex of) vector space(s) to any dualizable *DG* category by taking the counit and the unit and then seeing where k goes. I learned that this construction sends $A-Mod$ to the vector space $A \otimes_{A\otimes A} A$, recovering the usual definition of Hochschild homology, and furthermore I learned that the Hochschild homology of the category of D-modules on a smooth variety X is the *Borel Moore homology.*

(8/3/2019) Today I learned the actual construction of the characteristic polynomial, which is in particular not given by the map $\mathfrak{g} \to \mathfrak{g}^* \to \mathfrak{t}^* \to \mathfrak{t} \to \mathfrak{t}//W$, which is more like a map that simply projects onto the diagonal. Specifically, the characteristic polynomial is actually the map given by the Chevalley restriction theorem, which says that the ring map $\mathbb{C}[\mathfrak{t}^*]^W \to \mathbb{C}[\mathfrak{g}^*]^G$ given by inclusion is an isomorphism.

(8/5/2019) Today I learned that the map of sheaves which maps $\delta_{N^-,\psi}$ into the sheaf associated to the counit of the BBM adjunction cannot be the splitting. This is because the adjunction involves a shift by two, whereas the counit does not.

(8/6/2019) Today I learned how to define a more general stack quotient than just that coming from a group action. Specifically, one can view the relation on X, at the set level, as a subset of $X \times X$. We can then define the quotient as the colimit of the two maps $R \rightrightarrows X$, which, through the simplicial category, can be upgraded to geometric realizations to define quotients such as $\mathfrak{t}//W^{aff}$.

(8/7/2019) Today I learned that the differential equation $u''(t) = tu(t)$ does not admit any elementary solutions. This is because its differential Galois group is $SL_2(\mathbb{C})$.

(8/8/2019) Today I learned what I think is a heuristic of the fact that if a right adjoint $R : \mathcal{C} \to \mathcal{D}$ has a W equivariant structure, for (at least a finite) group W, then so does the left adjoint. Namely, for an $s \in S$, a group acting on $\mathcal{C}$ can be viewed as a functor $F_s \in End(\mathcal{C})$, and then one can identify $LF_s \to LF_sRL \cong LRL \to L$, which is an equivalence.

(8/10/2019) Today I learned a point about groups acting on categories. For example, considering $w \in W$ as acting by its skyscraper sheaf $\delta_w * -$, we have that even though the action of δ_w may fix an object, that does not mean it acts as the identity on the morphisms of that set.

(8/11/2019) Today I learned that the W action on the unit of the BBM adjunction is trivial when restricted to the closed subset $N^- \subseteq G$. This is because you can first use equivariance to show that it suffices to compute this isomorphism when restricted to the identity, by N^- equivariance. Then you can base change a few times to reduce the identification to the fact that in W acting on $D(T)$, and then using the fact that the inclusion of the identity into T is W equivariant, we reduce both sheaves to this.

(8/13/2019) Today I learned that if an algebraic group G acts on some scheme X, then the functor Av_* (:= the right adjoint to the forgetful functor) is actually $(X \to X/G)_{*,dR}[-2dim(G)]$. This follows from a general fact that the left adjoint to !-ing by $q : X \to X/G$ (which we identify with the forgetful functor) is actually includes the above shift.

(8/14/2019) Today I learned that there is a equivalence between $D-$modules on the space Ran_X, the space of finite subsets of X, called *factorization algebras*, and a certain class of algebras called *chiral algebras*. The way that this equivalence is realized (in one direction) is given by taking our $\mathcal{A} \in D(Ran_X)$, taking the fiber at X, shifting it by 1, and then using the excision sequence associated to the diagonal and off diagonal inclusion into $X \times X$ to get an algebra map on the shifted sheaf.

(8/15/2019) Today I learned the likely unit for the BBM adjunction is given by the fact that the sheaf $m_*(k \boxtimes \psi \boxtimes k)$ contains the sheaf δ_{1N} as a subsheaf. I also learned that a conjecture of Witten which says that the *skien module* associated to a closed three manifold is finite dimensional, which has been proven using ideas of quantum groups.

(8/16/2019) Today I learned that for the perverse t-structure on a smooth variety X of dimension d, the constant sheaf is in degree d Furthermore, the canonical sheaf in degree $-d$ because we define us as the sheaf of forms *shifted to the left by d.*

(8/17/2019) Today I learned that it's not the case in derived categories that you can restrict a map of sheaves to a "subsheaf" (ie the counit map associated to a closed embedding) and argue that the restriction is nonzero. I also learned the notion of a *Lie algebroid* (essentially a Lie algebra with functions) and the notion of the universal enveloping algebroid, which in characteristic 0, for the tangent shear acting as a Lie algebroid on the space of functions, produces the usual differential operators; and in char p produces crystalline differential operators.

(8/19/2019) Today I learned about the existence of the Frobenius morphism of any algebraic variety, specifically, the affine map given to a scheme X over characteristic p given by $f \to f^p$. I also learned that this map on $\mathbb{P}^1$ pushes forward the line bundle $O(m)$ to a rank p vector bundle with line sub bundles either isomorphic to $O(q)$ or $O(q-1)$, where q is the quotient of m when divided by p.

(8/20/2019) Today I learned a heuristic for why the Bore-Weil theorem should be true. Namely, given any vector space V on which a reductive G acts simply, we can consider the line bundle $O(1)$ lying over the space $\mathbb{P}(V^*)$. Because any Borel subgroup will have an eigenvalue via the action of B on G/B, we can consider "G-ing around this line" which will descend to a map $G/B \to \mathbb{P}(V^*)$ on which we can pull back our line bundle to get an inclusion into global sections of some line bundle on G/B. I also learned the Bott part of the BWB theorem is false in characteristic p (and the other cohomologies aren't known).

(8/21/2019) Today I learned that the representations of $\mathfrak{sl}_2$ in characteristic p can actually jump multiplicities—for instance, for the prime 5 there is no 9 weight space of the rep of highest weight 17. I also learned an idea of Clifford theory which builds on the idea that you can restrict a irrep to a normal subgroup and (at least in the finite group case) get a semisimple rep of the normal subgroup. Also if the subgroup is unipotent, you can argue the restriction of a simple restricts to a trivial rep.

(8/22/2019) Today I learned a specific connection of the Hecke algebra associated to the local Langlands program. Specifically, given any compact open subgroup of $G(K)$ for a local field K, one can ask for the bi-invariant, compactly supported smooth functions $G(K) \to \mathbb{C}^\times$ and show that this has a convolution structure. I also learned that one has a bijection between the smooth representations of $G(K)$ which have some $G(\mathcal{O})$ fixed vector and the dominant weights of G, which follows from the Satake correspondence.

(8/23/2019) Today I learned that a certain plan of attack I had to show my sheaf is a direct sum fails, because two simple objects having nontrivial extensions can occur (eg M^0 is a nontrivial extension of two simples. Similarly, I learned that in an abelian category with at least two simple objects, no simple object is injective.

(8/24/2019) Today I learned that any local system defined on an open subset of a space whose complement has codimension larger than 1 automatically extends uniquely to the entire space. This is because local systems are representations of the fundamental group, and because of the fact that for any open subset $V \subseteq X$ of real codimension 3 or larger, the map $\pi_1(V) to \pi_1(X)$ is an isomorphism.

(8/26/2019) Today I learned that, for at least reductive groups G, the functor $(-)^{G,w} : G-Mod_w \to DGCat$ is conservative, which means the same thing as it means for the one categorical world. I also learned that this is false for strong invariants. I also learned one way to phrase the generalized Beilinson-Bernstein theorem of David Ben-Zvi and David Nadler, which says that the category $\mathfrak{g}-mod$ is equivalent to $(D(G/N)^{T,w})^{\mathcal{W}}$, where $\mathcal{W}$ refers to a certain endofunctor obtained by doing the corresponding global sections, then Loc, functor.

(8/27/2019) Today I learned a construction of the Weyl group acting on the space of functions on some semisimple group G. Specifically, for any simple root s, one can realize G/N as an open subset of a rank two vector bundle over the space $G/[P_s, P_s]$ (which is a point when $G = SL_2$ and then construct a function pointwise by integrating over this vector bundle, twisted by some nontrivial character.

(8/28/2019) Today I learned a fun proof that the de Rham cohomology of the delta sheaf is concentrated in degree zero and given by k there. Specifically, when one defines the constant sheaf as the pushforward of k via the the embedding $* \to X$ and defines deRham cohomology as the functor of pushing forward via $X \to *$, this becomes a tautology.

(8/29/2019) Today I learned a proof of the comparison theorem, which says that for a smooth manifold X, $H_{dR}(X) = H_{sing}(X, \mathbb{Z}) \otimes \mathbb{R}$. Specifically, this follows formally from the facts that for smooth X, the de Rham complex is quasiisomorphic to the constant complex and a result which says that $\Gamma(X, \Omega^i)$ has no higher cohomology.

(8/30/2019) Today I learned a formal way to construct, for at least a finite group G, the notion of the full G-subcategory spanned by given objects in a category $\mathcal{C}$. Namely, one uses the free, forgetful adjunction and constructs the essential image "by hand" as a colimit and then show that this category forgets to the category spanned by the collection of objects.

July 2019

(7/1/2019) Today I finally learned why the sphere spectrum $\mathcal{S}$ is not the integers, and why this should be interpreted as the true initial place to do arithmetic (at least homotopically), and why in characteristic zero everything vanishes. In particular, the first can be shown by taking the map $\mathcal{S} \to \mathbb{Z}$ (associated to the unit of the adjunction of π_0 : Spectra $\to$ Abelian Groups : Sp) and noting that $\mathcal{S}$ has nonzero homotopy groups, i.e. $\pi_1(\mathcal{S}) := colim(\pi_1(S^2) \to \pi_2(S_3) \to ...$ is nonzero, for example. The fact that, say, S_n acts on S^n by changing coordinates is what keeps track of the various identifications, and the fact that if $i \neq j, \pi_i(S^j)$ is torsion is why if you tensor with all primes, the groups (and thus the associated $\pi_n(\mathcal{S})$ for $n > 0$) vanish.

(7/3/2019) Today I learned that the ring $k\langle t, d_t\rangle$ has no nonzero finite dimensional representations (for k an algebraically closed field of characteristic zero). This is because any such representation is supported on finitely many points on the line, and by Kashiwara's lemma these can be viewed in terms of their restriction (i.e. the functor $i_{*,dR}$ is fully faithful) so in particular you can find some nonzero vector in any nonzero representation for which t acts by some scalar, and by shifting we can assume that scalar is zero. Then we can argue that $d^N(v)$ cannot be expressed as a $k-$sum of the previous $d^i(v)$'s.

(7/4/2019) Today I learned a new possible interpretation for the functor of convolving sheaves which are canonically W-equivariant for the Weyl group W, at least for $G = SL_2$. Specifically, given two $N-$equivariant sheaves on two nondegenerate categories $\mathcal{C}, \mathcal{D}$ which N acts on the opposite sides, you can create a category $\mathcal{C} \otimes \mathcal{D}$ which hopefully acquires a *nondegenerate* G action, which in particular implies that W canonically acts on the N invariants. Then, assuming that there is a G-map $D(G/N)_{nondeg} \to D(G/G) = Vect$, we obtain a way to average in a W-equivariant way.

(7/5/2019) Today I learned an inconsistency in my own thinking. Namely, I expected the convolution of two nondegenerate N-equivariant sheaves on G to be convolvable by taking the diagonal action on $D(G)_n$ and obtaining a W action canonically from the fact that this turns the product of the two categories into a nondegenerate category. However, this can't be true AND that a nondegenerate category always has no G equivariant objects.

(7/7/2019) Today I learned a step in the proof that Lie algebras are classified by their root systems—-namely, to show how the map taking a Lie Algebra and it's choice of Cartan subalgebra to its associated root system is injective. Specifically, this is done by writing out the Lie algebra in terms of generators and relations which only depend on the roots.

(7/8/2019) Today I learned an explicit computation of showing that the pullback of the exponential D-module on some line has no cohomology. This is an explicit computation (and the fact that $\partial_y - 1 : k[y] \to k[y]$ is an isomorphism). I also learned that this does not in fact prove that the de Rham global sections of $p_{1,-y}(\psi)$ vanishes because G/N is not $\mathbb{A}^2$, but is $\mathbb{A}^2 \setminus 0$.

(7/9/2019) Today I learned that the notion of the Geometric Satake Equivalence only holds in the usual form for abelian categories, and that there is a notion of the derived Satake equivalence. I also learned that I was computing a limit of categories when I probably wanted to compute a limit in categories.

(7/10/2019) Today I learned a more rigid way to define the category $Vect_\psi$. Specifically, using the fact that a comonoidal functor induces a functor on the respective module categories in the same direction, you can use the exponential D module, and the fact that it induces a monoidal functor $Vect \to D(\mathbb{G}_a)$ to produce a category called $Vect_{exp}$ (say) which is a comodule over the coalgebra object $D^!(\mathbb{G}_a)$ and then use the similar monoidal functor $D^!(\mathbb{G}_a) \to D^!(N)$ for any group homomorphism $\psi : N \to \mathbb{G}_a$ to obtain $Vect_\psi$, a comodule for the comonoidal category $D^!(N)$.

(7/11/2019) Today I learned that, letting $K := k((t))$, how to define the notion of a *level*, which corresponds to a bilinear form on $\mathfrak{g}$. Specifically, given any such bilinear form κ, there is a canonical central extension of $\mathfrak{g}((t))$ by κ. Furthermore, I learned that the Killing form provides a way to find a duality between categories $\mathfrak{g} - Mod_\kappa$.

(7/12/2019) Today I learned, or at least solidified, that taking the fixed points of some finite group W acting on the vector space k trivially has nontrivial cohomology in its fixed points. In other words, taking the fixed points of the trivial action recovers more than just k. I sort of knew this already, but didn't have it in my heart of hearts. In particular, I learned that for all odd positive i, $H^i(W, k_{triv}) = \{\pm 1\}$ for $W = \mathbb{Z}/2\mathbb{Z}$.

(7/13/2019) Today I learned that the functor $oblv : Rep(W) \to Vect$, for W a finite group and over an algebraically closed field of characteristic zero, admits a right adjoint. This is the decategorified analogue of the result of Gaitsgory.

(7/14/2019) Today I learned that given a category $\mathcal{C}$ on which a group G acts, the counit of the $oblv, Av_*^G$ adjunction is given by the functor of tensoring with the G representation $\Gamma(O_G)$ when identifying $Q(G) \otimes_{G,w} Vect \cong \mathcal{C}_{G,w}$.

(7/15/2019) Today I learned that there is an $\mathbb{A}^2$ inside SL_2 which can be realized as the closed embedding $N^- \times N \to SL_2$ via multiplication. I also learned the fundamental theorem of tropical geometry, which says that given a nonarchimedian field κ and a polynomial $f \in \kappa[x_1^{\pm 1}, ..., x_n^{\pm 1}]$, you can compute the closure of the image of $V(f)$ under the valuation map $(x_i)_i \in (\kappa^*)^n \to \mathbb{R}^n$ is sent to $(log(|x_1|_\kappa), ..., log(|x_n|_\kappa))$ by computing the tropical hypersurface associated to f in $(\kappa^*)^n$. This emphasizes the slogan of tropical geometry providing information about the original variety while sometimes being easier to study.

(7/16/2019) Today I learned that, while the map $exp : \mathfrak{g} \to G$ is surjective when $G = GL_n$, it is not surjective in general, even for surjective G. In particular, it is not surjective for $G = SL_2(\mathbb{C})$. However, it still provides a local diffeomorphism which, for a connected Lie group, is enough to determine any map from a group.

(7/17/2019) Today I learned a fun way to realize that $\mathfrak{so}_n$ is the group of skew symmetric matrices. Specifically, you can write the group SO_n as the matrices for which $M^T = M^{-1}$ and have determininant one. Then you can plug in the matrix $M = I + \epsilon N$ to see that the condition reads $I + eN^T = I - eN$, which specifically gives your skew symmetric matrices.

(7/18/2019) Today I learned the algebro-geometric construction called the *deformation to the normal cone* for a closed subscheme $Z \subseteq X$. Specifically, you can sheafify the affine case, for which if you have a ring A and an ideal I, you can form the Rees construction on the filtration whose $-i^{th}$ component is given by I^i. We then obtain the fiber at zero is given by $Spec(A/I \oplus I/I^2 \oplus ...)$, and if I is cut out by a regular sequence, this scheme is isomorphic to $Spec(Sym_{A/I}(I/I^2))$.

(7/19/2019) Today I learned that the multiplication map $N^- \times N \times N^- \to SL_2$ restricts to an open embedding when restricting to coordinates when the N coordinate is not the identity.

(7/20/2019) Today I learned the fact that the equivalence of categories $D(T)^W \cong QCoh(\mathfrak{t})^{\tilde{W}^{aff}}$ can be interpreted as a Fourier transform statement, and a result of Gus Lonergan's which says that this can be extended to say that $D(G)^{N^- \times N^-, \psi \times \psi}$, a full subcategory of $D(T)^W$, can be identified with the full subcategory of $QCoh(\mathfrak{t})^{\tilde{W}^{aff}}$ of objects with *trivial derived isotropy.*

(7/22/2019) Today I learned the specific axioms of a *root system*, the geometric object which classifies all semisimple Lie algebras. Specifically, a root system is a real Euclidean space with an inner product and a finite spanning set (the "roots") such that for all $\alpha \in R$, $n\alpha \in R$ if and only if $n = \pm 1$, such that the roots are closed with respect to reflections in the hyperplanes given by the $\alpha \in R$ and an "angle restriction condition" which requires $2\frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$.

(7/23/2019) Today I learned the general flow of how the basics of the theory of semisimple Lie algebras work. Namely, you first show that the Killing form is nondegenerate for semisimple (and only semisimple) Lie algebras, and from that, you can show that the representations preserve the

Jordan Decompositions and thus a Cartan subalgebras exist and are nonzero. This existence of a nondegenerate Killing form gives the existence of sub $\mathfrak{sl}_2$ algebras of the Lie algebras, and the fact that the roots span $\mathfrak{t}^*$ and that the roots are closed under negation.

(7/24/2019) Today I learned that, given a category $\mathcal{C}$ for which the loop group $G(K)$ acts, there are functors $Whit^{\leq n}$ and fully faithful embeddings $Whit^{\leq n}\mathcal{C} \to Whit^{\leq n+1}\mathcal{C}$ for which you can recover the $Whit(\mathcal{C}) = \mathcal{C}^{N(K),\psi}$ as the colimit of these fully faithful embeddings.

(7/26/2019) Today I learned the notion of a general $G-$local system on a punctured disk. Specifically, one can define it as the quotient of $\mathfrak{g}((t))$ by its Gague action by $G(K)$. Furthermore, I learned that for the group $\mathbb{G}_m$, the local systems are given by a product $\mathbb{A}^1/\mathbb{Z} \times colim_n \mathbb{A}^n_{dR} \times B\mathbb{G}_m$, and that the regular singular local systems give just the first and last factor in the product.

(7/27/2019) Today I learned two smaller things which are putting cogs in the correct places. Namely, I first learned that $Ext^1(L,N)$ classifies *extensions of L by N*, where in particular for our extension, N is our subobject. I also learned that the first congruence subgroup of $\mathbb{G}_m$ is not $t\mathbb{G}_a$.

(7/28/2019) Today I learned (solidified?) the definition of an affine Weyl group in my head. Specifically, the *affine Weyl group* is the group of reflections of a root system given by the semidirect product of the Weyl group and the translations given by the coroot lattice, i.e. the lattice given by coroots. I realized I can remember that the coroot lattice is the normal factor of the affine Weyl group because in the SL_2 case it has index two, whereas the Weyl group is not normal in the affine Weyl group!

(7/29/2019) Today I learned that topological field theories (i.e. symmetric monoidal functors from a bordism category to the category of vector spaces) map into finite dimensional vector spaces. This is because any symmetric monoidal functor preserves dualizable objects. I also learned a result of Ginzburg's, which he calls a dual to a theorem of BBM, which says that the functor $D(G)^{N^-,\psi} \to D(G) \to D(G)^N$ given by averaging is t-exact.

(7/30/2019) Today I learned another form of the result about the biWhittaker category which identifies it as a full subcategory of $D(T)^W$. Specifically, each $M \in D(T)^W$ canonically carries with it a W representaiton, and the functor specifically maps onto those objects $M \in D(T)^W$ for which the map $Sym(\mathfrak{t}) \otimes_{Sym(\mathfrak{t})^W} M^W \to oblv(M)$ is an isomorphism.

(7/31/2019) Today I learned the outline of Ginzburg's proof of the theorem that you can specialize to a spherical subalgebra of an affine nilhecke Algebra and get the bi-Whittaker differential operators. Namely, Ginzburg constructs a map $D_T^W \to$ and then constructs a bimodule where the group ring $D_T \times W$ acts on the left and the biWhittaker operators act on the right, and then extends this biaction to the left hand side being the entire spherical nilHecke Algebra. He then constructs the map on the spherical subalgwbra by showing that every element of this subalgwbra commutes with the canonical generator in "one and only one way."

June 2019

(6/1/2019) Today I learned that the computation of the category $(\mathfrak{g}-Mod)^N_{nondeg}$ (which still has a G action by Beilinson-Bernstein) and that it is given by $IndCoh(\chi^{-1}(0))$ where $\chi : \mathfrak{t} \to \mathfrak{t}//W$ is the characteristic polynomial map. I also computed this inverse in the case where $\mathfrak{g} = \mathfrak{sl}_2$ and it turns out to be the dual numbers, which more evocatively is the complex $k[t] \to k[t]$ given by multiplication by t^2. I also learned that for irreducible representations of a general Lie algebra $\mathfrak{g}$, say V, then there is a representation of $\mathfrak{g}_{ss} := \mathfrak{g}/Rad(\mathfrak{g})$, say W, of the same dimension, where $V = W \otimes L$ for some one dimensional L. This reduces a lot of the study of Lie algebra representations to that of semisimple ones.

(6/3/2019) Today I learned about the notion of $D-$affine ness for a smooth variety X. Specifically, a variety is *D-affine* if the standard global sections functor $D(X) \to D_X - Mod$ renders an equivalence. It is known that any partial flag variety G/P is always $D-$affine for a reductive group G, but it is not known whether any $D-$affine variety is of the form G/P for some reductive group G. I also learned about the *Levi decomposition* of a Lie algebra, which says that there is a maximal solvable subalgebra of any lie algebra $\mathfrak{g}$, called $Rad(\mathfrak{g})$ and that there is a semisimple $\mathfrak{l}$ such that as a vector space, $\mathfrak{g} \cong Rad(\mathfrak{g}) \oplus \mathfrak{l}$.

(6/4/2019) Today I learned the statement, and parts of how to prase the statement, of the quantum local geometric Langlands conjecture. Specifically, the conjecture makes reference to a κ in a canonical extension of the loop group $G(K)$ for a fixed $K = k((t))$ and says that for a reductive group G and $\kappa \neq \frac{1}{2}$, that there is a canonical equivalence of the modules of the "local Lie algebra" of G at level κ and the local Lie algebra at the Langlands dual group at level $\frac{2}{2\kappa-1}$.

(6/5/2019) Today I learned ideas behind *arithmetic D-modules*, which are modules over a ring $D^\dagger$, an analogue of $D-$modules in algebraic geometry which allows for infinite order operators such that the associated power series have a certain radius of convergence. This is often used instead of the finite order differential operators because the theory behind this is on *rigid analytic spaces*, which roughly speaking attempt to replace the theory of varieties for $p-$analytic spaces. But in viewing this $D^\dagger$ as giving a collection of functions on the rigid analytic space version of the cotangent bundle, we need infinite order operators. I also learned that there is an arithmetic D-module version of the Beilinson-Bernstein theorem.

(6/6/2019) Today I learned that "any category $\mathcal{C}$ on which a reductive group G acts are of highest weight," ie that $\mathcal{C}^N = 0$ implies $\mathcal{C} = 0$. I also learned what opers are, at least for the group GL_n (namely a certain complete flag where subquotients can be identified). I also learned that the functor $Rep(G) \to \mathfrak{g} - Mod$ is not fully faithful, in general, because while the trivial representation has a contractable endomorphism space, the corresponding space on the level of Lie algebras does not.

(6/7/2019) Today I learned how to interpret the category of $\mathfrak{t}$ Modules acting on some category $\mathcal{C}$ as a category over the base $\mathfrak{t}^\vee$. Specifically, by viewing this structure as coming in part from a functor $\mathcal{C} \to \mathcal{C} \otimes \mathfrak{t} - Mod$ we can take a point of the dual of $\mathfrak{t}$ and use the associated functor to get an endofunctor of the category $\mathcal{C}$.

(6/9/2019) Today I learned that if the generalized functor $Loc : D(G/N)^{T,w} \to U(\mathfrak{t}) \otimes_{Z(\mathfrak{g})} Sym(\mathfrak{t}) - Mod$ sends $D_{G/N}$ (with its canonical T weakly equivariant structure) to $U(\mathfrak{t}) \otimes_{Z(\mathfrak{g})} Sym(\mathfrak{t}) - Mod$, then Loc is fully faithful. Specifically, because the right adjoint is global sections, the fact that $U(\mathfrak{t}) \otimes_{Z(\mathfrak{g})} Sym(\mathfrak{t}) \to \Gamma(G/N, D_{G/N})^T$ is an isomorphism says that the unit of the adjunction is an isomorphism and thus the left adjoint (which I have confirmed is given by the fact that $id \to RL$ is an iso, i.e. R is a partial inverse) is fully faithful. I also learned that for nonabelain G, the stack Bun_G is not quasicompact and that there is a difference between the !-extension and the *-extensions.

(6/10/2019) Today I learned a possible way to control the infinitesimal action without affecting the action of a group on an underlying geometric object. Specifically, given a group acting on some smooth variety X, we obtain a moment map $T^*X \to \mathfrak{g}^*$. Taking the fiber at a specific $\lambda \in \mathfrak{g}^*$ we can (hopefully) dictate how infinitesimal action should operate without affecting the underlying action on X.

(6/11/2019) Today I learned the notion of *Tate cohomology*, which for a finite (or profinite) group G is a way of splicing between the group cohomology and the group homology. It's defined at every integer by either the cohomolgy and homology (except at levels zero and -1, where it's defined via the kernel and cokernel of the norm map). Explicitly, for a G module A, there are functors $A_G \to A^G$ which allow us to construct a doubly infinitely long exact sequence with respect to the Tate cohomology. I also learned that there are certain conditions on the Sylow p subgroups on which you can detect a "cohomology duality" of Tate cohomology with Tate cohomology shifted by 2.

(6/12/2019) Today I learned the idea of the local geometric Langlands conjecture, which says that there is some sort of approximate equivalence between the $D(K)$-Module Categories $\mathcal{C}$ and the $\mathcal{C}^\vee$-categories over regular singular local systems on the punctured disk. I also learned that the forgetful functor on the right hand side to $Vect$ is not expected to commute with the forgetful functor on the left hand side, but more like is expected to do so after applying Whittaker invariants, which turn out (by a theorem) to be a colimit of Whittakers with respect to the various congruence subgroups.

(6/13/2019) Today I learned about *ideles* and the *idele class group*, which are like the adeles (a restricted, topological product) but with respect to the units of the field with valuations given by how large of an N you can put your element in $1 + \mathfrak{p}^N$. I also learned that similarly that you can construct the idele class group by embedding the units into this restricted product.

(6/14/2019) Today I learned that, given a category $\mathcal{C}$ on which GL_2 acts, you can use the Fourier transform to write $\mathcal{C}$ as a comodule category over $(D(\mathbb{G}_a, \otimes^!)$ to show that you can recover at least the fiber of $\mathcal{C}$ at every point of $\mathbb{G}_a$ by just knowing the N invariant fiber and at one nondegenerate fiber, because all of the other fibers are related through the action of the torus.

(6/16/2019) Today I learned this sort of way to view the fact that, say, $Hom_G(D(G)^{G,w}, \mathcal{C}) \cong \mathcal{C}^{G,w}$ and how to apply it to adjoints. Specifically, if I want to show that a certain functor $R : D(G)^P \to \mathcal{C}$ of G-categories was fully faithful, I could simply show that the counit $LR \to id$ was an isomorphism. However, I can then identify both LR and id as biequivariant sheaves, i.e. $D(P\backslash G/P)$ and construct the counit as a map between them.

(6/17/2019) Today I learned one motivation for using $cdga^{\leq 0}$'s as a model for "homotopical rings." Specifically, these are the exact rings for which allowing derived tensor products would be closed under. Furthermore, I learned the other two models in use today for homotopical rings. Specifically, I learned that over a field of characteristic p, one either uses *simplicial rings*, i.e. simplicial objects in the category of rings, which implicitly come with their own homotopy theory, or $\mathbb{E}_\infty$ ring spectra.

(6/18/2019) Today I learned the complex analytic construction of the *nearby cycles functor*, which takes a closed codimension one subset of a complex manifold and a sheaf (or complex of sheaves) on it, restricts it to a tubular neighborhood of the closed subset minus the closed subset and pulls it back further along the map given by pulling back realizing this tubular neighborhood minus closed subset as a map over the punctured disk and pulling back by the universal cover of this punctured disk. After pushing back along this map to a sheaf on X, we obtain the nearby cycles functor, and by taking the cofiber (which I learned is analogous to the cokernel–i.e. co's match with co's) of this, we can obtain the *vanishing cycles functor*. I also learned that the analogue of this universal cover in algebraic geometry tends to be the process of *deformation to the normal cone.*

(6/19/2019) Today I learned an alternative characterization of a flat module over a derived ring. Specifically, an A module M is flat if and only if $M \otimes_A -$ preserves the heart of the t-structure in Mod_A. Alternatively, M is flat if and only if $\pi_0(M)$ is a flat $\pi_0(A)$ module and the induced map $\pi_n(A) \otimes_{\pi_0(A)} \pi_0(M) \to \pi_n(A)$ is an isomorphism for all n.

(6/20/2019) Today I learned that $N^-\backslash B^-N/N \cong T$, so in particular this obtains a Weyl group action. I also learned the universal property of the cotangent complex again (namely, that the space of maps characterizes maps into square zero extensions) but I also learned that a lot of the important properties of the cotangent complex are formal from this and playing an adjoint game. In particular, you can compute the cotangent complex on free modules and you can show that it distributes over any colimit of rings (so to speak), and this is in fact how to show existence for any (derived) ring.

(6/21/2019) Today I learned that if you restrict your two morphisms of the category of correspondences to be proper morphisms, there is a symmetric monoidal functor $IndCoh$ and the fact that it's symmetric monoidal exhibits the $f_! = f_!$ for proper maps. Furthermore, I learned that the unit for the monoidal structure on N, ψ biequivariant sheaves, for finite groups at least, is given by the delta function which on anything in the coset N takes the value e^{ψ} and otherwise is zero.

(6/22/2019) Today I learned about the *symplectic Fourier transform*, which says that given a symplectic form on a vector space V, you can use the identification $V^{\vee} \cong V$ induced by the symplectic form to create an Sp invariant transformation on $D(V)$ which squares to the identity (opposed to the Fourier transform which squares to minus the identity). With this, you can argue that at least for $G = SL_2$, we obtain an action of the Weyl group on $D(G/N)_{nondeg}$.

(6/24/2019) Today I learned an alternate description of the Fourier transform. In particular, you can view the Fourier transform as an identification of $D(V) \cong D(V^{\vee})$ via the identification of the associated rings by noticing that D_V is generated by the functions on V, namely $Sym(V^{\vee})$ and the vector fields of V, which act as derivations in the ring, and that these vector fields are, unsurprisingly, generated by vectors in the vector space. In particular, you can swap the roles of V and $V^{\vee}$ to obtain a ring isomorphism $D_V \cong D_{V^{\vee}}$.

(6/26/2019) Today I learned that if R is some right adjoint which is an equivalence of categories via its left adjoint L, then in fact R is the left adjoint of L as well. In particular, I learned that the theorem of BBM which states that $D(G/N)^{N^-,\psi} \to D(B^-)^{N^-,\psi}$ is an isomorphism in particular implies that the left adjoint of $j^!$, where $j : B^- \times N/N \to G/N$ is the open embedding, exists and is $j_{*,dR}$.

(6/27/2019) Today I learned that in the finite group analogy, the sheaf associated to the functor $\mathcal{C}^{N^-,\psi} \to \mathcal{C} \to \mathcal{C}^N$ is equivariant with respect to the symplectic Fourier transform. I also learned that it is a consequence of the above theorem that the functor $Av_*^{N^-,\psi}[2dimN] : \mathcal{C} \to \mathcal{C}^{N^-,\psi}$ is the left adjoint to the forgetful functor when applied to the full subcategory given by C^N.

(6/28/2019) Today I learned that the sheaf associated to the identity in $D(G)^{N^-\times N^-,\psi\times\psi}$ has support entirely on N^-, and there it is isomorphic to ψ itself. This is because by base change, the N^- multiplication action has a subrep given by N^- itself, so on N^- the description follows from the description of the unit as $act_{*,dR}(\psi \boxtimes \delta_1)$.

(6/30/2019) Today I learned an alternate interpretation of the objects in a category $\mathcal{C}$ on which $W := \mathbb{Z}/2\mathbb{Z}$ acts. Specifically, given the functor which specifies this action, because it must pull back on the identity on W, you can determine the entire functor by what it does on the other point. This in particular gives a "square identity" endofunctor $\mathcal{C} \to \mathcal{C}$ which allows you to say, at least on some one categorical level, that the objects of this category are certain designated pairs of objects with an identification between them. This also gives a notion for what a morphism of $D(W) - CoMod$ might be–namely a functor "commuting" with the endofunctors!

May 2019

(5/1/2019) Today I learned that there is a functor $-Mod : Alg_{\mathcal{E}_1}(Vect) \to Alg_{\mathcal{E}_0}(DGCat)$ which has a right adjoint given by taking the endomorphisms of the distinguished element in the category. This specifically codifies the fact that a (continuous DG) functor from $A{-}Mod$ to a DG Category C is determined by where it sends the object A and specifying how A acts as endomorphisms on that object. Furthermore, I learned this equivalence is symmetric monoidal, and in particular can be upgraded to an adjoint pair with left adjoint given by $-Mod : Alg_{\mathcal{E}_2}(Vect) \to Alg_{\mathcal{E}_1}(DGCat)$, which in particular via the counit says that for any reductive G the DG category $Hom_G(\mathfrak{g}{-}Mod, \mathfrak{g}{-}Mod)$ receives a map from $Z(\mathfrak{g}) - Mod$.

(5/2/2019) Today I learned that, given an algebraic group G over a field of characteristic zero which acts on some category $\mathcal{C}$, given any two objects $F, G \in \mathcal{C}^{G,w}$, there exists an object $\mathcal{H} \in Rep(G)$ such that $\mathcal{H}^G \cong Hom_{\mathcal{C}^{G,w}}(F, G)$ (which exists by the adjoint functor theorem), and furthermore if F is a compact object in the category then $\mathcal{H} \cong Hom_{\mathcal{C}} oblv(F), oblv(G)$. In other news, I also learned that the problem of "a farmer needs to make a cabbage, a goat, and a wolf cross the river–how many things does he needs to carry into a boat" can be generalized to a number called the *Alcuin number* of a graph (where an edge denotes things that must always have human intervention), and this number is the vertex covering number or one larger than it, and computing this specific number is NP hard.

(5/3/2019) Today I learned that the above method to compute the fixed points specifically allows you to compute the endomorphisms of the unit of the $\mathcal{E}_1$ algebra structure on $Hom_G(\mathfrak{g}-Mod, \mathfrak{g}-Mod)$ without specifically knowing all of the structure on the unit, just where it oblv's to in the original category $\mathfrak{g}-Mod$. I also learned that in the above group acting on category setup, a weakly equivariant object is compact if and only if its corresponding object in the original category is compact.

(5/4/2019) Today I learned a general construction known as the *convolution monoidal structure* you can place on a category like $IndCoh$. Specifically, given any map $X \to Y$, you can pull-push along $Z \times_Y Z \leftarrow Z \times_X Z \to Z$ where $Z := X \times_Y X$ (at least when Y is a point), and recover things like matrix multiplication, when X is a finite set.

(5/6/2019) Today I learned that any *Segal object* determines an algebra object in the category of correspondences, and that you can detect the unit by the correspondence $* \leftarrow c^0 \to c^1$, at least in the case where the Segal object is the Cartesian product of a finite set with itself. I also learned a theorem of Dennis Gaitsgory which says that if you take the category of $QCoh(G) - Mod$ with its convolution monoidal structure, then the functor $(-)^{G,w} : QCoh(G) - Mod \to DGCat_{cont}$ is conservative.

(5/7/2019) Today I learned an interpretation of the weak equivariants of the category of A-Modules can be interpreted as literally A-Modules equipped with a trivialization of the G action. I learned through this explicit formula that the weak invariants of $Vect = k-Mod$ can be interpreted as representations of G!

(5/8/2019) Today I learned a specific interpretation of the fact that any Segal object yields an associative Algebra object in the category of correspondences in the case of $H\backslash G/H \cong BH \times_{BG} BH$ whose multiplication specifically corresponds to the correspondence $(H\backslash G/H) \times (H\backslash G/H) \leftarrow H\backslash G \times^H G/H \to H\backslash G/H$ (and for orientation, I know when H is trivial I would like this to correspond to the monoidal structure on G and when $H = G$ this should correspond to the monoidal structure on representations on G). I also learned that you can show that the compactness of a weakly equivariant object is determined by the compactness of its image in the original category, roughly speaking, by noting that for an algebraic group a totalization can be computed after finitely many terms and so the the continuity of the functor $Hom_{\mathcal{C}^{G,w}}(F, -)$ can be computed via a bunch of things that commute with colimits (because I'm stable categories finite limits are finite colimits).

(5/9/2019) Today I learned something I wish I had learned a year or two ago. Specifically, the data of an adjunction can be given as the data of two functors and a unit and a counit functor which are, roughly speaking, inverses of each other (really, satisfying the duality or Zorro relations). You can use this to recover the traditional formulas that an adjunction of functors $L : \mathcal{C} \leftrightarrow \mathcal{D} : R$ is obtained from the traditional functorial bijection by sending a map $f : LC \to D$ to the map obtained by precomposing $RLC \to RD$ with the unit map $C \to RLC$.

(5/10/2019) Today I learned a few interpretations of the shifted symplectic ideas we've been working with. Specifically, I learned that the *expected dimension* of a complex should be reasonably be interpreted as the Euler characteristic of the complex, viewing the Euler characteristic as a thing that would not change perturbing, say, your base Calabi-Yau 3-fold a little bit. Incidentally, this is also why many results involve CY *three*-folds, because the map from the space to the stack with 2-shifted symplectic structure $Perf$ yields a -1-shifted symplectic structure, which in particular can often be "quantized" to yield an integer.

(5/12/2019) Today I learned that by an adjunction game, for a $\mathcal{C} \in D(G) - Mod$ categories, the map $\mathcal{C}^{G,w} \to C^{N,(T,w)}$ can be associated with a *kernel* $K \in \mathfrak{g} - Mod^{N,(T,w)}$ by first constructing this map for the case where $\mathcal{C} = D(G)$, then showing that the map obtained via $Hom_G(-, \mathcal{C}) : D(G) - Mod \to DGCat$ is monoidal, and then use adjunctions.

(5/13/2019) Today I learned that the functor $Hom_G(-, \mathcal{C}) : D(G) - Mod \to DGCat$ is contravaraint. Really, that was at least a mistake I made today and I learned that if I am going to use that functor to appeal to the universal case to show that $\mathfrak{g} - mog \to D(G/N)^{T,w}$ is $Z(\mathfrak{g})$-Mod equivariant, I am going to have to use the functor in the other direction. I also learned the real reason why you can use the fact that invariance is coinvariance to argue that $\mathcal{C}^{G,w} \cong Hom_G(\mathfrak{g} - mod, \mathcal{C})$. Specifically, by replacing $\mathfrak{g} - Mod$ with $D(G)_{G,w}$, you can write and subsequently pull out a colimit in the first argument, and then can rewrite all terms in the sequence and thus get the limit expression for $\mathcal{C}^{G,w}$.

(5/14/2019) Today I learned that there is a more general class of subsets of morphisms (annoyingly) called a good class of morphisms for which one can do more generalized homology theory. This relates to the fact that you can define intersection cohomology and perverse sheaves on complex manifolds. I also learned an example of a stratification of a manifold which is not a Whitney stratification–specifically, a Jesus fish extended forever to cover the plane (okay fine, a self intersecting elliptic curve making a loop) raised up divides $\mathbb{R}^3$ into a few regions, and then the Jesus fish head pushed into intersect the Jesus fish line. With the line and the regions as the stratification, there is a colimit of points whose limit of secant lines is not in the tangent plane, breaking the Whitney stratification property.

(5/15/2019) Today I learned that there is an induced map in the "other" direction, for say, an algebraic subgroup $i : B \hookrightarrow G$ acting on a category $\mathcal{C}$, that goes $\mathcal{C}^{B,w} \to \mathcal{C}^{C,w}$. This map is obtained by precomposing $\mathcal{C}_{B,w} \to \mathcal{C}_{G,w}$ with the two averaging functors which yield equivalence of weak invariance and weak coinvariants. I also highly suspect that this map is the adjoint to the "forgetful" map $\mathcal{C}^{G,w} \to \mathcal{C}^{B,w}$, which I learned is induced by sending each "factor" in the simplicial set $QCoh(G)^{\otimes s}$ to $QCoh(B)^{\otimes s}$ via i^*.

(5/16/2019) Today I learned a way to construct the characteristic polynomial using the Killing form. Explicitly, the characteristic polynomial of some Lie algebra $\mathfrak{g}$ is given by identifying $\mathfrak{g} \cong Spec(Sym(\mathfrak{g}^\vee))$ and then restricting via the dual of $\mathfrak{t} \subseteq \mathfrak{g}$.

(5/17/2019) Today I learned the statement of the Geometric Langlands conjecture, which says that given an algebraic group G and an algebraic curve Σ, there is a canonical equivalence of categories $QCoh(LocSys_{G^\vee}(\Sigma)) \cong D - Mod(Bun_G(\Sigma))$. I also learned that the full strength of this conjecture (which includes statements about correspondences of Hecke eigensheaves on the right hand side) is true literally when G is a torus, but is not true for $G = SL_2$ and $\Sigma = \mathbb{P}^1$.

(5/21/2019) Today I learned that identifying objects of the category $D(G_{dR}/G)$ as "objects of $D(G_{dR})$ with G-equivariant structure, you can show that if the pull back of $e : * \to G_{dR}/G$ by $G_{dR} \to G_{dR}/G$ is the quotient map $q : G \to G_{dR}$, you can show that by base change arguments that the "underlying object" of $U(\mathfrak{g}) \in D(G_{dR}/G) \cong D(B\breve{G}_1) \cong \mathfrak{g} - Mod$ pulls back to D_G via base changing and noting that q_*^{IndCoh} corresponds to the functor $Ind : QCoh(G) \to IndCoh(G_{dR}) = D(G)$.

(5/22/2019) Today I learned the notion of a *Bar resolution*, which often times allows one to compute facts about any A module M for which the fact is known about the A module A and it is known that the truth of the fact commutes with tensor products and geometric realizations.

Specifically, any A module M can be resolved as the geometric realization of $A \otimes A \otimes M \to\to A \otimes M$.

(5/23/2019) Today I learned how I can use the specific case of $\mathcal{C} = D(G)$ to show that, for a general $\mathcal{C} \in D(G)-Mod$, the functor $\mathcal{C}^{G,w} \otimes_{Z(\mathfrak{g})-Mod} Sym(\mathfrak{t})-Mod \to \mathcal{C}^{N,(T,w)}$ given by forgetting and then averaging is fully faithful. Specifically, assuming you have this for the case $\mathcal{C} = D(G)$, you can then note that the functor has a right adjoint. Fully faithfulness of a left adjoint is equivalent to the unit being an isomorphism, and you can note that if you tensor via $\mathcal{C} \otimes_{D(G)}$ then the adjoint of $id \otimes (\mathcal{A}v \circ oblv)$ becomes the $id \otimes$ the right adjoint. In particular, the unit is still an isomorphism.

(5/24/2019) Today I learned a way to represent the forgetful map $D(G)^{G,w} \to D(G)^{B,w}$ as a map of $D(G)-$modules. Specifically, one can note that the forgetful functor of usual $D(G)-$modules (as opposed to $D(G)$-bimodules) is induced by the map $(B \to G)^*$ on quasicoherent sheaves. Similarly, one can consider $D(G)$-bimodules as $G \times G$ modules and then pull back by the functor $(G \times B \to G \times G)^*$.

(5/25/2019) Today I learned a possible way to use the identification $Hom_G(D(G)^{G,w}, \mathcal{C}) \cong \mathcal{C}^{G,w}$ to obtain the underlying object in $\mathcal{C}$. Specifically, any functor $F \in Hom_G(D(G)^{G,w}, \mathcal{C})$ can be averaged and precomposed with the functor $D(G) \to D(G)_{G,w}$ to obtain a functor in $Hom_G(D(G), \mathcal{C}) \cong \mathcal{C}$. Furthermore, if this category $\mathcal{C}$ happened to be $D(G)^{B,w}$, we could also postcompose with the canonical functor $D(G)^{B,w} \to D(G)$ to obtain the underlying object.

(5/27/2019) Today I learned a way to construct the category $Vect_\psi$ given a character $\psi : N \to \mathbb{G}_a$. Specifically, the exponential D-module *exp* provides a functor $Vect \to D(\mathbb{G}_a)$ which is monoidal with respect to the $!-$tensor product, and we can use the monoidal functor $\psi^! : D(\mathbb{G}_a) \to D(N)$ to pull back the associated $D(\mathbb{G}_a)$-comodule.

(5/28/2019) Today I learned about the concept of a *nondegenerate category* $\mathcal{C} \in D(G)-Comod$, which is a category which contains no *partially integrable* objects, i.e. objects in the essential image of C^P for some parabolic P larger than the Borel. These are the categories for which our conjecture applies.

(5/29/2019) Today I learned that the normalizer of the torus does not act on the group G/N in the way one might expect. Specifically, I computed that $wnw = n^T$, where w is the nontrivial element of the Weyl group of $G = SL_2$ and n is the matrix with all ones except for the $2,1$ entry. This in particular implies that the Weyl group does not necessarily act on all N equivariant categories with a G action, only possibly nondegenerate ones.

(5/30/2019) Today I learned an example of the degenerate subcategory on a category $\mathcal{C}$ on which some algebraic group G acts. Specifically, one can check that the only partially integrable objects in SL_2 are those which come from G equivaraint ones, i.e. those in the essential image of the functor $Rep(G) \to (\mathfrak{sl}_2 - Mod)^N$. I also learned that you can either call the category itself nondegenerate or you can call the $N-$invariants of the category nondegenerate because the functor $\mathcal{C}^N \to \mathcal{C}$ is fully faithful.

April 2019

(4/1/2019) Today I learned that you can't just show a certain functor is linear by showing that the adjoint functor is linear–specifically, this only gives some kind of lax linearity. But I also learned that there are certain categories, known as rigid abelian categories, for which the adjoint being linear is enough to guarantee linearity.

(4/2/2019) Today I learned the computation of Chow rings for $\mathbb{A}^n$ and $\mathbb{P}^n$. Specifically, the rings are given by $\mathbb{Z}$ concentrated in degree zero and $\mathbb{Z}[x]/(x^{n+1})$ as a graded ring respectively. I also learned a recent theorem which says that if a variety has an affine stratification then the stratifying sets form a basis for the Chow ring.

(4/3/2019) Today I learned that you can classify any functor of categories on which N acts $C^{N,w} \to C^N$ via viewing this as a functor $Hom_N(\mathfrak{n}-Mod, -) \to Hom_N(Vect, -)$ and then using Yoneda's lemma to associate this to an N linear functor $Vect \to \mathfrak{n}-Mod$, which in turn associates to an element of $(\mathfrak{n}-Mod)^N = Rep(N)$.

(4/4/2019) Today I learned a stronger version of the tensor hom adjunction than the one I had in my head. Specifically, the stronger version says that if B is an right R, left S bimodule, then for any left R-module N and any left S module M, we have a canonical isomorphism $Hom_R(M \otimes_S B, N) \cong Hom_S(M, Hom_R(B, N))$.

(4/5/2019) Today I learned what a mapping stack of two (derived) stacks X and Y are– specifically, they can be defined as the internal mapping objects in derived stacks, i.e. the object M for which $Hom(Z, M) \cong Hom(Z \times X, Y)$ for any Z. You can also define it via functor of points as taking maps over the based change X and Y. You can also show that if F is any derived Artin stack with a shifted symplectic structure, then there is a certain notion of a derived orientation on a derived Artin stack X for which $Map(F, X)$ inherits a shifted symplectic structure.

(4/6/2019) Today I learned why for a general algebraic group G, the functor of forgetting the G invariants is not fully faithful. Specifically, I learned that if you pull back the constant sheaf to BG (to make what is also called the constant sheaf), then the $D-$Homs between the constant sheaves is also (by adjunction) given by the de Rham cohomology, which I learned is nonzero for, for example, all tori.

(4/7/2019) Today I learned that for a reductive group G with maximal Borel subgroup B, there is a resolution of singularities of the nilpotent cone given by the cotangent bundle of the flag variety T^*X by identifying the cotangent bundle of the flag variety with $\mathfrak{g}/\mathfrak{b} \cong \mathfrak{n}^-$, which in particular says that we can use symplectic methods on a resultion of singularities of the nilpotent cone to obtain things like a moment map.

(4/8/2019) Today I learned a possible method of attack for showing that for a general category C on which an algebraic group G acts, the functor $C^{G,w} \to (C^N)^{T,w}$ is $Z(\mathfrak{g})$-equivariant. Specifically, by using the fact that $(D(B)^N)^{T,w} \cong U(\mathfrak{t})$-modules, I hope that you can show that the two functors are given by pulling back $U(\mathfrak{t})$ and realizing it as a $\mathfrak{b}$-module via $\mathfrak{b} \to \mathfrak{b}/\mathfrak{n}$ and then Inding it up to be a $\mathfrak{g}$ module.

(4/9/2019) Today I learned a conjecture that says given a pair of adjoint functors $L : C \to D : R$, then the functor $Hom(C, -) \to Hom(D, -)$ given by pullback by R has a right adjoint given by pullback of L.

(4/10/2019) Today I learned that given an abelian category, you can create a new heart of thederived category by a process known as *tilting*, which takes in two additive subcategories F and T of the abelian category which have follow the pattern of free and torsion sheaves and you can set a new category (which is often genuinely different) by declaring that the zeroth cohomology is in T, the -1^{st} cohomology is in F, and every other cohomology in the category is trivial.

(4/11/2019) Today I learned that the tangent space of a mapping stack between two stacks can be computed on A valued points f as the global sections of the pulled back tangent bundle on the codomain by the map (using the internal Hom property). This is proven by computing a specific fiber product and realizing this fiber product in two separate ways.

(4/12/2019) Today I learned that the differential operators on some scheme X on which some group G acts is automatically weakly equivariant, because it is induced from O_X. I also learned that $U(\mathfrak{g})$ is a free $U(\mathfrak{b})$ module and $U(\mathfrak{t})$ is in general not, since this would have implications about the functor $(-)_{\mathfrak{n}}$.

(4/16/2019) Today I learned the definition of a monoidal structure on a category. Specifically, a *monoidal structure* is a functor $\Delta^{op} \to (\infty, 1) - Cat$ for which the set $[0]$ maps to the trivial object $*$ and the induced map from the n maps $[1] \to [n]$ together induce an isomorphism.

(4/17/2019) Today I learned the notions of straightening and unstraightening, which allow one to classify all functors $C \to (\infty, 1) - Cat$ by viewing them as certain (co?)Cartesian fibrations $F : D \to C$. The idea is that upon objects, $c \in C$ is sent to its fiber over $* \cong c \to C$.

(4/18/2019) Today I learned about the notion of a *Cartesian monoidal category*, which is an $(\infty, 1)$ category whose moniodal structure is given by taking products in the category. In such a category, it's a theorem that the "inclusion" functor from the coalgebra objects of the category to the category itself is an equivalence, which says that every object in a Cartesian monoidal category is a coalgebra (with comultiplication and counit given by the diagonal and the point map respectively). This result also says that these "trivial" coalgebras are the only possible coalgebra structures.

(4/19/2019) Today I learned in an $n-$shifted symplectic Artin stack F, I learned that the (derived) intersection of two Lagrangian Artin stacks obtains a canonical $(n-1)$ shifted symplectic structure, which in particular implies that the intersection of any Lagrangians in a classical symplectic manifold acquires a -1 shifted symplectic structure, which in particular takes the traditional theory of Lagrangians and views it in a derived world.

(4/21/2019) Today I learned that any mapping object in the category of $D(G)-$modules acquires a natural monoidal structure via the fixed point functor. This is because given the functor which exhibits the internal mapping object as a monoidal object, you can compose it with the fixed point functor. You can show that the fixed point functor preserves the final object of the category because it is a right adjoint, and you can argue that because right adjoints commute with limits, the induced maps are all isomorphisms.

(4/22/2019) Today I learned a recognition principal which tells you whether a given category is R modules for some E_1 module R. Specifically, if you have a compact generator G of a certain stable ∞ category C, then your category is equivalent to $End_C(G) - Mod$. I learned this as a generalization of this fact in vector spaces, which says that because any finite dimensional vector space is a compact generator of $Vect$, we have that $Vect \cong M_n(k) - Mod$, where $M_n(k)$ is the ring of $n \times n$ matrices for some $n \in \mathbb{N}$.

(4/23/2019) Today I learned the notion of a *lax monoidal functor*, which is a functor between two monoidal categories for which the "algebra morphism" diagram is allowed to commute only up to natural transformation. Using this notion, you can also define the notion of an algebra object in a monoidal category as a lax moniodal functor from the point category (with its unique moniodal structure) to the monoidal category. I also learned the notion of an *endomorphism object* in a category and that it automatically acquires an algebra structure.

(4/24/2019) Today I learned an outline to show that $Hom_G(\mathfrak{g} - Mod, \mathfrak{g} - Mod)$ has a monoidal structure. Specifically, to do this you can show that the associated endomorphism object in $D(G) - Mod(DGCat)$ has a moniodal structure (because it is an endomorphism object) and then show that the fixed point functor is lax monoidal, which in particular preserves monoidal structure. Similarly, you can show that the functor $Hom_G(-, C)$ for any category C is lax moniodal, which in particular preserves that $\mathfrak{g}$ is a module for $End_G(\mathfrak{g})$.

(4/25/2019) Today I learned a theorem of Harish-Chandra which allows us to classify representations of real reductive groups. Specifically, the theorem of Harish-Chandra says that you can recover a representation of the group from knowing how it operates on a maximal compact subgroup K and how it operates as a representations of the Lie algebra $\mathfrak{g}$.

(4/26/2019) Today I learned that there is a stack called $Perf$ which takes in a homotopical ring and returns the perfect complexes on it. I also learned that this stack can be viewed as a colimit/union of Artin stacks (just as the underlying space/$\infty-$groupoid of abelian category of Vector spaces can be viewed as an increasing union of BGL_n's) and that it acquires a two shifted symplectic structure.

(4/27/2019) Today I learned that there is a notion of a *monoidal DG category*, which is literally by definition an associative algebra object in the monoidal category $DGCat^{\otimes}$. This clarified a lot of the confusion I had regarding certain functors of things that were clearly associative algebra objects of $DGCat^{\otimes}$, which before I realized this "equivalence" held weren't monoidal categories, being talked about as if they were monoidal categories. The philosophy I picked up was to view $DGCat^{\otimes}$ as sort of the "base" category. I also learned that given an $(\infty, 1)$-category C, there is a notion of the *center of the category.*

(4/29/2019) Today I learned about the notion of the *center* $Z(f)$ of a morphism $f : A \to B$ between two $\mathcal{E}_k$ maps of a symmetric monoidal infinity category C. Specifically, the center is universal with respect to the existence of a diagram of $\mathcal{E}_k$ algebras for which $A \to A \otimes Z(f) \to B$ over the map f. Furthermore, I learned that if f is a unit of some the $\mathcal{E}_k$ structure on B for $k > 0$, we can then use this to obtain a monoidal map to B.

(4/30/2019) Today I learned a few things about the Lurie tensor product. Specifically, the Lurie tensor product is a monoidal structure on the category of stable categories, whose unit is the category of Spectra, an algebra object whose unit is the sphere spectrum. I also learned that functors of the Lurie tensor product of two categories to another stable category correspond to functors which are exact and continuous in both variables.

March 2019

(3/1/2019) Today I learned a lot about simplicial sets. In particular, I learned that you can define the product of two simplicial sets to have n cells given by the product of the respective n cells, and it turns out by some magic that the product of the geometric realizations is the geometric realization of the product. I also learned, while trying to compute the right adjoint to the trunctation functor to truncated simplicial sets, that one way to compute what such an adjoint should be is to use Yoneda embedding on the generating objects and then just simply define the right adjoint on the generating objects to be what comes out via the adjunction. This in particular allows you to show that the right adjoint to the truncation functor is given by "gluing exactly one higher cell in when you can."

(3/2/2019) Today I learned the specific model category structure you can put on the category of bounded by above chain complexes of a ring R. Specifically, you can declare to be the weak equivalences to be as usual, the cofibrations to be the injections with projective kernel, and the fibrations to be those maps of chain complexes which surject in positive degrees. With this structure, you obtain the *projective* structure on the category of chain complexes. I also learned that any two of the classes of maps determines the third, so you can't change cofibrations to just be injections without changing what the weak equivalences are.

(3/3/2019) Today I learned a theorem called *Brown Representability Theorem*, which says that not only is there a way to take a spectra and associate to it a generalized cohomology theory, but to each generalized cohomology theory there is an associated spectra. However, I also learned that the category of cohomology theories is not equivalent to the category of spectra, because there are more maps in the category of spectra.

(3/5/2019) Today I learned an alternative interpretation of graded mixed complexes which has a more geometric flavor. Specifically, for reasons that I haven't totally figured out yet, $B\mathbb{G}_a$ is a group scheme, and through the pullback functor (and the trick of viewing graded modules as modules over the ring $\mathbb{Z}[t^{\pm 1}]$), you can view the infinity category of graded mixed complexes as quasicoherent sheaves on $B(\mathbb{G}_m \ltimes B\mathbb{G}_a)$, which in particular admits a functor $B(\mathbb{G}_m \ltimes B\mathbb{G}_a) \to B\mathbb{G}_m$ given by projection which can be interpreted as the negative cyclic complex.

(3/6/2019) Today I learned an alternate interpretation of the averaging functor. Specifically, fixing a character $\psi : N \to \mathbb{G}_a$, you can interpret the averaging functor $a_*(\psi^*(exp) \boxtimes -)$ as taking in a sheaf F and taking it to the "function" taking $g \in G$ to $\int_n e^{\psi}(n)F(gn)$.

(3/7/2019) Today I learned that the above interpretation likely lends itself to the fact that there are no nontrivial averaging functors on N equivariant sheaves. I also learned what a Bridgeland stability condition is, and that the set of Bridgeland stability conditions has a topology which makes it a complex manifold.

(3/8/2019) Today I learned that the grading on the sym of any chain complex is a grading of complexes, and the grading of complexes preserve this differenial. In particular, this perspective shows why zero shifted p-forms are the usual p-forms.

(3/9/2019) Today I learned a theorem of groups acting on categories which says that given any algebraic group G acting on a category $\mathcal{C}$ then there is a right adjoint to the forgetful functor $\mathcal{C}^G \to \mathcal{C}$ which is given by "averaging." This functor exists both in the case of a weak action and a strong action, and it plays analogy to the map taking $v \in V \in G - Mod$ to $\sum_{g \in G} gv$ in the finite group case.

(3/11/2019) Today I learned a neat little trick to show that certain diagrams commute. Namely, you can show a diagram commutes possibly by showing all functors involved in the diagram are right adjoints, and then show that for all the left adjoints replaced with the diagram commutes.

(3/12/2019) Today I learned the picture (basically a cross in the plane) that corresponds to the product of two $\mathbb{P}^1$'s in toric geometry. I also learned what a fan is, which is just the requirements that the faces of a polyhedral cone give you the gluing data you expect, and that the product of two cones yields the product of two fans.

(3/13/2019) Today I learned a way to cofibrantly replace the module k when regarded as a mixed graded complex. Specifically, you can take the complex $S := k[\epsilon]$ with ϵ in degree -1 and take the projective resolution to be $\oplus_{j \geq 0} S[2j]$. I also learned that with this projective resolution, the zero shifted zero forms are really given by the closed forms, because being a map of ϵ modules really does require all of the other arrows to be zero and the form to be closed.

(3/14/2019) Today I learned about the Dold-Kan correspondence, which specifically gives an equivalence of categories between complexes with cohomology concentrated in nonpositive degrees and simplicial sets such that the associated space to a given complex has the same cohomology in degree $-i$ as its i^{th} homotopy group.

(3/15/2019) Today I specifically computed the closed n-shifted $p-$forms on a smooth discrete ring A. You can compute this and realize that these forms are the *space* realized via the Dold-Kan correspondence above is given by the stupidly truncated de Rham complex $\Omega^{\geq p}[n]$. From this, you can show, for example, that the 0-shifted $0-$forms in the derived sense carry much more data if the de Rham cohomology is nontrivial.

(3/17/2019) Today I learned what a *projective cover* is (namely, it's the unique up to non canonical isomorphism projective object in an Artinian category with enough projectives for which no proper submodule of it maps surjectively onto the object) and what the *big projective* object is in BGG's category $D(G/B)^N$, is–specifically it's the projective cover of the big cell. I learned you can also realize this sheaf via pulling back the exponential module on $\mathbb{G}_a$ and then pushing it forward from N^- to G/B and then averaging with respect to the N action.

(3/18/2019) Today I learned what translation functors are and why they are defined. Specifically, a translation functor on the category O is defined by choosing two weights whose difference is integral, and takes the block associated to one weight, tensors with the unique simple object in O associated to the difference, and then projects onto the other weight. The switching of the weights give an adjoint, and this gives an equivalence of categories with certain blocks of O.

(3/19/2019) Today I learned why the center of a category acts on the weak invariants of any category on which a fixed algebraic group acts. To show this, you can reduce to the case that your category is $D-$modules of your algebraic group, since the weak invariants are also the strong G invariant maps from $\mathfrak{g}-$modules to your category, and then you can use a general argument about monoidal categories to show that the endomorphisms of $U(\mathfrak{g})$ in the Harish-Chandra category is precisely the center, which gives the functor $Z(\mathfrak{g}) - Mod \to End_G(U(\mathfrak{g}))$.

(3/20/2019) Today I learned that any continuous functor of DG categories $A - Mod \to C$ for some DG category C is determined by where the object A is sent and where the endomorphisms of A ($\cong$ A) are sent to as endomorphisms of the object that A is sent to.

(3/21/2019) Today I learned that filtered graded modules can be interpreted as quasicoherent sheaves on $\mathbb{A}^1/\mathbb{G}^m$, and the pushforward to the point map can be interpreted as taking the associated graded.

(3/22/2019) Today I learned a fact which is probably true which is that the convolution of $U(\mathfrak{g})$ with the constant sheaf on N yields the universal Harish-Chandra module $U(\mathfrak{g}) \otimes_{U(\mathfrak{n})} k$, which in turn shows that for the category $C := D(G)$, the associated map $C^{G,w} \to C^{N,(T,w)}$ given by forgetting to B invariants and then averaging is $Z(\mathfrak{g})$, where the right hand side inherits a $Z(\mathfrak{g})$ action via the action of $Sym(\mathfrak{t})$ pulled back by the Harish-Chandra map.

(3/25/2019) Today, in addition to the fact that Harish-Chandra is one person, I learned about Bridgeland Stability Conditions. Specifically associated to a triangulated category C and a map $\nu : K_0(C) \to \Lambda$, a BSC is a slicing of the triangulated category and a central charge $Z : \Lambda \to \mathbb{C}$ (i.e. group hom) which maps each slicing to that angle and is subject to some support condition which is equivalent to there being a certain symmetric bilinear form on the lattice.

(3/26/2019) Today I learned a neat little trick to show that you can swap out $Hom_G(F, Hom_G(I, J))$ for $G-$categories F, I, J. Specifically, you can use the tensor Hom adjunction to move the I onto the other side, and then switch the symmetry via changing a left action to a right one. You can then move the F over to the other side and use the fact that the Hom of two left G modules is also the Hom of them as right G modules.

(3/27/2019) Today I learned a fact about showing a functor F from some category $A - Mod$ to some DG category is fully faithful. Specifically, to show the (continuous) functor is fully faithful you can show that the object A is sent to a compact object, so the functor preserves compact objects, and then you can show that the map $A = End(A) \to End(F(A))$ is an isomorphism. You can show this by writing any object in the image as a colimit of the A's, and I also learned the existence of a better proof.

(3/28/2019) Today I learned a general overview of a proof technique in modern intersection theory. Specifically, if you want to know say how many twisted cubics are tangent to n quadrics in general position, you take the moduli space of all quadrics and consider the first n for which (in general position) the intersection of n general ones yields a zero dimensional scheme, and then count the number of points. I learned specifically that this number is larger than ten digits and that *Hilbert's 15th problem* was to provide rigorous foundations for these calculations in "Schubert Calculus."

(3/29/2019) Today I learned that $k[\epsilon] - Mod$ is equivalent to $k[u] - Mod$, where ϵ has degree -1 and u has degree 2. I also learned that $k[\epsilon]$ is the Koszul dual of $k[u]$ but it doesn't in general imply that you can take the Koszul dual of a cdga and get an equivalent module category.

(3/31/2019) Today I learned more about the translation functors. In particular, I learned that under the Beilinson-Bernstein correspondence, the translation functor which translates down by an integral dominant weight μ is the same as, on the twisted $D-$module on G/B, as tensoring with the dual of the BWB line bundle on G/B.

February 2019

(2/1/2019) Today I learned an idea behind replacing the technique of using Kodaria Vanishing in characteristic p, where the theorem does not hold. Specifically, since Kodaria vanishing gives the vanishing of higher (than zero) cohomology groups of an ample line bundle tensored with the canonical sheaf, and often times this can be worked around by showing that the Frobenius gives a similar vanishing on global sections.

(2/3/2019) Today I learned a cool interpretation of symplectic topology which probably was the original motivation for studying it. In particular, given some kind of (mechanical, say) system, you can argue that there is a smooth manifold of M of possible states that your system can be in, and that each individual system can be given by a function $M \to \mathbb{R}$. You can also use the fact that in physics, we hope that the present has some ability to predict the future to show that this predictability is given by a nondegenrate map $\omega : TM \to T^*M$, and furthermore you can use the fact that this prediction should be independent with respect to time to show that this ω must actually be antisymmetric and thus a two form. This is often given by a particle in motion where the cotangent bundle gives both the position and the velocity and there's a theorem that says any symplectic manifold is locally this canonical one.

(2/4/2019) Today I learned that the category of flat $k[\epsilon] := k[\epsilon]/(\epsilon^2)$ is equivalent to a vector space together with a self extension of that vector space. I also learned that you can further try to soup this up by using derived categories, but the naive notion of this fails because k is not a perfect object in the (derived) category of $k[\epsilon]$ modules. This is resolved by replacing the category $QCoh(Spec(k[\epsilon])$ with $IndCoh(Spec(k[\epsilon]))$.

(2/5/2019) Today I learned a basic, basic overview of how Gromov-Witten invariants work. Essentially, curves are input and a Hochschild cohomology class is returned which characterizes the curves of a certain genus. I also learned the gist of a definition of the functor category of points on which some $N \subseteq G$ acts trivially. Specifically, given a nondegenerate character $\psi : N \to \mathbb{G}_a$, we can use the Fourier-Deligne transform and show that $(D(\mathbb{A}^1), \otimes^!)$ acts on the category $Vect$, and we denote this category via $Vect_\psi$. Then given any category C on which N acts, we can define the category where N acts by ψ as $Hom_N(Vect_\psi, C)$.

(2/6/2019) Today I learned that using the *Eilenberg-Maclane* construction, which embeds commutative rings into the category of E_∞ ring spectra fully faithfully. Furthermore, today I learned that the category of $\mathbb{W}$ modules is equivalent to the category of D modules on G which have a certain twisted equivariant structure.

(2/7/2019) Today I learned that the weaker notion of the Darboux theorem, which says that any symplectic structure on any symplectic manifold is locally isomorphic to the canonical one on the cotangent bundle (and in algebraic geometry, formally locally isomorphic) can be extended to the setting of derived shifted symplectic geometry. In particular, there are two possible definitions for what the canonical structure should be, and the theorem is true for the weaker one.

(2/8/2019) Today I learned that any vector bundle on $\mathbb{P}^1$ is a direct sum of line bundles. To see this, you induct on the rank of the vector bundle and consider the injection given by the line bundle $O(a)$ where a is the maximal integer for which Serre vanishing doesn't hold for the sheaf. You can then show the inclusion of the line bundle into your vector bundle must have a locally free cokernel as well, lest otherwise the torsion subsheaf be supported at a point which you can show violates the maximality of a. You can then show your sequence must split by explicitly computing $Ext(O(a), O(b))$ (and use of the induction hypothesis).

(2/9/2019) Today I learned about the Brauer group of a field k, which by definition is the group of equivalence classes of finite dimensional simple central algebras over the ground field up to isomorphism (and addition given by the tensor product). I also learned that any central simple algebra has an index, which is the rank of the unique up to isomorphism central division algebra that the csa is a matrix group of, and that it is known the order of the element in the algebra divides its index which divides its order to some power, and it is an open problem in general to answer whether a uniform bound on this power exists in general.

(2/10/2019) Today I learned that if you have an algebraic group acting on a ring A, there is a canonical obstruction to lifting the associated weak action on the category $A-Mod$ to a strong action. Specifically, the obstruction is the existence of an associated map of Lie algebras from the Lie algebra of the group to the (Lie algebra) A such that commutating by any vector in the Lie algebra is the same as the induced Lie algebra action on A.

(2/12/2019) Today I learned that the moduli space of semistable degree d and rank r vector bundle on a curve of genus g is entirely classified. Furthermore, I learned in almost all cases you can detect the strictly semistable line bundles in this moduli space by looking at the singular locus of the moduli space of semistable points.

(2/13/2019) Today I learned an alternative notion of a finite type algebraic group acting on a category. Specifically, the group structure allows one to view the category of $D-$ modules or quasicoherent sheaves on G as a monadic category under convolution, and you can define the notion of the group acting on the category as a module over this monad. I also learned that, even ∞-categorically, $End(Vect) \cong Vect$.

(2/14/2019) Today I learned that you can generalize the notion of stability to the notion of a *stability function*, which is a function from the Grothendieck group of some abelian category to the upper half plane and nonpositive real line of the complex numbers. You can then define the generalized rank, degree, and slope, as in the cases of quivers and curves, and also you can show that the stability functions for which all objects admit Harder-Narasimhan filtrations are those for which the image of the real part is discrete, because then you can induct on the rank of each individual object.

(2/15/2019) Today I learned the category of Spectra has a t structure, and under this t structure, the heart of the category is the category of abelian groups. I also learned that you can consider the category of Spectra over some topological space, and this notion fits into the six functor formalism. I also learned that you can use Ind and $CoInd$ to fit into the six functor formalism in Representation Theory.

(2/16/2019) Today I learned a cool little proof that the category $D(G/B)^{N,\chi}$ is equivalent to the category of vector spaces, where χ is a nondegenerate character $N \to \mathbb{G}_a$. Specifically, you can use the Beilinson-Bernstein correspondence to note that this is equivalent to the category of $\mathfrak{g}$ modules with central character zero, and show that the (N,χ) equivariance of this category is in particular the category for which the associated Lie algebra $\mathfrak{n}$ acts locally nilpotently. You can then use a fact of Kostant that any k linearly independent Whittaker vectors are actually $U(\mathfrak{g})$ linearly independent and split as a direct sum to show that at least any object in the category is given by some cardinality.

(2/18/2019) Today I learned about *Skyrabin's equivalence.* Specifically, Skyrabin's equivalence says that given a nondegenerate character of the unipotent radical of some reductive group, say $\chi : N \to \mathbb{G}_m$, then the category of $\mathfrak{g}$ modules for which $\mathfrak{n}$ acts locally χpotently has a compact generator $Ind_{\mathfrak{n}}^{\mathfrak{g}}(k)$, and so by formal nonsense, this category is equivalent to modules over $Ind_{\mathfrak{n}}^{\mathfrak{g}}(k)$ and furthermore Skyrabin's equivalence shows that the map from the endomorphism ring of the center is in fact an isomorphism.

(2/19/2019) Today I learned that $G(k[[t]])$-equivariant $D-$modules on the affine Grassmannian are automatically regular and holonomic. In particular, a rephrasing of the Geometric Satake theorem would then say that these equivariant modules are equivalent to the category of representations of the Langlands dual group of G.

(2/20/2019) Today I learned more about the proof of the fact that the category $End(Ind_{\mathfrak{n}}^{\mathfrak{g}}(\psi))-modules$ is equivalent to the category of $\mathfrak{g}$-modules for which a nondegenerate character $\psi : \mathfrak{n} \to \mathbb{G}_m$ acts locally nilpotently is equivalent. Specifically, you can use the nilpotence of $\mathfrak{n}$ to show that $Ind_{\mathfrak{n}}^{\mathfrak{g}}(\psi)$ is a compact generator of your category (and this uses the fact that $H^1(\mathfrak{n}, W) = 0$ where W is a module of Whittaker vectors, and this gives you the conservativity part of the Barr-Beck construction.

(2/21/2019) Today I learned that given any model category (or furthermore, any category with a notion of weak equivalences) you can localize the weak equivalences in the infinity category of infinity categories (using a universal property) and then obtain an infinity category. This can be done to something actually concrete, like the category of chain complexes of modules over a ring or something! I also learned that if you have a *model category*, as opposed to some category with just weak equivalences, then you can take the simplicial nerve of the model category and obtain an infinity category which is equivalent to this, but more easily computable.

(2/24/2019) Today I learned that the affine Weyl group of SL_2 is the semidirect product of the integers with the cyclic group of order two (and I don't feel like learning how to type the semidirect product symbol). I also learned that this is a Coxeter group, as in particular any element with nontrivial "coordinate" in the group of order two is itself of order two, and that the group can be "Coexter generated" with two elements.

(2/25/2019) Today I learned the actual ∞-categorical definition of a generator of the category. Specifically, an object generates the category, say $C \in \mathcal{C}$, if and only if for all objects D that $Hom(C[-i], D) \cong *$ for all nonnegative i implies that $D = 0$ itself. I also learned why only nonnegative i's are used–specifically, the negative i's are covered in the homotopy groups!

(2/26/2019) Today I learned one form of Koszul duality. Specifically, one version of Koszul duality says that Lie algebras are the Koszul Dual to commutative Algebras. I also learned that there is a notion known as the Mellin transform which identifies D-modules on $\mathbb{G}_m$ with quasicoherent sheaves on the quotient $\mathbb{A}^1/\mathbb{Z}$ and furthermore this identification can be extended with D modules on any torus.

(2/27/2019) Today I learned the idea behind a closed n shifted p form is and solidified the definition. Namely, a closed n shifted p form is the category of maps from the ground field k, shifted to weight $-p$ and of degree p, as graded modules equipped with a square zero map from the complex to the complex shifted by one. This last part is what encodes the closedness, and the idea here is to require the closedness to be witnessed by some element in the complex, but in order for that element to be homotopically meaningful, that element must have zero (complex) differential, and there must be an idenfitication of those which must be homotopically meaningful, and much more.

(2/28/2019) Today I learned one reason that the "rightward" direction in commutative differential graded algebras is often considered the stacky direction. Specifically, by declaring our cdga's to be nonpositively graded, we can only test any of those objects. In particular, because stacks are strongly related to the presence of nontrivial automorphisms/identifications, we see that, for example, a cdga concentrated in degree one can only provide identifications of the "-1st" homotopy group, there are no nonzero objects in cohomological grading 1 by assumption!

January 2019

(1/2/2019) Today I learned an alternative way to view the map in the filtration of the proof that the global sections functor is conservative. Specifically, you can filter by whether over the point $gB \in G/B$, the vector $v \in L^-(\nu)$ is in gL^i where L^i is a B submodule in a specific B filtration of

$L^-(\nu)$. Specifically, there is an isomorphism $G \times^B L^-(\nu) \to G/B \times L^-(\nu)$ which is given by taking $(g, v) \to (g, gv)$ and roughly straigtens or destraightens the vector bundle. You can show that the filtration here then corresponds to $G \times^B L^i$.

(1/3/2019) Today I learned how to get a $\mathfrak{g}$ action for a map of $G-$equivariant vector bundles. Specifically, first note that for global sections of a G-equivariant vector bundle, there is a G action given by sending $\sigma \to g\sigma(g^{-1})$. This formula can be applied to any section (and used to show that $\mathfrak{g}$ maps any open set U of the base into itself) to take a section σ and an $\xi \in \mathfrak{g}$ and produce a map from the base space times $Speck[\epsilon]/(\epsilon^2)$ which restricts to the same section. This, through Yoneda esque arguments, gives a map to sections tensored over k with $k[\epsilon]/(\epsilon^2)$, and you can define the derivation to be the ϵ coordinate. In particular, this shows that any G equivariant map of $G-$equivariant vector bundles preserves this $\mathfrak{g}$ action on each section.

(1/4/2019) Today I learned there is a notion of a (finite type algebraic) group acting on a category in two different ways. Specifically, a group can act *weakly* or *strongly* on a category. The weak action on a category is defined as a comodule structure over the coalgebra given by multiplication of the group under pullback of the multiplication map on quasicoherent sheves, which can be dualized to convolution and a module structure, and the strong action is the same but with the category of $D-$modules under $!-$ pullback. With this, you can translate using BG's and formal neighborhoods to show that G acts strongly on its Lie algebra $\mathfrak{g}$ and this rigidly categorifies it.

(1/5/2019) Today I learned what the adjoint action is not. Namely, you have a map of algebraic groups given by $G \times G \to G$ given by conjugating the second element by the first, and the adjoint action is not simply plugging in $D = k[\epsilon]/(\epsilon^2)$ points everywhere–this actually would simply yield the identity if you just restricted to $\mathfrak{g}$. More specifically, the adjoint action is the action given by differentiating the "conjugate by" map $G \to Aut(\mathfrak{g})$. Specifically, in the first map, there are epsilons that multiply to zero that shouldn't, roughly speaking.

(1/7/2019) Today I learned another part in the proof that you can compute the multiplicity of a simple inside a Verma using the associated Kazhdan-Lusztig polynomial. Specifically, I learned that you can compute the pushforward of the Demazure resolution via an isomorphism of the Demazure resolution with gluing a bunch of various $G-$equivariant copies of Schubert cells which are $\mathbb{P}^1$'s, and through this, you can use the base change formula to show that the pushforward of the constant sheaf (which is also the IC sheaf, by smoothness) is the convolution of the associated constant sheaves.

(1/8/2019) Today I learned the full statement of the decomposition theorem and an immediate consequence regarding maps which are resolutions of singularities. Specifically, the decomposition theorem says if you push forward the IC sheaf along a proper map the object you get is still a semisimple object—in other words, the object you obtain is a direct sum of pushforwards of IC sheaves of locally closed subsets, possibly twisted by local systems. Using this, you can argue that given a resolution of singularities, i.e. a proper birational map, then only one sheaf in the direct sum can have support on the big open for which the map is an isomorphism. You can show that, because IC sheaves are defined by how they behave on big opens, that this implies that the IC sheaf on the codomain of a resolution of singularities is a direct summand of the pushforward of the IC sheaf of the domain.

(1/9/2019) Today I learned that the year is 2019. Also, I learned an alternative way to recognize the Hecke algebra. Specifically, if G is a finite group and B is some subgroup, one can use a certain duality and other isomorphisms to show that functions on the double coset space $k[B\backslash G/B]$ (for a fixed field k) can be realized as the $k[G]$-equivariant endomorphisms of $k[G/B]$. In particular, since the right hand side has an algebra structure, so does the left hand side, and this can be used to give you the algebra structure on the Hecke algebra.

(1/10/2019) Today I learned about the *Geometric Satake Theorem*, which is a method for computing solely algebro-geometrically the Langlands dual of a given reductive group. Specifically, the Geometric Satake Theorem says that you can realize the category of representations of the Langlands dual group as the category of perverse sheaves (i.e. certain $D-$modules) on the *affine Grassmannian*, which is is a certain ind-projective scheme $G(K)/G(O)$, which has a symmetric monoidal structure (which is a theorem in and of itself!)

(1/11/2019) Today I learned the gist of the proof of the Geometric Satake Theorem. Specifically, you can use some sort of "Tannankian theorem" which says that given a certain symmetric monoidal category with a certain functor satisfying a laundry list of conditions, you can show that it's actually the category of representations of some group, and you can use the functor to determine which group that it is. Of course, you have to show your category of perverse sheaves on the affine Grassmannian/your spherical Hecke category actually is symmetric monoidal, which is difficult in and of itself, but once you do that you can show the category you get is semisimple (and thus the group it is representations of is reductive) and then use root data to compute that it is the Langlands dual of your group.

(1/12/2019) Today I learned the statement of the Tannakian formalism, well, formally. Specifically, the Tannakian formalism says that given a symmetric monoidal $k-$linear abelian category C such that the endomorphisms of the unit are the ground field and such that it admits a "forgetful like" i.e. exact faithful functor of symmetric monoidal categories to vector spaces then that category is representations of the tensorial natural automorphisms of that functor, which (and this is part of the statement of the theorem) is itself a representable algebraic group. One computation I saw in particular to get this was to note that the natural transformation given by "multiplication by some $g \in G$" is a tensorial map but based on the group structure on the tensor product, multiplication by $g_1 + g_2 \in kG$ for distinct g_i is not tensorial. This gives some justification for why you can recover the group.

(1/14/2019) Today I learned another idea in the proof that the convolution product on the category of $G(O)$-equivariant perverse sheaves can be uniquely given a symmetric monoidal structure. Specifically, you can show that this category naturally embeds into a certain category of sheaves on the Beilinson-Drinfeld Grassmannian on $\mathbb{A}^1$, which has a symmetric monoidal product itself, called the fusion product, and show that the identification maps convolution to the fusion product.

(1/15/2019) Today I learned a very irritating fact about symmetric monoidal functors. Specifically, if you call a functor symmetric monoidal, not only do there have to be morphisms (or isomorphisms, depending on context) $F(A) \otimes F(B) \to F(A \otimes B)$, but also, these identifications must commute with the "symmetric identifications" of $A \otimes B \cong B \otimes A$. Specifically, the symmetric structure of chain complexes stems not from the "obvious" choice of map (which isn't a map of chain complexes), but instead that map multiplied by negative one if both homogeneous terms come from odd degrees. This must be accounted for in the proof that global cohomology on the equivariant perverse sheaves on the affine Grassmannian is symmetric monoidal.

(1/16/2019) Today I learned another key fact in the proof of Geometric Satake. Specifically, you can show that the convolution product on the perverse sheaves on the affine Grassmannian $G(K)$ can be globalized to a construction involving the global loop group. Using this, one can construct the *convolution Grassmannian* for which there is a natural multiplication map which, on the fiber of any $k-$point $x \in X$, specializes to the multiplication map in the convolution case. From this, you can show that the pushforward via this multiplication map of the tilde of the box product of some product of sheaves is also the intermediate extension of those sheaves on the convolution Grassmannian.

(1/17/2019) Today I learned another fact in the proof of Geometric Satake. Specifically, I learned that the category of $G(O)-$equivariant perverse sheaves on the affine Grassmannian is a semisimple category, whose simple objects are the intersection complexes of the various $G(O)$

orbits. The reason that the simple objects are the IC complexes is by a theorem which says that any simple perverse sheaf is the intermediate extension of some local system on some locally closed subset, and you can use the $G(O)-$equivariance and the fact that each of the $G(O)$ orbits are etale simply connected and the fact that the stabilizers are simply connected to show that any local system on our $G(O)$-equivariant closed subset is the constant sheaf.

(1/18/2019) Today I learned what a local system actually is and the two equivalent ways to define it. The algebro-geometric way of defining it is just a sheaf of vector spaces for which each point has an open subset such that the restriction maps "below" that open set are isomorphisms. Alternatively, one could define it as a representation of the fundamental group (at least for a manifold, but this translates to etale π_1). The reason that these two are equivalent is through the *monodromy representation*, where given a local system and a point on a certain fiber we can draw trivializing local systems around points and complete it to a linear map of vector spaces.

(1/19/2019) Today I learned about the *nilpotent cone* and some of its uses. Specifically, I learned that the nilpotent cone is the set of all elements nilpotent in your Lie algebra, which, as a matrix is nilpotent if and only if all of its eigenvalues are zero (over an algebraically closed field) can be defined by algebraic equations. Furthermore, you can show that it is also the pullback of zero under the map $\mathfrak{g} \to \mathfrak{g}//G$ for a reductive algebraic group G.

(1/21/2019) Today I learned you can recover a reductive algebraic group from just knowing its Grothendieck semiring (i.e. no additive inverses). This is because you can use the bijeciton of dominant weights and irreducible representations to recover the dominant weights, and then you can use the fact that "the IC of the sum of roots appears in the direct summand of the convolution of the IC's" to show that you can recover a weaker ordering and the additive structure of the weights, and then you can use a lemma that says you can recover "what you have to mod out by to get the actual partial ordering" using just the addition structure of the roots and the multiplication in the Grothendieck semiring.

(1/22/2019) Today I learned that the group SO_4 can be double covered by the product of two Sp_1's. This is because given any two vector spaces with a Hermetian structure, you can tensor the Hermetian structures to obtain a real structure which in particular gives you a map $Sp_1 \times Sp_1 \to SO_4$. I also learned what a conformal structure is–it's an $\mathbb{R}^{>0}$'s worth of inner products, which in particular gives you the notion of angle but not the notion of length. But in particular, there is a $\star$ operator (called the Hodge star) on an oriented vector space with an inner product which, for the middle dimension if the dimension of the vector space is even, only depends on the conformal structure.

(1/23/2019) Today I learned how to recover the addition structure on the semigroup of dominant weights by just knowing the structure of the Grothendieck semiring of the category of representations. Specifically, you can recover the sum of two dominant weights λ, μ by finding the highest dominant weight ν such that the representation associated to ν is a subrep of the tensor of the reps associated to λ and μ respectively.

(1/24/2019) Today I learned a bunch of basic facts of Geometric Invariant Theory. Specifically, it's a theorem that for any geometrically reductive group (of which any reductive group is—another theorem) acting on some vector space, the ring of polynomial invariants on that vector space is finitely generated (and further, the ring of equivariant functions with respect to any other character too) and so in particular you can take Proj of the ring. Further, I learned the main theorem of geometric invariant theory, which says that there is a surjective map from the semi simple elements of your original vector space to the quotient. Further, I learned that the quotient of the action of GL_2 acting on the space of $k \times 2$ matrixes is the Grassmannian.

(1/25/2019) Today I learned how to show that for any given dominant coweight μ of a reductive algebraic group G, there are no nontrivial extensions of IC_μ by itself in the affine Grassmannian

Gr_G. Specifically, you can show using the distinguished triangle associated to the dense open associated to μ on which the constant sheaf is trivial that the group Ext^1 fits into a short exact sequence of extensions whose supports are in the open and closed. You can then show that there are no nontrivial extensions on the open by its simply connectedness, and use perverse degree arguments to show that there are no extensions supported on the closed either.

(1/28/2019) Today I learned that there is a notion of a sheaf of categories, and that you can consider the Beilinson-Bernstein theorem in a more general context by viewing the weakly T equivariant D-modules on G/N as a sheaf of categories over $\mathfrak{t}^*$, for which you can take a fiber at a coweight and obtain the category of the usual Beilinson-Bernstein D-modules.

(1/29/2019) Today I learned that there are two notions of stability and semistability which relate to the representations of quivers. Specifically, when given a certain class of allowable characters θ of the product of the GL_n's acting on your various maps, we obtain that the semistable points are precisely the maps for which $\theta(N) \geq 0$ for all subreps N, which does not use any GIT in its formulation.

(1/30/2019) Today I learned more in the representations of quivers. Specifically, I learned that fixing a quiver Q and a finite dimensional vector space for each vertex, the numerical criterion for (semi)stability says that, fixing a character $\theta \in \mathbb{Z}^{Q_0}$ (noting that all characters of GL_n are products of powers of the determinant), to have any semistable point M we necessarily must have $\theta(M) = 0$. Furthermore, fixing a θ I learned that the set of all representations of our quiver Q, say M, for which $\theta)M) = 0$ forms an abelian category.

(1/31/2019) Today I learned a universal property of the cotangent complex of a given simplicial ring R. Specifically, you can define the cotangent complex $L_{R/k}$ so that for any ring map $R \to S$ and any S module M, we have that as a topological space $Hom_{S-Mod}(L_{R/k}, M) \cong Hom_{/S}R, S \oplus M)$. Furthermore, this implies that $\pi_0(L_{R/k}) = \Omega^1_{\pi_0(R)/k}$ if (at least) k is a field.

December 2018

(12/1/2018) Today I learned about the concept of the *continuous dual* of a profinite or discrete vector space V. In particular, given a profinite vector space $V = lim_{\to} V_i$, the continuous dual is defined to be $lim_{\to} V_i^{\vee}$, and similarly the arrows flip viewing a discrete vector space as the colimit of its finite dimensional vector subspaces. With this definition, the continuous dual of $k((t))$ is itself, which provides the first evidence for local Rieman-Roch, which says for smooth curves with closed point x, residues provide an isomorphism $K_x = \Omega_{K_x}$.

(12/3/2018) Today I learned what the *trace class* of the endomorphisms of a Tate vector space is, and why it is called that. Namely, given a continuous endomorphism of a Tate vector space, you can talk about the bounded maps whose kernel contains a lattice of the Tate vector space, and you can choose two such lattices and take the endomorphism restricted to this subquotient and then take the associated trace. After some "localization" esque arguments, you can show the independence of choices boils down to the fact that if you added more to your lattice, it would contribute nothing to the trace because your transformation would only map into things in the old smaller lattice you chose.

(12/4/2018) Today I learned the statement of *Zariski's Main Theorem*, which says that given any birational proper map of locally Noetherian schemes such that the target is normal, then the map is O connected, that is, the map on the structure sheaves is an isomorphism. This theorem can be used to show that the associated map $\tilde{\mathfrak{g}} \to \mathfrak{g} \times_{\mathfrak{t}//W} \mathfrak{t}$ is an isomorphism by showing it is an isomorphism over the codimension two subset of regular elements of the Lie algebra.

(12/5/2018) Today I learned a universal property of the relative *Spec* of a quasicoherent sheaf of algebras over a scheme X, say $\mathcal{B}$. Namely, given a map $\xi : W \to X$, maps $W \to Spec_X(\mathcal{B})$ over X are in bijective correspondence with maps $\mathcal{B} \to \xi_* O_W$ of O_X algebras.

(12/6/2018) Today I learned an alternate definition of what it means for an integral (i.e. irreducible) scheme X to be normal. Specifically, a scheme X is normal if and only if all finite birational $Z \to X$ are isomorphisms. In the forward direction, this can be shown directly by showing the map is an isomorphsim via the integral closure, and the reverse direction can be shown through the normalization of X.

(12/7/2018) Today I learned what an adele is! Namely, if X is a smooth curve, an element of the adele associated to X is an element of $\Pi_{x \in X} K_x$ where all but finitely many elements are in O_x. I also learned that the rational functions viewed inside this adele are a closed, discrete subset, which you can do by taking any nonzero element and creating a coset for which any element in the coset will have all its valuations sum to something larger than 0, which can't happen for rational functions.

(12/9/2018) Today I learned the outline of the proof that for a smooth projective curve X and a fixed one form, the sum of the residues of the one form at each point is zero. This is because you can consider the sum of each of the residues as a globlal residue of the ring of adeles on the smooth curve X, which is in particular a Tate vector space. You can then show that the rational functions are a discrete subspace of the adeles $\mathbb{A}_X$ and furthermore show that $\mathbb{A}_X/K(X)$ is a profinite vector space. Thus because we have a splitting as vector spaces with a $K(X)$ action for which the residue is trivial on both components (as residue is trivial on profinite and discrete vector spaces) the residue of the adeles at the one form, which is the sum of the residues at the one form at each point, is zero.

(12/10/2018) Today I learned the basic idea of an outline for why there is an isomorphism from the functions on the Grothendieck resolution to the (pushforward of) functions on $\mathfrak{g} \otimes_{Z(\mathfrak{g})} \mathfrak{t}$. Namley, you can first prove this on H^0 and then argue that the homotopy kernel of the associated map is concentrated in degree 0, argue that the homotopy kernel of the map pushed forward is in fact a $\mathbb{G}_m$ equivariant coherent sheaf on $\mathfrak{g}$ for which the pullback at 0 is zero, then the actual sheaf itself is zero. The pullback is argued via reducing to the degree 0 case.

(12/11/2018) Today I learned that the $A := k[x_1, ..., x_n]$ module $k = A/(x_i)_i$ is not flat for any i. Intuitively, this is because inclusion of a point does not yield a continuously varying family of points at each fiber (because at the zero point there is a jump!) but furthermore you can show this explicitly using the A module map $A \to A$ given by sending $1 \to x_1$, which maps to the zero map under tensoring with k.

(12/12/2018) Today I learned a notion of morphism that is super similar to finite flat morphisms of schemes, called *finite locally free* morphisms. These are morphisms for which the pushforward sends the structure sheaf to a locally free sheaf. In fact, if the morphism is also locally of finite presentation the notion of finite flat and the notion of finite locally free are equivalent. This equivalence stems from the fact that any finite, flat A module is locally a free A module, which you can show using Nakayama's lemma.

(12/13/2018) Today I learned an alternative construction of the Harish-Chandra morphism. Namely, noting that $U(\mathfrak{g})$ has a commuting left and right action both by $U(\mathfrak{g})$, the algebra $U(\mathfrak{g})_{\mathfrak{n}}$ still has a right action by $U(\mathfrak{t})$–roughly speaking because $\mathfrak{t}$ and $\mathfrak{n}$ commute if you then kill off then $\mathfrak{n}$ part. This in particular gives a algebra map $Sym(\mathfrak{t}) \to End_{\mathfrak{g}}(U(\mathfrak{g})_{\mathfrak{n}})$, and another "Harish-Chandra theorem" says that this map is an equivalence. Using this fact, you can define the Harish-Chandra map to be the composite of the inverse of this map with $Z(\mathfrak{g}) \to End_{\mathfrak{g}}(U(\mathfrak{g})_{\mathfrak{n}})$.

(12/14/2018) Today I learned the basics of *Hodge theory*, which roughly says that for certain varieties defined over a field of characteristic zero, the n^{th} cohomology admits a direct sum decomposition $\oplus_{i+j=n} H^{i,j}$ where $H^{i,j}$ is the i^{th} cohomology of the j^{th} exterior power of the cotangent sheaf. In particular, I learned that for flag varieties, the cohomology is concentrated in $H^{i,i}$ (and in particular the odd degree cohomologies are zero).

(12/15/2018) Today I learned another naitve, but helpful, interpretation of $\mathfrak{g}//G$. Namely, you can imagine that $\mathfrak{g}$ being $Spec(Sym(\mathfrak{g}^\vee))$ and that you can distinguish points of $\mathfrak{g}$ by testing on all possible functions. But if you restrict to only the functions which are invariant under the conjugation action of the inputs, then you can't distinguish two things in the same orbit. This gives a plausibility check that $Sym(\mathfrak{g}^\vee)^G$ should be the ring of functions for some kind of quotient space.

(12/17/2018) Today I learned that there's a local ring A (namely, A is the local ring of $\mathbb{A}^3$ at the origin) and a vector bundle on an open subset of $Spec(A)$ (namely, removing the closed point of the origin) which does not extend to a vector bundle on $Spec(A)$. To show this, consider the quasicoherent sheaf on $Spec(A)$ given by the kernel of the map $(x, y, z) : A^3 \to A$. This is a vector bundle on $D(x), D(y)$, and $D(z)$ because you can explicitly write out its two free generators. However, this does not extend to any vector bundle on $Spec(A)$, since this would imply that k has projective dimension 2 instead of projective dimension 3 in A.

(12/18/2018) Today I learned an outline of the proof that the Beilinson-Bernstein global sections functor is conservative, which completes the proof that the Beilinson-Bernstein global sections functor yields an equivalence of categories. Specifically, given a D module, you can tensor it by a vector space associated to a highest weight module and show that there's an isomorphism with another twisting. But you can also show that this other twisting is *ample*, which in particular by a theorem of Serre implies that if you tensor with it with enough powers, it yields global sections.

(12/19/2018) Today I learned a more specific outline of the above proof. Namely, one can show that for any antidominant weight which does not pair to zero with any positive root, the associated Borel-Weyl-Bott line bundle is ample. Moreover, the global sectons of this line bundle has a canonical filtration which induces some nice ordering properties on the roots of the representation. This can be used to show that any twisting by the associated Borel-Weyl-Bott line bundle is actually generated by global sections by showing a certain map splits.

(12/20/2018) Today I learned an even more specific outline of the above proof. Namely, for any antidominant weight $\nu \in \mathfrak{t}^*$, I learned that the lowest weight representation $L^-(\nu)$ has an associated filtration for which the subquotients are certain weights which are ordered. Furthermore, you can translate this through the multiplication map to a filtration on the total space of the trivial bundle $G/B \times L^-(\nu)$, say, $U^1 \subseteq ... \subseteq U^r$, and then you can use the fact that these themselves are total spaces of certain vector bundles to get a filtration on $O_X \otimes_k L^-(\nu)$ which have subquotients of the line bundles of the same weights.

(12/21/2018) Today I learned theh specifics of the Borel-Weyl-Bott line bundle construction. Namely, given any weight $\mu \in \mathfrak{t}^\vee$, one can construct the space $G \times^B k_\mu$, which is the quotient of the product space $G \times k$ by the diagonal action of B, where B acts diagonally on k through the character μ and the quotient map $B \to B/N$, where N is the maximal unipotent subgroup. This $\times^B$ notation also allows us to represent the associated map on global sections, I think. Namely, for the lowest weight representation L_μ associated to μ, the map $G/B \times L_\mu \to G \times^B k_\mu$, where the second map is given by the coefficient on the lowest weight vector, allows us to pull back sections of total space.

(12/22/2018) Today I learned that any $U(\mathfrak{g})$-module V which is locally finite as a $Z((\mathfrak{g})$ module be decomposable as a direct sum of its generalized $Z(\mathfrak{g})$ generalized eigenspaces. This is because these direct sums definitely inject into V, and then to show surjectivity, you can note any $v \in V$ has a finite dimensional subspace $W = Z((\mathfrak{g})v \subseteq V$, and since $Z(\mathfrak{g})$ are all commuting matrices, and so they are simultaneously diagonalizable (well, Jordan Normal Form-able).

(12/23/2018) Today I learned the definition of the Hecke algebra and how they relate to Verma modules. Namely, given a Coxeter system (W, S) you can construct an algebra over the polynomial ring $\mathbb{Z}[t, t^{-1}]$ (where t is often written as $u^{\frac{1}{2}}$) which, as a vector space, is generated by elements of the form T_w for each $w \in W$, and a weird, weird multiplication structure. The reason it's used though is that there is an involution known as the *Kazhdan-Lusztig involution* which is a ring morphism mapping $t \to t^{-1}$ and $T_{w^{-1}}^{-1}$ (where by the weird multiplication, each T_w is a priori invertible). Then there are unique elements indexed by $w \in W$ which are invariant under this involution. These are known as the *Kazhdan-Lusztig polynomials*, and these give the multiplicity of simples in Vermas!

(12/24/2018) Today I learned more about why the proof of Kazhdan-Lusztig's theorem about their polynomials in a Hecke algebra and why this gives multiplicities in a Verma. Specifically, I learned that the Grothendieck group on category O is freely generated as an abelian group by the simple objects and that given an object in the category you can read off its composition series by knowing its direct sum of simples in the Grothendieck group. Furthermore, I learned that it's possible to use the *de Rham functor* to determine which coefficients of $D_{G/B} \otimes_{U(\mathfrak{g})} M_{ww_0\star-2\rho}$ the unique simple sub has, and use an inversion like formula to go the other way. Today I also learned that Harish-Chandra is one person, dash and all.

I also learned what *BGG duality* is today. Namely in the BGG category O, for each weight $\lambda \in \approx^\vee$ you can first show that there is a projective module P_λ which surjects onto the unique simple quotient of the $\lambda-$verma, L_λ, such that no submodule of P_λ surjects onto L_λ. This yields a "projective cover." You can then show this module admits a *standard filtration*, meaning a filtration for which each subquotient is a Verma, and show BGG duality, which in particular says that for any two weights $\lambda, \mu \in \approx^\vee$, the multiplicity of L_μ in the λ-Verma is the number of times that μ-Verma appears in a (fact: for any) standard filtration of P_λ.

(12/26/2018) Today I learned what a perverse sheaf is. Namely, it is a D_X-module F such that for all integers $j, dim(supp(H^j(F)) \leq -j$ and $dim(supp(H^j(\mathbb{D}F))) \leq -j$. This in particular implies that the cohomology is concentrated in negative degrees. Furthermore, the above notion gives a nontrivial t structure on the category of D modules, and the *Riemann-Hilbert correspondence* (i.e. tensoring over D_X with the canonical sheaf) provides an equivalence of this category and the category of bounded (complexes of) coherent D_X modules.

(12/27/2018) Today I learned that there is a notion of a G-equivariant vector bundle which yields the notion of a G equivariant sheaf. Namely, a G-equivariant vector bundle is a G-equivariant morphism $E \to B$ that is a vector bundle and for every $g \in G$, the map induced by g yields a linear isomorphism on fibers $V_x \to V_{gx}$ for all $x \in B$. Any G-equivariant vector bundle also has G acting on sections via $(gs)(x) \to gs(g^{-1}x)$.

(12/28/2018) Today I learned that any global section of a G equivariant sheaf admits a G action on global sections. Namely, on points, you can map the function f to the function $(gf)(x) := g(f(g^{-1}x))$ via G. With this, you can show that the set of sections are G equivariant and thus show a certain filtration of $O_{G/B} \otimes_k L^-\mu$ is actually G-equivariant.

(12/29/2018) Today I learned that the Schubert cells which partition the flag variety into various cells can be equivalently described in a G equivariant way–namely, the diagonal space $G/B \times G/B$ is partitioned by the orbits of each $w \in W$ under $\dot{G}(B/B, wB/B)$. This in turn has a certain convolution product and also has an analogue of the Demazur resolution, which I also learned about today, which is a resolution of singularities of Schubert varieties which may not be smooth. All of this in turn goes into showing that you can describe a multiple of the KL polynomial C_w in terms of the dimensions of the cohomology of stalks of the IC sheaf of the G-Schubert variety.

(12/30/2018) Today I learned more about what the decomposition theorem says and how it factors into the proof that the multiplicity of simples in Vermas are given by the coefficients of the

Kashdan-Lusztig polynomials. Namely, the decomposition theorem says that because the map from the Demazur resolution is proper, you can write the pushforward of the IC sheaf of the (smooth) domain as the direct sum of IC sheaves of the $G-$Schubert cells tensored with a finite dimensional graded vector space. Because the Verdier duality functor $\mathbb{D}$ commutes with proper maps, you can also show that this is self dual and thus these graded vector spaces are symmetric, which thus is used to argue that a certain polynomial is KL-involution invariant.

(12/31/2018) Today I learned more about why the given filtrations of certain sections of $G/B \times L^-(\nu)$ are $Z(\mathfrak{g})$ invariant, at least in part. Namely, the sheaf of sections of this vector bundle can be identified with $O_{G/B} \otimes_k L^-(\nu)$, which acquires a $\mathfrak{g}$ algebra structure via the product rule. However, you can use the fact that the center acts by the character zero to show that it suffices to show that the center preserves certain filtrations.

November 2018

(11/1/2018) Today I learned an extension of the Beilinson-Bernstein localization theorem and some generalizations that go into making the statement correct in the infinite dimensional case. Specifically, I learned about the critical extension of a Lie algebra $\mathfrak{g}((t))$ for a reductive G, which can be defined as the unique extension $\mathfrak{g}_c$ for which $0 \to \mathbb{C} \to \mathfrak{g}_c \to \mathfrak{g}((t)) \to 0$ which has a nontrivial center, and this is the extension of the "central character zero" part of the statement of the Beilinson-Bernstein localization theorem.

(11/4/2018) Today I learned that you can use the Barr-Beck theorem to show that quasicoherent sheaves on a scheme X satisfy fpqc descent, at least one categorically. The reason for this is that given a faithfully flat quasicompact $f : \mathcal{U} \to X$, you can use base change (which our flat f satisfies) to show that the descent data is equivalent to comodules over the comonad f^*f_* on $\mathcal{U}$. You can then check that all the conditions of Barr-Beck are satisfied (for example, f^* is conservative because the map is faithful, i.e. surjective) to show that the 1-category of quasicoherent sheaves on X is also equivalent to comodules over the comonad f^*f_* on $\mathcal{U}$.

(11/5/2018) Today I learned that you can recover every possible thing you could ever want to know about a smooth projective irreducible curve X by looking at its function field. In particular, I learned that you can recover the topology on the space by looking at all possible integrally closed domains whose fraction field is the entire function field of the curve, and that you can recover any maps of curves by looking at the maps of the function field by extending the valuative criterion for properness to the generic point.

(11/6/2018) Today I learned a heuristic for why the tangent space of BG for a reductive group G is $\mathfrak{g}[1]$. Namely, if $G = GL_n$, you can translate through the definition of the tangent bundle to see that, using the functor of points interpretation, the tangent space of BGL_n at its point is the set of principal $D := k[\epsilon]/(\epsilon^2)$ bundles which restrict to the trivial bundle on $Spec(k)$. This in particular means that we are given a D module M and an isomorphism $M \otimes_D k \cong k^n$, which is equivalent to the data of a class in $Ext^1_D(k^n, k^n)$, which you can compute explicitly is $\mathfrak{gl}_n$.

(11/7/2018) Today I learned that the valuative criteiron for properness says. Namely, I proved that given any map from an open subset of a smooth curve X (or its generic point!) to a projective scheme Y, then there exists a unique map from the curve to Y extending this map. This is roughly because if the associated open set is $SpecA_f$, then a map to projective space is a certain line bundle L and a choice of nonvanishing sections. Then you can consider the A module spanned by those sections and show that your choice gives a line bundle on $SpecA$ and the sections still don't vanish.

(11/8/2018) Today I actually learned something about quantum physics which was sort of interesting. Namely, electrons always have angular momentum, even if they are not "moving" in some sense. Furthermore, the position of a given electron cannot be determined without measuring the position, so before measuring, the position of that electron is given by a probability distribution. Then by measuring it, you project the electron into a particular eigenspace.

(11/9/2018) Today I learned that you only need the target to be separated to talk about the locus of points where two morphisms $f, g : S \to T$ agree. This is because you can base change the diagonal morphism with the morphism $(f, g) : S \to T \times T$, which yields the locus on which they agree, and this is a closed embedding because closed embeddings are closed under base change. I also learned the statement of the fact that if you have a morphism from a separated integral scheme that is locally an open embedding on the source, then it is a local embedding.

(11/10/2018) Today I learned that any algebra object in the category of algebras must necessarily be commutative. The observation is that A being an algebra object in the *category* of algebras says that our "new" multiplication map must be a morphism of algebras with respect to the old algebra map, so in particular the map given by new multiplication $A \times A \to A$ must be an algebra morphism, where $A \times A$ is an algebra by the "product algebra" structure. Running through this and the other diagram, one would see there is hope to force commutivity, and this is forced by the fact that using the diagrams you can show the new and old units agree.

(11/12/2018) Today I learned why the homotopy product of a point by a point over a pointed topological space X, x must be loops on that point. Specifically, I learned that if you want to take the homotopy fiber product, you can (co?)fibrantly replace the point with the contractible space of paths on X, and because the map ev_1 is a fibration, the homotopy product is the regular product, and thus the homotopy product is the space of paths for which the start and endpoint are the basepoint, i.e. are loops. I also cleared up a misconception. Namely, the sheaf of derivations of $k[t]$ is a free $k[t]$ module of rank 1–in particular, there is a derivation sending $t \to t^2$, say.

(11/13/2018) Today I learned something kind of cool. I finally learned why $(D_{G/B})_{1B} \cong M^0$ as $U(\mathfrak{g})$ modules, where M^0 is the 0^{th} Verma module. This is literally because you can argue that $T_1(G/B) \cong \mathfrak{g}/\mathfrak{b}$ and thus you can show that both $\mathfrak{n}$ and $\mathfrak{t}$ act as derivations by their image under the map $\mathfrak{g} \to \mathfrak{g}/\mathfrak{b}$ and you can furthermore show this map is filtered and is an injection on associated graded.

(11/14/2018) Today I learned a way to compute the global sections the cotangent bundleof a quotient space X/G for at least a smooth affine algebraic group G acting on a space X. Namely, there is a map called the *moment map* $\mu : T^*X \to \mathfrak{g}^*$ which comes from the dual of the map $\mathfrak{g} \to TX$ given by the infinitesimal action, and the global sections of the cotangent bundle of this is given by taking the (derived) preimage of zero and modding out by the action of G, i.e. $\Gamma(X, O_{T^*X}) \cong \mu^{-1}(0)/G$.

(11/15/2018) Today I learned/solidified an example about the map $U(\mathfrak{g}) \otimes_{Z(\mathfrak{g})} Sym(\mathfrak{t}) \to \Gamma(G/N, D_{G/N})$ using the $G = SL_2$ example. In particular, today I literally learned that G/N is quotenting G by the *right* action of N on it (which is something I had to go through a lot, because sometimes math is annoying!) and in particular G still acts on the right by it. Furthermore, since T normalizes N, we have that T still acts on G/N on the right.

(11/16/2018) Today I learned a way to show that every projective $k-$curve has finite dimensional k global sections, and furthermore the global sections of any coherent sheaf on a projective $k-$curve is finite dimensional as a $k-$vector space. Specifically, you can reduce to the smooth case, compute it directly for $\mathbb{P}^1$ and show that for a smooth projective curve X, the claim is equivalent to showing it for global sections of the structure sheaf (by using the fact that any line bundle on a smooth curve is $O(D)$ for a divisor D on the curve) and then using that to generalize to vector bundles (using a Jordan like decomposition theorem), and then show it for global sections by taking a finite map to $\mathbb{P}^1$.

(11/18/2018) Today I learned the idea of regular and regular semisimple elements of a Lie algebra $\mathfrak{g}$. In particular, a *regular* element of a Lie algebra is an element whose G centralizer is of minimal dimension, which turns out to be the dimension of G, i.e. the dimension of its maximal torus. This is because generically any element is fixed by precisely a Cartan (because generically all elements have distinct eigenvalues) and so the generic fiber has dimension the torus.

(11/19/2018) Today I learned the idea behind the concept of a *deformation*, which is defined to be the pullback of $Spec(k) \to Spec(k[\epsilon]/(\epsilon^2)) \leftarrow Y$. The idea here is to imagine the deformation X as lying in a moduli space which parametrizes flat families of something, and to consider the map $X \to Spec(k)$ as some k point of the scheme, whereas the $Y \to Spec(k[\epsilon]/(\epsilon^2))$ point yields a $k[\epsilon]/(\epsilon^2)$ point of the moduli space, i.e. tangent vector of the moduli space.

(11/20/2018) Today I learned that given a map of complexes $f : F \to G$ with homotopy cokernel π, there is a canonical map $F \to hker(\pi)$ which is a homotopy equivalence. This is because you can note that the identity map $hcoker(f) \to hcoker(f)$ yields a canonical nullhomotopy of the map πf, and so you can obtain a map $F \to ker(\pi)$ by using the fact that a map to $ker(\pi)$ is equivalent to the data of a map $F \to G$ (since the domain of π is G), for which you can use f, and a nullhomotopy of the composite πf, for which we can use literally the *same nullhomotopy* as before. Pretty cool!

(11/21/2018) Today I learned that there is such a thing as the *valuative criteiron for separatedness*, in addition to the valuative criterion for properness. This in particular says that a map $X \to Y$ is separated (resp. proper) if and only if for any discrete valuation ring R (where the "valuative" part comes from) whose fraction field is K, then any map $Spec(R) \to Y$ with a lift of $Spec(K) \to X$ can be lifted to a map $Spec(R) \to X$. I don't want to Tex up the diagram, but it's that the obvious square has at most one lift (resp. exactly one lift). Through this criterion I worked through why $\mathbb{P}^n$ is separated.

(11/22/2018) Today I learned about why cohomology is well defined! In particular, you first argue that it suffices to show for a refinement of a given open cover, and then you can show that the homotopy fiber product preserves quasi-isomorphism (in the same sense that if you have commuting isomorphisms of three objects in a category then their products, if they exists, are isomorphic) and use this to reduce to the affine case. You can then use the fact that you can argue a map is a quasi-isomorphism if and only if its homotopy cokernel is acyclic, and that you can check this on an affine open cover. You can then use the fact it's well defined when one of the sets in the open cover is the set itself directly.

(11/23/2018) Today I learned one new insight as to why every line bundle L of a smooth irreducible curve is $O(D)$ for some divisor D. Namely, choosing a nonzero section s of the line bundle, you can construct the line bundle $O(D) \to L$ which is called "multiplication by s", but is better called "multiplication on each affine open subsets by the identification of s", and show that this gives an isomorphism on each affine open and is thus an isomorphism.

(11/24/2018) Today I learned the finished outline of the proof that for smooth projective curves X, the cohomology of any coherent sheaf is finite dimensional. This is because we have a result that on smooth curves, being torsion free implies that you are a vector bundle, so in particular it suffices to show the theorem for torsion sheaves and for vector bundles. You can show it on the vector bundle part using induction on the rank and an equivalence relation on whether the theorem holds for a given line bundle, and you can show it on the torsion subsheaf by directly computing that any torsion coherent sheaf is a finite direct sum of sheaves of the form $O_X/O_x(-nx)$ for $n \in \mathbb{N}$.

(11/26/2018) Today I learned an interpretation of cohomology on a smooth projective curve X. Namely, given a vector bundle on a smooth projective curve X, say E, and a choice of closed point $x \in X$, you can use the $E(\infty x) := \lim_{\to,n} E(nx)$ construction and the fact that $R\Gamma E(\infty) = R\Gamma(X \setminus x, E)$, you can use the long exact sequence and the fact that $X \setminus x$ is affine to show that $H^1(X, E)$ is the obstruction for a formal Taylor series to be formally defined as a function on $X \setminus x$.

(11/27/2018) Today I solidified the fact that the Beilinson Bernstein localization theorem is false for right D modules. In particular, fo r$G = SL_2$, $G/B \cong \mathbb{P}^1$ and there you can take global sections of the dualizing sheaf, which just so happens to be the cotangent bundle in this case. You can compute explicitly or use Serre duality to see that this has nonzero cohomology in degree 1 (or more generally, degree $dim G/B$) and so this isn't exact. Moreover, δ_0 is a D module which has cohomology in degree 0, i.e. has global sections, so this can't be remedied with a shift.

(11/28/2018) Today I learned a corollary of the BB localization theorem. Namely, the D modules on $\mathbb{P}^1$ the same as the $\mathfrak{sl}_2$ modules for which the central character acts as zero. I explicitly realized the isomorphism $U(\mathfrak{sl}_2)_0 \to D_{\mathbb{P}^1}$ and showed this kernel is the center of the universal enveloping algebra with no degree zero part by showing explicitly that at every point $gB \in \mathbb{P}^1$, the localized $D_{\mathbb{P}^1}$ module $D_{\mathbb{P}^1}/\mathfrak{m}D_{\mathbb{P}^1}$ is, up to an adjoint action by g, the zeroth Verma and thus the center acts on it by zero.

(11/29/2018) Today I learned a rigorous formation of the halting problem, which in particular says that there is no Turing machine which takes as an input a Turing machine and an input state and (always) tells you whether the program will run forever or not. This is because you can explicilty make a new Turing machine which takes in a Turing machine and input, uses the old Turing machine to tell you whether it will halt or not, and if it halts, set the new machine to run forever, and if the old Turing machine says it will loop, then set the machine to end. Then you can feed the machine itself and find a contradiction!

(11/30/2018) Today I learned that given any smooth affine $k-$curve, say $X = Spec(A)$, and closed point x in X with residue field k' and associated maximal ideal $\mathfrak{m}$, the local ring $O_x := \lim_{\to,n} A/\mathfrak{m}^n$ is isomorphic to the formal power series $k'[[t]]$ for some uniformizer t. This is because you can show that any map $A/\mathfrak{m}^n \to A/\mathfrak{m} = k'$ must split uniquely as rings, which provides a map $k' \to O_x$, which provides a map $k'[[t]] \to O_x$ which you can show is an isomorphism modulo t^n and $\mathfrak{m}^n$ and thus conclude it's an isomorphism.

October 2018

(10/1/2018) Today I learned a few theorems about functors admitting adjoints that are helping me understand indcoh. For example, I learned that a functor $F : C \to D$ between categories closed under all small colimits is continuous if and only if it admits a right adjoint (this is known as the *adjoint functor theorem*) and you can show using this theorem that a functor between categories which is continuous has *continuous* right adjoint if and only if the functor F preserves compact objects. If one wishes to work in the context of functor which preserve colimits (i.e. are continuous) then one can't work with quasicoh, since pushforwards do not preserve compact objects.

(10/2/2018) Today I learned a few functorial constructions that you can make associated to the IndCoh construction. The easiest functor you can make given any map of (DG Noetherian) schemes $f : X \to Y$ is the *pushforward* map $IndCoh(X) \to IndCoh(Y)$. You can show that this is t exact with respect to the t structures.

(10/3/2018) Today I learned one interesting insight on the idea of Noetherian vs. finite type. The former is a property of the space/scheme/ring tiself, whereas finite type is more of a property of a morphism (even the trivial morphism to the point). I also learned that for eventually coconnective maps of Noetherian DG schemes there is also the notion of a pullback.

(10/4/2018) Today I learned that the canonical relative Spec map $Spec_X(A) \to X$ is an affine morphism of schemes for a quasicoherent sheaf of algebras A on X. After weeding through the definitions and such, some of the things I took as the "major points" were the fact that the relative Spec has $Spec(B)$ points that are pairs of $\pi \in X(B)$ and an section of the pullback $\pi^*(A)$, which in particular shows that given some $Spec(B) \to X$ and some map ξ with domain $Spec(R)$ going to the product, this map is determined uniquely by the part of the section map given by $1 \otimes_B \xi^*\pi^*(A)$, roughly due to the fact that the ring maps are determined on B which live in "both sides" of the tensor product.

(10/5/2018) Today I finally learned what a good use for homotopy limits are. Namely, given a scheme X which is covered by two open sets U, V, a long time ago I showed that $QCoh(X)$ was the "same thing" as a quasicoherent sheaf on U, a quasicoherent sheaf on V, and an identification of their restrictions. But of course, this is the precise definition of the limit of the categories (or the infinity categorical limit, anyway). In particular, $QCoh(X)$ is not the usual limit because that would imply "equality."

(10/6/2018) Today I learned why any two integral weights linked by the dot action of the Weyl group have the same central character. This is something that can be shown by induction on the length of the Weyl group element "connecting" the two words, and for a simple root, one can note that the choice of simplicity gives the "picture" and in particular one can show that after raising the appropriate f to an appropriate power you can realize one maximal weight vector in the other. By an algebro geometric density argument, one can also extend this to say that any two weights linked by the dot action have the same central character, and the Harish-Chandra theorem gives a converse to this statement.

(10/7/2018) Today I learned that the global sections of any projective $A-$ scheme X for a Noetherian ring A must be finitely generated. Furthermore, I learned that there is an easy proof of this using properties of cohomology groups (in positive degrees!) Namely, since affine morphisms induce isomorphisms on homologies, one can show the statement for $X = \mathbb{P}^n_A$ and then you can show that since any coherent sheaf $F \in Coh(X)$ can be written into a short exact sequence of line bundles you can look at the "end" of the cohomological long exact sequence to show that the n^{th} cohomology of any coherent sheaf is finitely generated. You can then just continue going down cohomology group after cohomology group until you get to the zeroth.

(10/8/2018) Today I learned (solidified?) the idea that any $U(\mathfrak{n})$ finite nonzero space in the BGG category $\mathfrak{O}$ has a highest weight vector, although you can't necessarily say that there's anything like a "highest" highest weight vector unless you have some sort of finite generation going for you. Namely, you can take any nonzero vector v and consider its $U(\mathfrak{n})$ subspace and use what Fulton and Harris call the "Fundamental Calculation" to show that if α is a positive root the associated vector e_α sends the λ eigenspace to the $\lambda + \alpha$ eigenspace. Since there's an ordering on these you can argue that there's a maximal one, specifically one killed by all $\mathfrak{n}$.

(10/9/2018) Today I learned that in the category O one has the notion of a *Dual module*, which is taken by taking the usual dual Verma module and requiring the Lie algebra to act on it by the transpose/Cartan involution instead of the usual negation. Doing this and using something called *formal characters*, you can show that the dual of a Verma module has a unique sub which is isomorphic to the unique quotient of the Verma in question. Further, you can show that this (up to scaling) is the only nonzero map from a Verma module to its dual.

(10/10/2018) Today I learned the method used to make IndCoh with the pushforward and the ! pullback into one general framework. The framework that is used here is the *category of correspondences* of certain sorts of DG schemes. This category in particular consists of the usual objects, but now the morphisms $X \to Y$ are diagrams $Z \to X$ and $Z \to Y$ with composition being cartesian product. Through this, one can construct an $IndCoh$ functor from the category of correspondences to $DGCat$ andn then right Kan extend to make it a functor on a general prestack!

(10/11/2018) Today I learned another motivation for derived algebraic geometry! In particular, one knows that any two distinct lines in $\mathbb{P}^2$ (over some base field k) intersect at precisely one point. However, one benefit of derived algebraic geometry is it offers the framework on which you can talk about the same line intersecting itself at one point. Namely, the framework is that if you take, say, $V(y) \subseteq \mathbb{P}^2$ twice, you can realize the relationship in the product "$y = 0$" as happening twice by drawing two separate one cells into the picture.

(10/12/2018) Today I learned another statement that can be viewed as the "computational" version of Serre duality, which namely says that for a smooth $f : X \to Y$, one can determine $f^!$ of the structure sheaf (as a functor on IndCoh(Y)) and explicitly write this in terms of Serre duality. One can combine this with the statement that for smooth proper maps, $f^!$ is a right adjoint for f_* and thus use this to prove the "classical" statement of Serre duality.

(10/13/2018) Today I learned a neat way to prove that being locally free is the same as being projective on any quasicoherent open set. Specifically, to show that "projective and finitely generated implies locally free" for an A module M, note that M can be viewed as a quasicoherent sheaf on $Spec(A)$. In particular, for each maximal ideal $\mathfrak{m} \in Spec(A)$, localizing away from $\mathfrak{m}$ we see that (since being projective is preserved under localization) $M_{\mathfrak{m}}$ pulls back to a projective $A/\mathfrak{m}$ module, but this is a field, so we can choose generators and use Nakayama's lemma to find open sets where the cokernel and kernel of our generating set is zero.

(10/15/2018) Today I learned that, for a fixed $\lambda \in \mathfrak{t}^*$, for any two $w_i \in W, dim(Hom(M^{w_1}, \mathbb{D}M^{w_2})) \leq 1$. This is because you can show that if there is any such map, it must map the w_1 highest weight vector to a w_1 highest weight vector, and furthermore, because duals use the "transpose" map action, must kill all lower terms. In particular you can use this argument to show that any morphism $M^w \to \mathbb{D}M^w$ must factor through the unique simple quotient of M^w which you can thus show must be the unique simple sub of $\mathbb{D}M^w$.

(10/16/2018) Today I learned a slightly different definition of a Noetherian ring. Namely, a ring A is Noetherian if and only if for any increasing sequence of injections of ideals of A, say $I_0 \hookrightarrow I_1 \hookrightarrow I_2 \hookrightarrow ...$ which correspond to a series of inclusions of the ideals in A (i.e. they are isomorphic to their images and the maps correspond to inclusions) then this chain eventually stabilizes, i.e. these maps eventually correspond to surjections. On the other hand, I learned that the slightly stronger converse is false. Namely, there is a Noetherian ring A for which there is an infinite strictly increasing (i.e. never surjective) sequence of maps!

(10/17/2018) Today I learned the outline of the proof of Weyl's complete reducibility theorem using the Casimir element. Namely, you can use the Casimir element to show that, given a semisimple Lie algebra $\mathfrak{g}$ and a codimension one irreducible $U(\mathfrak{g})$ subspace W of a $U(\mathfrak{g})$ space V, you can use the Casimir element and the fact its trace is nonzero to show that the kernel of "multiplication of the Casimir" can't be inside W and therefore it is a completementary subspace. From there for an arbitrary $W \subseteq V$, one can put a $U(\mathfrak{g})$ module structure on $\mathcal{V} :=$ the subspace of $Hom_k(V, W)$ which when restricted to W act as a scalar. You can show this splits and then choosing some nice element $f \in \mathcal{V}$ that your splitting map is given by $id - f$.

(10/18/2018) Today I learned a potential proof that $U(\mathfrak{t})$ acts generalized semisimply on any given finitely generated $M \in O(\lambda)$. This is because by hypothesis, $U(\ltimes)v$ is finite dimensional, and I'm 99 percent sure that you can show that it follows that $U(\ltimes)U(\approx)v$ is finite dimensional. This is preserved by $U(\approx)$ and you can use the fact that "commuting matrices are simultaneously trigonalizable" to show that you can write v in a basis of generalized eigenvectors.

(10/19/2018) Today I learned an important example illustrating the point that smoothness is a relative notion. Namely if k is a field that has characteristic p and some $\alpha \in k$ which does not have a p^{th} root, such as in the field $\mathbb{F}_p(\alpha)$, then if k' is the field obtained by adjoining a p^{th} root, then $Spec(k')$ is not smooth over $Spec(k)$.

(10/20/2018) Today I learned that in the category O'_0, the module M^1 (i.e. the Verma module for which everything in the center acts trivially) is projective, at least in the heart of the above category. This is because in the heart of the abelian category, you can check projectivity by simply showing that given any map $M^1 \to N_2$ and any surjection $N_1 \to N_2$, that there is a lift of that map to M^1, and you can check this by showing that M^1 must have a highest weight vector which $U(\mathfrak{g})$ hits some choice of lift of the vector $v^1 \in M^1$ hit, and then show that this highest vector is acted on by a central character (as opposed to a generalized central character) and by assumption on the category and the Harish-Chandra theorem, this weight space must be in the $W, \bullet$ orbit of 1, but the only way that is possible by ordering of W is if the weight space itself is one–i.e. the highest weight vector has weight 1.

(10/21/2018) Today I learned the broad outline of the proof that the BB localization functor is t exact. Namely, you can factor the functor as the composite of the pull back of the projection, which maps to something that is B equivariant, which is free because the quotient is free, and then the usual (exact) global sections functor, exact since G is affine, and finally the B fixed points, which you can show is exact by showing that the B fixed points are also the functor $Hom_{O_0'}(M^1, -)$ which uses the fact that the category O is the sum of the generalized central character eigenspaces and the fact that M^1 is projective above.

(10/22/2018) Today I learned that for a local k-algebra A with maximal ideal $\mathfrak{m}$ such that $A/\mathfrak{m} \cong k$, there is a bijective correspondence between the set of derivations of $A \to k$ (viewing k as an A module via the quotient map) and the set of k linear maps $\mathfrak{m}/\mathfrak{m}^2 \to k$. This correspondence is taken by taking a derivation and restricting it to the maximal ideal, and in the other direction is given by taking a k linear map ϕ and defining the derivation $a \to \phi(a - \pi(a))$ where $\pi : A \to \mathfrak{m}$ is the quotient map. The proof that this actually gives a derivation uses the fact that the domain uses $\mathfrak{m}^2$ cosets.

(10/23/2018) Today I learned a quicker definition of a 1-Artin stack. Namely, a $1-$Artin stack is a stack $\mathcal{Y}$ such that the diagonal morphism is *0-representable*, i.e. for any map from a disjoint union of affine schemes into $\mathcal{Y}$ the base change of the map is a map of affine schemes and furthermore there is a map $\mathcal{U} \to \mathcal{Y}$ from a disjoint union of affine schemes which is a smooth surjection.

(10/25/2018) Today I learned the idea behind the proof that the classifying space is a quotient stack. Namely, you can take something called the *Hilbert Stack* $\mathcal{H}_d(*/BG)$, which seems to be a generalization of the Hilbert scheme, and show that that's a scheme by general nonsense/the fact that $* \to BG$ is a map of stacks to show that the Hilbert Stack actually is a stack, and then show that the coproduct of the canonical maps also form a stack, and that the associated composite map from the covering to BG is a smooth surjection. This reduces the problem to showing that the morphism is 0-representable and fppf, i.e. locally of finite presentation and flat.

(10/26/2018) Today I learned there is a functor between two triangulated categories (in fact, the homotopy category of an abelian one) which is additive but does not send distinguished triangles to distinguished triangles. Namely, if you take F to be the functor which is the identity on complexes whose negative entries are zero and F to be zero otherwise, you can show that it takes the cone of the "multiplication by two" map to zero, which does not have the appropriate cohomology to be the cone of the image of the map, the multiplication by two map.

(10/28/2018) Today I learned that there is a module in the category O_λ for some λ which is not genereated by highest weight vectors. Specifically, if $\lambda = 0$ and we take $\mathbb{D}M^1$, then due to the Harish-Chandra theorem, we can argue that if any Verma module maps to $\mathbb{D}M^1$, it must be in the $W, \bullet$ orbit of 0, and since 0 is a dominant and integral weight, the map itself must be from an M^1. But any such map factors through the trivial rep, so this is not generated by highest weight vectors.

(10/29/2018) Today I learned some ideas used in the proof of the Beilinson-Bernstein theorem. Namely, you can show that if $\mathcal{A}$ is a *Grothendieck Abelian Category*, i.e. an abelian category with certain set theoretic conditions holding, such that $\mathcal{A}$ admits a "forgetful functor" $F : \mathcal{A} \to Vect$, where $Vect$ denotes the abelian category of vector spaces, meaning a functor that is exact, commutes with all colimits, and is conservative, then in particular it has a left adjoint G and you can show that "over the category of vector spaces" then if $A := FG(k)$ where k is the ground field, then $\mathcal{A} \cong A - Mod$.

(10/30/2018) Today I learned more into why BG is a stack. Namely, you can show that any principal G bundle over an affine scheme is affine, so you can show that the map $* \to BG$ is an affine map, and it's smooth because we've been assuming that G is a smooth group so in particular any principal G bundle is locally just projection onto the factor that isn't G. Furthermore, it's a stack itself because given any etale covering one can recover a principal G bundle on the whole space from that covering data.

(10/31/2018) Today I learned the *valuative criterion of properness*, which says that given any map from a dense open subset of a smooth curve C to a proper scheme P, say $U \to P$, this map can be uniquely extended to a map $C \to P$. For the case of a projective P, this can be proven using a uniformizer argument at every point and the fact that a map to a projective variety can be viewed as a line bundle and choice of nonvanishing sections.

September 2018

(9/1/2018) Today I learned a way to identify finite groups "into" algebraic groups, in a sense. Specifically, I learned that over an algebraically closed field k of characteristic zero, there is an equivalence of categories between finite groups and finite étale algebraic groups over k. This equivalence is given by realizing each finite group as subgroup of S_n, which in turn can be be viewed as a subgroup of GL_n via permutation matrices. The other direction of the equivalence is obtained by noting that one characterization of an étale morphism is given by the fiber at each point y being a disjoint union of points whose residue fields are finite, separable extensions of y. In particular, for $y = Spec(k) = Spec(\overline{k})$, we obtain that all points are simply $k-$points, and the finite hypothesis implies that the fibers are finite.

(9/2/2018) Today I learned a neat little fact about which simplicial sets K are equivalent to the nerve of some category. Specifically, a simplicial set is equivalent to the nerve of some category if and only if for each $n > 1$ and for each $i \in \{1, ..., n-1\}$, any map from the i^{th} horn of the n simplex to K can be extended in a unique way to a map from the entire n simplex Δ^n. Essentially what this is saying is that if you're given the data of maps $X_0 \to ... \to X_n$ except for the entire composition and some map "in the middle" (i.e. $X_0 \to X_1$ and $X_{n-1} \to X_n$ are required), you can recover in a unique way what the composite of all of them has to be.

(9/4/2018) Today I learned that you can tell whether the (co)unit of an adjunction is an isomorphism by equivalently checking whether the right (left) functor of the adjunction is fully faithful. You can use this in the $D-$ module setting to show that for an open embedding $j^! j_{*,dR}$ without the base change theorem, which allows you to more easily define the map $j_!(F) \to j_{*,dR}(F)$ and show that $j^!$ of it, is the identity, and by exactness, $j^!$ of the intermediate extension $j_{!*}(F)$ is F itself.

(9/5/2018) Today I learned that the pullback of the intermediate extension of a simple D_U module F is again simple where $j : U \to X$ embeds a smooth scheme into a not necessarily smooth one. This is again because any subobject of $j_{!*}(F)$ can be pulled back via $j^!$, which is exact, and thus any subobject must either pull back to 0 or F. In the first case, you can show that because $j_{!*}(F)$ is a subobject of $j_{*,dR}(F)$, if $i : Z = U^c \to X$ is the associated closed embedding then $i^!$ of our subobject is zero too, and thus our object is supported nowhere and thus is zero. In the second case you can apply the same/dual arguments to the quotient.

(9/6/2018) Today I learned about a construction called *Witt vectors*, which to an $\mathbb{F}_p$ algebra A

associates to it the set of formal power series $W(A) := \{\sum_{n=1}^{\infty}[a_n]p^i : a_n \in A\}$, where $[a_n]$ is the associated *Teichmüller lift*. It turns out that there is a unique algebra structure on this satisfying certain "natural" things you'd expect something like this to have, and that $W(\mathbb{F}_p) = \mathbb{Z}_p$.

(9/7/2018) Today I learned a few different ways to describe an open cover of a general "space" and why they are equivalent. Namely, you can describe an open embedding as the "compliment of a closed embedding," where a compliment is simply the maps which fiber product with the inclusion to the empty set. Then you can describe a collection of open maps as a covering if for all points (or equivalently all field valued points), the product with some open embedding is nonempty, or you can for affine schemes realize the closed subsets as cut out by some (choice!) of ideals, and show that an open cover is one where the sum of those ideals sum to the "unit ideal," i.e. the entire ring.

(9/8/2018) Today I learned a pretty cool construction which shows you that the functor $\mathbb{A}_1$ is a Zariski sheaf. Essentially, you can show that localizaiton is exact by realizing localizing an endomorphism of some B module M as the functor $-\otimes_B [t]B[t^{\pm 1}]$. This shows you that localization is right exact, but it's also exact because it's localization by the filtered colimit of flat modules (because $B[t^{\pm 1}] = \lim_{\rightarrow} B[t] \rightarrow B[t] \rightarrow ...$, where each map is given by multiplication by t. Because the filtered colimit of a collection of flat modules are flat, we see that the injectivity or surjectivity of a map $M \rightarrow N$ of A modules can be checked on some distinguished affine open cover $D(f_i)$ (well, at least one direction of the checking is covered this way–the other is by the fact that the ideal generated by the f_i is actually A!)

(9/10/2018) Today I learned the construction of an infinity categorical limits and colimits over some diagram. In particular, there is some notion of an over and under category in an infinity category which is constructed via a universal property, analogous to classical category theory, which says that $Hom(D, C_{/p}) = Hom(D \star [0], C)$ where $\star$ is the categorical notion of the join of two categories. This can't be done through the naive definition, though, because morphisms of the traditional over/under category involve the equality of the commutivity of a certain diagram. Through this construction (the same universal property in infinity category theory as it is in regular category theory) one can obtain the notion of limits and colimits as the final and initial (respectively) objects in the over category of the diagram.

(9/11/2018) Today I learned what I am 99 percent sure is the proof that any holonomic D_X module is locally a vector bundle. The reason for this is that you can use a variant of Bernstein's inequality to say that the dimension of any irreducible component of the singular support is no less than the dimension of X, and then you can use a theorem which says that for irreducible varieties the map from the singular support to X is "almost everywhere" given by fibers of dimension zero. You can then argue if your fibers had any point but the zero vector, you would have "one dimension" worth of information... I think.

(9/12/2018) Today I learned some basics of derived algebraic geometry. Namely, in derived algebraic geometry, we need to take certain kinds of limits, and using only functors to sets its not known how to define G equivariant sheaves for an algebraic group G acting on a scheme X. However, in the derived category sense, it is very easy to compute–it's just a certain limit. I also learned "triangulated categories do not glue well."

(9/13/2018) Today I learned some kind of overview of DAG again. Basically, I learned that any derived scheme X has a canonical map from its "classical" scheme, which locally can be computed by its degree 0 homology, and this embeds algebraic geometry into the theory of derived algebraic geometry. I also learned that if X is a smooth scheme that this map is an isomorphism, so the derived complex essentially detects new sorts of "cohomological nilpotents" that can occur on a scheme. Finally, I learned that the obstruction preventing a morphism of derived schemes being an isomorphism on classical schemes is how it acts on the Kahler differentials ${}^L\Omega^1$.

(9/14/2018) Today I learned more facts behind the notion of infinity categories adding cohomological nilpotents but not changing the quasicoherent sheaf. This is because the modules on a field k are not the modules on the ring $k[\epsilon]/(\epsilon)^2$, but however, if ϵ is assigned to have degree -1 and we have a k vector space V with $\epsilon : V \to V[-1]$ acting on it, we see that we have a map between something concentrated in degrees ≤ 0 with something in degrees > 0, so in particular this map is canonically nullhomotopic. Today I also learned what a *Verma module* is, which is essentially the infinite dimensional generalization of what's going on when analyzing finite dimensional representations of Lie algebras.

(9/15/2018) Today I learned that the (de Rham) pullback of the exponential module on the line Ψ via the addition map $a : \mathbb{A}^2 \to \mathbb{A}^1$ is also given by $\Psi \boxtimes \Psi$. This is because any D_X module whose underlying O_X module is O_X is given by a connection, and any connection can be written as $d - \omega$ for some one form ω (which in this case on $\mathbb{A}^1 = Spec(k[t])$ was dt), and the pullback is given by pulling back the one forms. You can show that pulling back the one form via a gives you the same action on $\mathbb{A}^2$ as the action given by the exterior tensor product Ψ, and so in particular the modules are isomorphic.

(9/17/2018) Today I learned about left fibrations. Namely I learned that using a process called *straigtening*, you can show that any left fibration to a category gives you a left fibration, and furthermore I learned things like the Hom functor (which I haven't really defined yet) actually gives a left fibration.

(9/18/2018) Today I learned what a topological quantum field theory (TQFT) is. Namely, it is a symmetric monoidal functor from the category of manifolds with maps given by cobordisms between them to the category of complex vector bundles. Specifically, as any closed oriented manifold can be regarded as a cobordism from the empty set to the empty set, you can view a TQFT as a map which assigns to any closed oriented manifold of the proper dimension a complex linear map $\mathbb{C} \to \mathbb{C}$, i.e. a complex number. One can ask whether any two manifolds are distinguished by TQFT subject to two "physical conditions" observed by the TQFTs that come up in physics, and the answer is yes if the dimension is 1, 2, or 3, no in 4 or higher, but for 5 simply connected manfiolds are distinguished.

(9/19/2018) Today I learned that another way to characterize locally free G equivariant sheaves is by realizing them as bundles with the relative **Spec** construction, and then noting that it is equipping the bundle with a G action such that the bundle projection map is a group morphism. With this, and using the fact that the tangent space functor in algebraic geometry can be realized as $Hom(\mathbb{D}, -)$, you can easily show that TG is a group for any algebraic group G, TX obtains a TG, and thus a G action by realizing it as the zero section.

(9/20/2018) Today I learned that the map obtained by adjointing a Cartesian diagram and using the isomorphism of pullbacks is not always an isomorphism. This can be shown with the example of $X = Spec(\coprod_{n \in \mathbb{N} Spec(k)}$ and $Y = \mathbb{A}^1$, which dissolves to the usual proof that the tensor product does not commute with arbitrary limits.

(9/21/2018) Today I learned what all the different possible types of maps of Verma modules for $\mathfrak{sl}_2$ are. Namely, you can first show that for any possible weight λ, which can be identified with an integer, that the Verma module associated to λ, say M_λ, is irreducible if and only if λ is not a nonnegative integer. Furthermore, you can show that if λ is a nonnegative integer, the unique maximal subobject is the irreducible Verma module $M_{-\lambda-2}$, so that the quotient is the unique irrep of $\mathfrak{sl}_2$ of highest weight λ. This irreducibility classifies all the possible maps, since nonzero maps from M_n must be injections when restricted to M_{-n-2}.

(9/22/2018) Today I learned that there is an "obvious" filtration $Z(\mathfrak{g})$ for a semisimple Lie algebra $\mathfrak{g}$ and this filtration, combined with the PBW theorem, gives an isomorphism of vector spaces that says $Z(\mathfrak{g}) \cong Sym(\mathfrak{g}^*)^G$. There is then a result known as *Chevalley's restriction theorem* which says that the restriction $Sym(\mathfrak{g}^*)^G \to Sym(\mathfrak{t}^*)^W$ is an isomorphism. I also learned that there is a map known as the *central character* which maps $Z(\mathfrak{g}) \to Z(\mathfrak{t}) = U(\mathfrak{t})$ and that this map can be "twisted" to be an isomorphism of algebras.

(9/23/2018) Today I learned a bunch of representation theory things more in depth into the Harish-Chandra morphism, but I also learned the statement of Arrow's voting theorem, which is very scary. The basic idea of it is that you can view any voting system as a function which takes in people's individual preferences and outputs a societal preference list. Now you might hope that a voting system satisfies two things. For one, you would hope that if everyone puts the exact same list in, then the same output would come out. Another is that if the same people choose someone (say x) over someone else (say y), the outcome shouldn't depend on where other people are in the list (this is called *independence of irrelevant alternatives*). Anyway, Arrow's Voting Theorem says the only voting system with more than two alternatives for which these two properties are satisfied is a dictatorship.

(9/25/2018) Today I learned the difference between perfect complexes and (complexes of) coherent sheaves, which was illustrated by means of a proposition. This proposition said that if R is a local ring whose maximal ideal contains only zero divisors, then any R module is either projective or, if not, any projective resolution of the module must be of infinite length. You can show this by using some categorical arguments to show that if your module has a finite projective resolution, you can show there is an R module which has a projective resolution of length 1. You can then argue, in fact, that this kernel had to be trivial in the first place.

(9/26/2018) Today I learned what it means for a map from a line bundle to a vector bundle to be nonvanishing and a few equivalent conditions on them. The most obvious one in my opinion is that for all "points" $Spec(B) \to$ your scheme, the pullback of the map doesn't vanish. Another condition is that the associated dual map in the other direction is an epimorphism.

(9/27/2018) Today I learned the idea of a *k Artin stack*, which is essentially the idea that inductively a zero Artin stack is given by a scheme and then inductively a k Artin stack is given by the quotient of a $k-1$ Artin stack by a group object.

(9/28/2018) Today I learned the idea of *quantization*, which in the setting of an algebra is using the grading to take a certain associated graded or something like that, which you can do via the *Rees construction*. I also learned important ideas about the theory of IndCoherent sheaves. Namely, IndCoherent sheaves can be viewed as an object which quotients to quasicoherent sheaves, and is a nice framework for which Serre duality holds.

(9/30/2018) Today I learned more precisely what Ind Coherent sheaves need to be quotiented by to get the category of quasicoherent sheaves, and that the functor is an equivalence only for eventually coconnective schemes. Namely, there are examples of indcoherent sheaves which have cohomological degree $-\infty$, i.e. for all n their cohomology is concentrated in degree $\leq n$, and there are nontrivial examples of ind coherent sheaves which satisfy this. I also learned a few formal constructions you can do with indcoherent sheaves, such as taking their tensor product (writing the quasicoherent sheaf as the colimit of perfect ones).

August 2018

(8/1/2018) Today I learned that for any affine group scheme, the associated map of rings of a representation must send degree one polynomials to degree one polynomials. This is because any map is determined by its coaction map, and conversely, the coaction map determines the representation. In particular, because one can construct an isomorphism of functors one can run through this isomorphism applied on the ring $Sym(V^*) \otimes A$ (where our group is $Spec(A)$ to verify this.

(8/2/2018) Today I learned what a formal completion of a scheme along a closed subscheme is. Namely, it is the functor given on each $Spec(B)$ by all of the maps $Spec(B)$ to your scheme such that the induced map $Spec(B/N(B))$ factors through the closed subscheme. I learned that for any compact scheme that the functor restricted to Noetherian rings is an indscheme (namely, it's the colimit of what can roughly be called "the closed subscheme with m^{th} order derivative information for each m), and furthermore you can use this to classify representations of the formal completion of the group $\mathbb{G}_a$ at the identity.

(8/3/2018) Today I learned how to define the Lie bracket on the Lie algebra of a Lie group, or at least the first sketch of it. Namely, given any vector in the Lie algebra (the tangent space at the identity, which I also learned can be considered as "dual numbers" $k[\epsilon]/(\epsilon^2)$valued points of the k scheme) you can use the group action to get a vector at each point. I'm not sure how precisely yet, but analogously to the real/complex case you can use this to construct a vector field and then take the standard Lie bracket of the vector field (alternatively, viewing vector fields as derivations you can take the commutator of two derivations and get a derivation) and then project down to the vector at the identity.

(8/4/2018) Today I learned that, given two vector spaces M, N on which a Lie algebra $\mathfrak{g}$ acts trivially, if $\mathfrak{g}$ is semisimple then there are no nontrivial extensions of M and N, because $\mathfrak{g}$ is it's own commutator so you can show $\mathfrak{g}$ must always act via multiplication by zero. On the other hand, if $\mathfrak{g} = k$, I learned that there are $Hom(M, N)$ many extensions, the idea being that "multiplication by 1" is well defined up to choice of lift of an element of M and furthermore this determines the extension, essentially because any splitting of vector spaces turns out to be an extension of $\mathfrak{g}$-modules.

(8/5/2018) Today I learned that any affine algebraic group G can be realized as a closed subgroup of GL_n for some n. A sketch of the argument goes as follows–pick a finite generating set for the algebraic group and note that, as a vector space, by the way indschemes work there is a G-equivariant, finite dimensional subspace containing all of the generators. Thus you can use this to argue that any element in G acts by multiplication by putting the matrix coeffcients in. I also learned that $H^i(\mathbb{P}^n, O_{\mathbb{P}_n} = 0$ if $i > 0$, which contrasts with the algebraic topology case.

(8/6/2018) Today I learned that given any reductive group G, choosing some maximal torus (analogous to diagonal matrices in GL_n), there exists some pairing of roots and coroots making a *root system*, which in particular classifies all representations of the reductive group.

(8/7/2018) Today I learned what the *Langlands dual* of a reductive group is. Specifically, each reductive group has a set of data called the root data inside $Hom(T, \mathbb{G}_m)$ where T is its maximal torus, and some canonical way to embed these roots inside the dual lattice $Hom(\mathbb{G}_m, T)$, and this classifies the group. The Langlands dual group is the group for which if you switched the lattice with its dual would hit that root data. I learned also that the Langlands dual of GL_n is GL_n.

(8/11/2018) Today I learned what a G-equivariant quasicoherent sheaf is on a scheme X on which G acts. Essentially, the definition is a quasicoherent sheaf F on X and the data of an isomorphism to the pullback of F via the action map $G \times X \to X$ and to the pullback of F via the projection map $G \times X \to X$. Essentially if you view F as some kind of functions, you can say the definition is saying that you are requiring $f(gx) \cong f(x)$, but in a way such that the cocycle condition holds for $f((g_1g_2)x)) = f(g_1(g_2(x))$ and sends the identity to the identity.

(8/12/2018) Today I learned the fact that there is an isomorphism $\Gamma(T_G)^G \otimes O_G \to T_G$ for an algebraic group G. Specifically, because G-equivariant sheaves on G correspond to vector spaces via the fixed point functor, this says that T_G itself is a G-equivariant sheaf.

(8/14/2018) Today I learned the full, definitive proof (and the correct statement!) of the fact that if Y is cut out by a finitely generated sheaf of ideals of a quasicompact, quasiseparated scheme X, then the formal completion $X_Y^\vee$ is an indscheme. Essentially you do what you would think, but my main problem was that given an affine open $Spec(A)$ of X and a B valued point $\pi : Spec(B) \to X$, it wasn't clear why $pi^{-1}(Spec(A))$ had to be a finite union of affine schemes. However, the quasiseparatedness of X gives that the inclusion of $Spec(A)$ is quasicompact, and thus by base change nonsense the inverse image must be!

(8/15/2018) Today I learned about the *Borel-Weil-Bott* construction, which, given some dominant weight of a reductive algebraic group G, gives the representation of G as the global section of the flag variety G/B. Specifically, doing this construction for bundles on $SL_2(\mathbb{C})$ yields that the irreducible representations of $SL_2(\mathbb{C})$ are in one to one correspondence with the global sections of the line bundle $SL_2(\mathbb{C})/B \cong \mathbb{P}^1$ given by the induced representation on G/B that the pulled back weight $B \to T \to \mathbb{G}_m$ gives.

(8/16/2018) Today I learned what a quasicoherent sheaf is. I didn't think this late in my graduate school career I would have to say that, but, well, here we are. Anyway, a quasicoherent sheaf on a space X is the limit $lim(A - Mod)$, where the limit is taken over maps $Spec(A) \to X$ and each map of rings $B \to A$ over X yields a pullback map of the A modules. In this way, you can view quasicoherent sheaves on X as a map $X(A) \to AMod$, and using this definition, given a map $f : X \to Y$ it is clear that from this assigmnent the pullback takes $Y(A) \to AMod$ to $X(A) \to Y(A) \to AMod$.

(8/17/2018) Today I learned the difference between requiring an etale cover of a principal G bundle vs. requiring a Zariski open cover. In particular, the map of rings $\mathbb{C}[x^{\pm 1}] \hookrightarrow \mathbb{C}[x^{\pm 1}, y]/(y^2 - x)$ yields the analogue of a double sheeted cover, and in particular the group $\mathbb{Z}/2\mathbb{Z}$ acts on it. On the other hand, you can't take a Zariski open neighborhood of $\mathbb{A}^1 - \{0\}$ and hope that this map is locally trivial because if it were, this would say that the irreducible space $Spec(\mathbb{C}[x^{\pm 1}, y]/(y^2 - x))$ is the disjoint union of two empty infinite subsets.

(8/18/2018) Today I learned that GL_n-torsors on a scheme X in the Zariski topology are equivalent to locally free sheaves on X of rank n, or at least the broad outline of it. In particular given a locally free sheaf, you can construct its *space of trivializations*, which on A valued points is simply the maps $Spec(A) \to X$ along with an identification of your vector bundle pulled back by the map. Additionally, given a GL_n-torsors on a scheme X, one can construct a vector bundle by taking the open cover given in the definition of a GL_n torsor, defining the vector bundle to be locally trivial on the same open cover of X, and defining transition functions on $V = Spec(R) \subseteq$ of their intersection by seeing where the R valued point "identity, identity matrix" goes in the composite of the isomorphisms $V \times GL_n \cong U \times_X P \cong V \times GL_n$.

(8/20/2018) Today I learned the intuition behind the quotient stack X/G, which is often denoted $X//G$ or $[X/G]$ by people. Namely, if the group G acts freely on a space X, then there is such a thing as the quotient sheaf, which I'll temporarily call X/G, with a map of spaces $X \to X/G$. Namely, given a homomorphism $Y \to X/G$ you can immediately base change among the two morphisms you have to X/G and inmmediately you obtain some map $P \to Y$, and P gains a G action by acting on the X factor in the product. You can show that these pairs of bundles, equivariant maps are in bijection with the actual maps for quotient sheaves, and apparently, these work better for what the quotient should be always.

(8/21/2018) Today I learned that the quotient stack $\mathbb{A}^{n+1} \setminus 0/\mathbb{G}_m$ can be identified with the functorial definition of $\mathbb{P}^n$, that is, we can identify the quotient stack on some ring A as the set of invertible sheaves on $Spec(A)$ as well as the choice of $n + 1$ sections that have no common zero. This is because you can immediately note that a $\mathbb{G}_m$ torsor $P \to Spec(A)$ is equivalently the data of a line bundle on $Spec(A)$, and the $\mathbb{G}_m$ equivariant map $P \to \mathbb{A}^{n+1} \setminus 0$ immediatley gives you global sections of P which you can take to be global sections of the line bundle and you can show that if these had a common zero than the map $P \to \mathbb{A}^{n+1}$ would have zero in its image.

(8/22/2018) Today I learned that as stacks, if H and K are closed subgroups of an algebraic group G then the double coset quotient $H\backslash G/K$ is isomorphic to $\mathbb{B}H \times_{\mathbb{B}G} \mathbb{B}K$. When H or K is a point, this fact follows from chasing the definitions (and using the fact that a principal H bundle $P \to B$ can be made into a principal G bundle using the *Ind* construction, where $Ind_H^G(P) := P(B) \times^{G/H} G(B)$l, where the notation means that you can carry elements of $H(B)$ across the pair).

(8/23/2018) Today I learned that the quotient of an affine scheme X by a reductive algebraic group G acting freely on X, all over a field of characteristic zero, is again affine. This result follows from a result of Serre which says that if the global sections functor on a scheme is exact, then the scheme itself is affine. You can then show that the global section functor on quasicoherent sheaves, i.e. the pushforward $X/G \to *$, factors as $X/G \to \mathbb{B}G \to *$, which says that the global section functor is the sheaf $RHom(triv, \Gamma(X, \pi^*F))$ for our quasicoherent sheaf F on X/G, where $\pi : X \to X/G$ is a projection. We get that the derived functor $\Gamma(X, -)$ has no cohomology since X is affine, and since G is reductive over a field of characteristic zero, $RHom_G$ also has no cohomology.

(8/25/2018) Today I learned that given a reductive group G, you can associate to it its *Weyl Group* W, which is the group generated by reflections about the hyperplane perpendicular to each root. This group turns out to be a finite group, and furthermore I learned about the *Bruhat decomposition*, which says that the cosets BwB (where B a choice of Borel subalgebra) partition the group G (where it turns out that $W \cong Nm(T)/T$ so this coset is well defined) and furthermore there is an ordering on W (from the more general theory of Coxeter groups) which is reflected in the closure of each BwB, namely, $Bw'B \subseteq \overline{BwB}$ if and only if $w' \leq w$.

(8/26/2018) Today I learned that the length of an element in the Weyl group can be determined by the number of positive roots the element turns negative. Translated to the specific case of S_n, the Weyl group of GL_n, this says that in particular the maximal length element $(i \to n+1-i)$ cannot be expressed as a product of fewer than $\frac{n(n-1)}{2}$ transpositions of the form $(i, i+1)$.

(8/27/2018) Today I learned about *Verdier Duality* $\mathbb{D}$, which for smooth k varieties X maps D_X to the shifted canonical sheaf and in general is $RHom(-, \mathbb{D}(D_X))$. I also learned the fact that this function provides an equivalence of categories of the category of bounded chain complexes with finitely generated cohomology with its opposite category.

(8/28/2018) Today I learned a version of the statement of Poincaré duality, which in particular says for a proper scheme X of finite type over a field of characteristic zero that there is an isomorphism $\Gamma_{dR}(IC_X) \cong \Gamma_{dR}(IC_X)$. Also I learned while talking to Rok Gregoric that the original statement of Poincaré duality actually gives this only for oriented manifolds, so in partiular this implies that all proper schemes of finite type over a field of characteristic zero have some sort of orientation attached to them. Furthermore, I learned a simple, easy to prove but useful fact that a simple R module M is an R module for which $RHom(M, M) = R$.

(8/29/2018) Today I learned how to prove the aforementioned Poincaré duality yesterday! In particular, for any proper scheme X,, the map $p : X \to *$ is proper, so we have that $p_{*,dR} = p_!$ (or, said a different way, the left adjoint to the pullback functor on D_*-modules, i.e. vector spaces, $p^!$ exists and it is $p_{*,dR}$). Because of this and the fact that $\mathbb{D}(IC_X) = IC_X$, we have that $\Gamma_{dR}(IC_X) = p_{*,dR}(X) = p_!(IC_X) = \mathbb{D}_* p_{*,dR} \mathbb{D}_X(IC_X) = \mathbb{D}_* p_{*,dR}(IC_X) = Hom(p_{*,dR}(IC_X), \mathbb{D}_*(k)) = Hom(p_{*,dR}(IC_X), k) =$ $\Gamma_{dR}(IC_X)^*$.

(8/30/2018) Today I learned a potential proof that for any holonomic $D-$module F and an open embedding $j : U \to X$ of a smooth set into a scheme of finite type over k of characteristic zero that $j^! j_{!*}(F) = F$. Namely, you can immediately unravel the definition to note that $j_{!*}(F)$ will be a sub of $j_{*,dR}(F)$ and so taking $j^!$ of it will immediately, with the base change theorem, give an isomorphism $j^! j_{*,dR}(F) = F$. Hopefully you can take stalks to give the isomorphism you need. I also learned the idea of optimal transport, which seeks to find a "transport function" T which maps the set where one distribution has positive mass to the other minimizing "cost" and learned that you can use this to provide a neat proof that the ball is the smallest surface with fixed volume minimizing surface area.

(8/31/2018) Today I learned that for an open embedding j as above, $j^!$ is an exact functor. I also learned an alternative way to define an open embedding of an affine scheme, namely, you can define the *complement* of a closed embedding as a space $X \setminus Z(A)$, defined to be the set of all points for which the empty set is the fiber product. In this context, you can define $\mathbb{A}^{n+1} \setminus 0$ on A points by the choice of $n+1 a_i's$ which generate A.

July 2018

(7/1/2018) Today I learned what the direct image of an O_X module is given some map $f : X \to Y$ of schemes–namely, $f^*M = O_X \otimes_{f^{-1}O_Y} f^{-1}M$ where f^{-1} denotes the inverse image functor, which is the sheafification of the presheaf whose sections on an open set U are the colimit of sections of the given sheaf on open sets containing $f(U)$. I also learned that this can be given a D_X module structure by derivations acting as a variant of the product rule.

(7/2/2018) Today I learned what the tensor hom adjunction on chain complexes actually does. Namely, it takes a map $\phi : X \otimes Y \to Z$ and rewrites it as a set of maps of the form $\phi^{(j,k)} : X^j \otimes Y^k \to Z^{j+k}$ and then applies the tensor hom adjunction to that original map to obtain a map $X^j \to Hom(Y^k, Z^{j+k})$ which determines a map $X^j \to \prod_k Hom(Y^k, Z^{j+k})$ which determines a map of chain complexes $X \to \underline{Hom}(Y, Z)$.

(7/3/2018) Today I learned another way to compute what a differential of a complex should be assuming that you have a morphism of complexes involving your morphsim in a tensor product. Specifically, I used the fact that there is a canonical morphism $H \otimes X \to Y$, where H is the hom complex, and the fact that I knew it had to be a morphism because it was a canonical morphism that I could think of that relates to the identity in the adjunction $Hom(H, H) = Hom(H \otimes X, Y)$ and then used the fact that I knew two of the three differentials involved (or three of four depending on if you separately count the tensor product) to compute the differential of H!

(7/4/2018) Today I learned that, given a map $f : X \to Y$ of chain complexes, a map $g : Y \to Z$ and two nullhomotopies $h_1, h_2 : coker(f) \to Y$ of the composite gf, the two induced maps $\epsilon_1, \epsilon_2 : coker(f) \to Z$ need not be homotopic. However, if you use the "correct" definition of a homotopy between h_1, h_2 using the differential of the complex $\underline{Hom}(X, Y)$ you do get an induced map, which is an argument for defining a homotopy of homotopies this way.

(7/5/2018) Today I learned the idea behind a stable infinity category. Namely, an infinity category is a category where you are no longer allowed to make statements about "equality". Instead, essentially every sentence you form must be invariant under the homotopy underlying the objects you are talking about. Also, a stable category is a category with a zero morphism for any pair of objects (called a *pointed* category) and one where each square of morphisms is a pushout if and only if it is a pullback.

(7/6/2018) Today I learned that the property of a Lie algebra being solvable or semisimple is closed under quotients. It follows then that any semisimple Lie algebra $\mathfrak{g}$ is *perfect*, that is, $[\mathfrak{g}, \mathfrak{g}]$. I also learned that any representation of a nilpotent algebra maps entirely to the set of nilpotent matrices, and furthermore I learned about *Engel's Theorem* which says that for any nilpotent Lie algebra representation there is a vector which is killed by all elements in the Lie algebra. Accordingly, this shows that any representation of a nilpotent Lie algebra has a basis for which the representation maps to strictly upper triangular matrices, and you can beef this up using *Lie's theorem* to show any representation of a solvable Lie algebra has a basis making every element in the image of the representation upper triangular.

(7/7/2018) Today I learned about the notion of the heart of the a stable category with a $t-$ structure on a triangulated category $\mathbb{C}$, which is defined to be the intersection of the subcategory of elements $\mathbb{C}^{\leq 0}$ with the elements in nonpositive degree, i.e. the cokernel of the objects whose hom sets with all elements in $\mathbb{C}^{\leq 0}$ is zero, is actually a one category (i.e. there is at most one morphism between any morphisms, and that only happens if the domain and codomain agree.)

(7/8/2018) Today I learned the toy model of the Fourier Deligne transform, which says that given a prime power q and a nontrivial character $\chi : \mathbb{F}_q \to \mathbb{C}^\times$ (or really, any field with a p^{th} root of unity) you can construct an invertible function on the set of all functions on V, defined as a finite dimensional vector space over $\mathbb{F}_q$, to the set of all funcions on the dual space V^* by "integrating over the projection $V \times V^* \to V^*$ of the pullback of the other projection map times the character pulled back. I learned that this is invertible (specifically, the opposite fourier transform takes $Four(f)(v) \to f(-v)$) through some coordinate changes which will suggest how the functorial version is "almost invertible".

(7/9/2018) Today I improved the above toy model to a statement of conjectures/hopes about how pullback and pushforward of $D-$ modules work. In particular, I took my proof of the above statement that the Fourier Deligne toy model transform is its own inverse and converted it into "functorial" language.

(7/10/2018) Today I learned another proof of the fact that if $w \in V$ is a nonzero vector and χ is a nontrivial character then $\sum_{\lambda \in V^*} \chi(\lambda(w)) = 0$. By assumption that w is a nontrivial vector, there is a ρ for which $\rho(w) \neq 1$. By symbol manipulation, you can show that this implies that $\sum_{\lambda \in V^*} \chi(\lambda(w) = \rho(w) \sum_{\lambda \in V^*} \chi(\lambda(w))$, and since that sum isn't one, you can conclude your sum is zero. This proof generalizes to the $D-$module case!

(7/12/2018) Today I learned that a Lie algebra is semisimple if and only if every representation is semisimple, that is, every invariant subspace has a complimentary subspace. I also learned one way to see the "only if" part. Namely, if your Lie algebra $\mathfrak{g}$ is semisimple, it has an abelian ideal $\mathfrak{a} \neq 0$ inside, and so the adjoint representation yields $\mathfrak{a}$ invariant. Therefore we obtain a quotient map $\mathfrak{g} \to \mathfrak{a}$.However, composing this with a representation $\mathfrak{a} \to V$ which isn't semisimple (which can be done, say, by adjusting the trick of the $\mathbb{R} \to M$ trick where $M_1 = e_1$ and $M_2 = e_1 + e_2$).

(7/13/2018) Today I learned Kashiwara's lemma, which says that the de Rham pushforward of a closed embedding is fully faithful embedding of categories. I also worked through the example of the embedding $\mathbb{A}^n \hookrightarrow \mathbb{A}^m$ and showed it, which the proof essentially stems from the fact that I suspect embedding a module is just making multiplication by all the other variables and differential operators to be zero.

(7/14/2018) Today I learned the definition of the de Rham pushforward of a closed embedding on D modules! Namely, you can locally choose coordinates so that your closed embedding is locally given by the vanishing of the first n of them, say, and then the push forward of a D module M is given by $\mathbb{C}[\partial_1, ..., \partial_n] \otimes_{\mathbb{C}} M$, with action given by "the only action it could be"–factoring out all of the ∂s on left tensors and commuting them, we define any function or old partial to act on the $1 \otimes M$ component via the restriction of function or the old differential operator. This immediately gives why de Rham pushforward on maps (which just tensors the maps by identity) must be injective since we are tensoring over vector spaces. Furthermore I learned why the map is surjective. Namely, any map of D modules is forced to send elements of the form $1 \otimes m$ to a unique element of the form $1 \otimes n$.

(7/16/2018) Today I learned that the equivalence of categories induced by a closed embedding can (probably) be viewed as a consequence of the base change theorem. Namely, one can note that the map $i_{*,dR}$ is in particular proper, so therefore there is a map $Hom(C, D) \to Hom(C, i^! i_{*,dR} D)$ which you can show by the base change theorem must be $Hom(C, D)$. I strongly suspect that this composition gives Kashiwara's equivalence.

(7/17/2018) Today I learned a bit about the representations of $\mathfrak{sl}_3$. Namely, using the adjoint representation one can determine that $\mathfrak{sl}_3$ is the sum of its diagonal subgroup $\mathfrak{h}$ and other subspaces $\mathfrak{g}_\alpha$ where each $\alpha \in \mathfrak{h}^*$, namely you can explicitly compute each α as $L_i - L_j$ for linear functionals where L_i is the indicator on the matrix with zero entries except for the (i, i) one. Then you can set up a lattice and use a similar idea to show that with the six nonzero functionals $L_i - L_j$ you can construct a "highest weight vector" and your representation is $\mathfrak{sl}_3$ spanned by the three "negative weight" functionals repeatedly applied to the highest weight vector.

(7/18/2018) Today I learned what the cotangent bundle is in algebraic geometry. Namely, given a locally free sheaf E on a scheme X, one can form the O_X algebra $Sym(E)$ that is also a quasicoherent sheaf. Then there exists a unique scheme W with map $f : W \to X$ such that $f^{-1}(Spec(B)) = E(Spec(B))$ for every affine open B with maps between any two given by restriction. The *cotangent bundle* is this construction (written **Spec** in Harthshorne) applied to the cotangent sheaf–i.e. the sheaf of derivations.)

(7/19/2018) Today I learned that on a smooth variety the singular support of any vector bundle is cut out by the variety inside its cotangent bundle via the zero section, and how to prove this. Namely, to show this, you can reduce to the local case where you can explicitly write out the model of the tangent bundle. Then you can show that the "obvious" choice for the associated graded of your vector bundle yields an associated graded that is only in degree zero. After that, you can note that at any point, any differential must then send the associated graded to zero, and so if any of them are allowed to be units the stalks must be a point.

(7/20/2018) Today I learned most of the proof that if $f : X \to Y$ is a qcqs map of schemes then the map $f_* : QCoh(X) \to QCoh(Y)$ is a continous map, i.e. it commutes with all colimits. Essentially the first thing you do is you show that it suffices to show it when $Y = Spec(B)$, and then you write X as a finite union of affine schemes with the intersection of any two affine schemes a finite union of affine schemes. Also I determined what the colimit of a quasicoherent sheaf is–on affine opens, you can compute it as the limit in modules.

(7/21/2018) Today I learned that the singular support of the delta module at a point x on a smooth variety X is cut out by the derivations at that point (as a subset of the cotangent bundle). This is because you can compute the singular support locally, and you can use a local isomorphism to show that your module identifies $\partial/\partial x_i$ with x_i^{-1} and kills any polynomial which has any variable in positive or negative degrees. Therefore you can show that if any x_j is invertible you can write $1 = x_j x_j^{-1}$ to kill any element–thus if x_j is invertible at your point, it can't be on the support, and conversely you "evaluate at zero" to show if all the x_i's aren't invertible, $1 \neq 0$.

(7/23/2018) Today I learned the full statement of Kashiwara's equivalence, which actually says that given a closed embedding $i : Z \hookrightarrow X$, not only is the map $i_{*,dR}$ a left adjoint to $i^!$, but furthermore $i_{*,dR}$ induces an equivalence of categories from the derived category of coherent D_Z modules with bounded cohomology to the full derived subcategory of the derived categor coherent D_X modules with bounded cohomology. I also learned this is how arbitrary D_Y modules are defined for not necessarily smooth Y–simply embed it into a smooth object.

(7/24/2018) Today I learned that if $i : Z \hookrightarrow X$ is a closed embedding and j is an open embedding that embeds the complement of Z into X, then you have maps $i_{*,dR}i^!(F) \to F \to j_{*,dR}j^!(F)$ which turn out to form a distinguished triangle in the sense of triangulated categories. I also learned that this can be formalized in the sense of *recollment.*

(7/25/2018) I just learned the *coolest* thing today. Namely, I was searching for a distinguished triangle of the form (with yesterday's setup) $i_{*,dR}i^!(F) \to F \to j_{*,dR}j^!(F) \to (+1)$. And I first found out that you can convert exact sequences of injective sheaves to distinguished triangles in the derived category using some isomorphisms involving the homotopy cokernel and the cylinder of a map. But then, I noticed on sheaves that you can explicitly compute $j_{*,dR}$ and $j^!$, which are the regular pushforward and the regular pullback explicitly, so in particular for a sheaf, $j_{*,dR}j^!(F)$ is just *the restriction to the open set of that sheaf.* Since, very roughly speaking, Kashiwara's equivalence says that $i_{*,dR}i^!(F)$ is the "part" of F that is supported on the closed subset Z, this says we can basically break D_X modules into two parts and get info from both those parts and piece them together to get info about your D_X module!

(7/26/2018) Today I learned what an ind-scheme is–essentially, in the category of "Spaces," i.e. functors $AffSch \to Set$, they are the functors which can be expressed as the colimit of actual schemes. I also learned with this interpretation what a representation of an algebraic group G is–it's simply a vector space V with a map $G \times \underline{V} \to \underline{V}$ where $\underline{V}(A) = V \otimes A$ (is the "indschemification"). Then I learned the fact that any one dimensional connected algebraic group over an algebraically closed field is either an elliptic curve, $\mathbb{G}_a$, or $\mathbb{G}_m$, and then I learned that the representations of $\mathbb{G}_m$ are in one to one correspondence with $\mathbb{Z}$ graded vector spaces–in fact, this is an equivalence of categories.

(7/27/2018) Today I finally proved that the map sending a scheme to "Spaces", $X \to Hom(-, X)$, is a fully faithful embedding of categories. Essentially what you do is first note that affine schemes fully embed into this category via Yoneda's lemma, and then you try to mimic the proof of that to show the functor is full. Specifically, for the affine case, you argue that any natural transformation $\eta : Hom(-, X) \to Hom(-, Y)$ actually is pullback by $\eta(Spec(A))(id)$, but for the scheme case you have to write the identity as a limit of the inclusions of some open cover.

(7/28/2018) Today I learned that, in the above notation, $\underline{V}$ is not representable by an affine scheme for any infinite dimensional vector space V. Roughly speaking, this is because Yoneda's lemma tells us that any natural transformation $\underline{V} \to Hom(-, Spec(A))$ is entirely determined by where the $Spec(A)$ part of that transformation sends the identity. However, because V is a colimit of its finite dimensional subspaces, $\underline{V}(A)$ is a colimit of A tensored with the finite dimensional subspaces. Thus Yoneda's lemma in this case would tell us that all of the data is contained in a finite dimensional piece, which you can use to show that A must be finitely generated as a ring, and then do a dimension argument (noting that the natural transformation must give a vector space morphism by naturality) to show this can't happen.

(7/30/2018) Today I learned that any map from a quasicompact scheme to an indscheme must factor through an affine scheme. Because inschemes are explicitly filtered colimits, you can first off immediately reduce to the case where the domain is actually affine. There, you can use the Yoneda principal of all of the information is contained in the identity morphism to show that "all the information is contained in" one of the X_i's.

(7/31/2018) Today I learned that representations of $\mathbb{G}_a = Spec(k[t])$ are equivalently any vector space with a locally nilpotent endomorphism, i.e. one that for each vector there is a power of the transformation which is zero. Given a representation of $\mathbb{G}_a$ you can recover the transformation T acting on a vector v by projecting onto the t coordinate after applying the coaction map induced by the representation. Then you can show that due to the fact that the coaction map must commute with addition (sort of the "coassociativity axiom") you can show the other coordinates are uniquely determined by this transformation, and that your transformation is simply $\sum_{n=0}^{\infty} \frac{T^n v}{n!} t^n$.

June 2018

(6/1/2018) Today I learned the idea behind quantum computing, which essentially uses the ideas of quantum physics to assign a probability to each output, and then sort of tries all the possibilities at once (but not really–this is the literal big lesson of Scott Aaronson's blog), but you only get to see one of them, chosen at random but with weighted probabilities. You can use this to find the period of certain special numbers, and you can use these to factor quickly. I also learned about the general cartoon idea of the Grothendieck-Ihara program, which essentially attempts to study the absolute Galois group $G_{\mathbb{Q}}$ by embedding it into the group $\pi_1^{et}(\mathbb{P}^1 - \{0, 1, \infty\})$.

(6/2/2018) Today I learned what a modular form of weight k is! It's a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ satisfying the identity $f(\gamma\tau) = (c\tau + d)^k f(\frac{1}{\tau})$ for any $\gamma \in SL_2(\mathbb{Z})$ where γ has entries a, b, c, d in the places I know they are but don't feel like typing, and such that as $\tau \to i\infty$, f remains bounded. The motivation for the first part apparently comes from hyperbolic geometry. Because the matrix of negative the identity is in $SL_2(\mathbb{Z})$, the set of modular forms of odd weight is zero.

(6/3/2018) Today I learned about *Haar measure*. One way to characterize it is to say that there exists a unique measure on any compact topological group G for which G has valuation one and is inner and outer regular with respect to the open sets of the topology. I also learned one of the parts of Tate's thesis. Essentially Tate first constructs additive and multiplicative characters for complete local fields (and multiplicative "quasicharacters" as he calls them) and uses algebra to determine a notion of positivity on the quasicharacters. He then defines an analogous ζ function to $\mathbb{C}$ for each character for "positive" characters and then shows there is a functional equation to define it everywhere.

(6/4/2018) Today I learned why the naive notion of degree for divisors on one dimensional varieties doesn't extend to higher dimensional examples in the way we want. In particular, a d dimensional hypersurface is linearly equivalent to d times a hyperplane class, so this notion of degree won't extend down in the way we want. Instead, we use intersection pairing to make the general Riemann-Roch theorem for surfaces. Namely, the Euler characteristic of the cohomology is a polynomial of the divisor and the canonical divisor. I also learned (for the first time ever, surprisingly!) what the Fourier transform really does. The idea is to almost "average" a periodic function of time to see what the periods are.

(6/6/2018) Today I learned what the Grothendieck group is. Namely, on any sheaf X, it is defined to be $K^0(X)$, which is the free abelian group generated by coherent sheaves on X modulo the relation that if you have a short exact sequence of any three sheaves, the middle is the sum of the other two. This plays a strong role in the generalized proof of the Riemann-Roch theorem as it is meant as some sort of a generalization of the Picard group. I also learned about Chow Rings which are just like the divisor group but instead allow any dimension/codimension.

(6/7/2018) Today I learned a proof for why the dual isogeny of the sum of two isogenies is the sum of the duals. Namely, you first look at the equality you want to show and show that their difference is zero. One can do this by passing to the group Div^0 on the target of the elliptic curve, but with values on the curve not just in the ground field, but in the new ground field of rational functions on the domain elliptic curve. You can then show that the dual isogeny associated to the associated divisor is the pullback on Div, which is an alternative definition of the dual isogeny, and then show the result there.

(6/8/2018) Today I learned the outline of the proof of the Grothendieck Riemann Roch theorem. Essentially you use a "moving lemma" to reduce the problem to factor your proper map of smooth varieties into a closed embedding and a projection map. For the closed embedding map, you use the idea of deforming to the normal cone to reduce your problem to showing it for a very certain map where you can compute the proof explicitly. For the projection map, the proof is essentially a diagram chase.

(6/9/2018) Today I learned an alternative proof that classifies the representations of S_3, and a way to determine the multiplicity of each irrep that appears in an arbitrary representation of S_3. The proof uses the fact that S_3 has a relatively large abelian subgroup and the classification of irreps of abelian subgroups is absolutely trivial. Then, using generators and relations, you can see what an element not in the abelian subgroup does, and that completely classifies the three irreps and gives you the multiplicity of each irrep.

(6/11/2018) Today I learned a bunch of things about homological algebra and representation theory. In particular, I learned what it means for two chain complexes to be homotopic. However, I also learned that the multiplication map $0 \to \mathbb{Z} \to \mathbb{Z} \to 0$ is not chain homotopic to $0 \to 0 \to \mathbb{Z}/2\mathbb{Z} \to 0$ so we need a new notion of equivalence, which we call *quasiisomorphisms*. However, this notion isn't symmetric since you can show quasiisometry in one direction of $0 \to \mathbb{Z} \to \mathbb{Z} \to 0$ (with the identity map) to the zero map but there is... oh wait. That chain complex is the one that isn't quasiisomorphic. Oh well. I learned it now.

(6/12/2018) Today I learned a really freaking cool proof of the fact that if V is a faithful representation of a finite group G, then every irreducible representation of G appears as a subrepresentation of $V^(\otimes n)$ for some $n > 1$. The idea is to use characters and let a_n be the inner product of the character of $V^(\otimes n)$ with your favorite representation. Then you can consider the power series $\sum_{n=0}^{\infty} a_n t^n$ which you can combine terms and argue via geometric series arguments that you can write this as a finite sum of inverses of certain geometric series, and you can argue using the faithfulness assumption that this sum cannot be a constant, and therefore the original sum cannot be a constant either.

(6/13/2018) Today I learned a few cool things about the functor Ext. First of all, I learned deeper into the fact that $Ext^n(M,N)$ classifies the $n-$extensions of N by M (or the other way), by which I mean possible E_i which fit into the exact sequence $0 \to N \to E_{n-1} \to ... \to E_0 \to M \to 0$, up to isomorphism of these sorts of sequences. First of all, I learned that there is an explicit sum you can construct for at least Ext^1, called the *Baer sum*, which makes this an isomorphism of abelian groups. I also learned that the map of a sequence is given by taking some projective resolution of M, say $... \to P_1 \to P_0 \to M \to 0$, noting that you can pull back the morphism $P_0 \to M$ to one $P_0 \to E_0$ by projectivity, and then you can note that pulling back *that* morphism from P_1 yields the zero morphism, so you get a map $P_1 \to E_1$ and then you keep repeating that to get your map $P_n \to N$ which pulls back to zero!

(6/14/2018) Today I learned where the connecting morphism on each page of a spectral sequence comes from. Essentially what you do is you first take an element in your homology and then take preimages in the homology of your filtration all the way down to the place which maps to the cohomology group you want. It's hard to explain without things that I am absolutely not going to tex up. I also learned that a character is real if and only if a representation is isomorphic to its dual. This makes sense for finite groups through character theory, and gives the existence of a bilinear map on your representation.

(6/15/2018) Today I learned the existence of the Leray-Hoschild-Serre spectral sequence which is a spectral sequence on group cohomology which converges to regular group cohomology given a short exact sequence of groups $1 \to N \to G \to G/N \to 1$. I also learned a more general lesson from this–namely, given a group G, we can put the discrete topology on it and define BG to be the unique up to homotopy space with $\pi_1(BG) = G$ and zero for higher π's. Then you can show that the homology (in topological spaces) of BG agrees with the group cohomology of G acting trivially on $\mathbb{Z}$ and can use that to construct a fibration. This allows you to use Serre's spectral sequence (which is about a fibration in topological spaces) to give you information about group cohomology.

(6/16/2018) Today I learned an interpretation of twisting of a quasicoherent sheaf and how it relates to grading shifts. In particular, you can view the module $\widetilde{M(n)}$ over $Proj(S_\bullet)$, where $S_\bullet$ is a graded ring generated in degree one, by viewing it on each open set $D(f)$ as $(M_f)_0 f^n$, where $f \in S_\bullet$ has degree one. I also made this interpretation explicit by showing $\widetilde{M}(n) := \widetilde{M} \otimes O(n) \cong \widetilde{M(n)}$.

(6/18/2018) Today I learned an extension of Nakayama's lemma that Vakil calls "Geometric

Nakayama's Lemma." Essentially assuming that you have a finite type quasicoherent sheaf and for some point you have a finite number of elements that generate the fiber (i.e. the stalk tensored with the local field), not only does normal Nakayama's lemma give you generators for the stalk, but you can also add the bonus finite typeness to get an open set on which those generators generate every fiber on the open set! I also learned a few basic things about the ring of differentials. For example, on the line, I learned that the ring of differentials on the line is the ring $\mathbb{C}\langle x, \partial\rangle/(x\partial - \partial x - 1)$. The fact that you have this relation follows from the product rule, and if you have a sum $\sum_{i=0}^{n} a_i(x)\partial^i$ you can feed in x^n to conclude that $a_n(x) = 0$ and then proceed inductively.

(6/19/2018) Today I learned what almost commutative filtered rings are (namely, a ring with an exhaustive filtration such that the associated graded ring is commutative) and a lemma about when the filtration placed on a graded module over the ring is finitely generated. In particular, one can construct the Rees ring, which is the direct sum of each of the filtered components, and one can show that the Rees module being finitely generated over a Rees ring is equivalent to knowing precisely what the filtration on the module is–namely for some generators the i^{th} piece is the sum of $A_{i-d_j}m_j$ for some m_j in the module.

(6/20/2018) Today I learned how to define the tangent sheaf on a general variety in two different ways. In one way, you can define it on each affine open set as the set of smooth k derivations from that ring to itself, where *derivations* refer to additive maps which send the field to zero and satisfy the product rule. You can also define it as the hom sheaf from the "universal differential" $\Omega_{A/k}$ where A is your ring, and by universal property nonsense these turn out to be equivalent. You can then define the sheaf D_X on a general variety by declaring it on each affine open set for which there exists a finite unramified map $X \to \mathbb{A}^{dim(X)}$ to be the subsheaf of $\mathbb{C}$ linear endomorphisms generated by multiplication by O_X and T_X, where T_X is generated by the pullbacks global differentials $\partial_1, ..., \partial_{dim(X)}$.

(6/21/2018) Today I learned why derivations can be regarded as vector fields. In particular, after localizing a derivation $d : A \to A$ to a point, then you can show any element in the square of the maximal ideal, say $\mathfrak{m}^2$, remains in $\mathfrak{m}$ due to the product rule. Thus the localization induces a map $(\mathfrak{m}/\mathfrak{m}^2) \to A/\mathfrak{m}$, which is a tangent vector!

(6/22/2018) Today I learned the motivation for the Lie bracket. In particular, because a connected Lie group is generated by any open neighborhood of the identity as a group, it's probably a good idea to check out what happens at the identity. Then you can check, given a morphism of Lie groups, what the map on the tangent space of the identity does, and you can draw the appropriate diagram and take the derivative to show that any morphism of Lie groups must preserve the *Lie bracket*, which, when your group is embedded in any GL, is the commutator.

(6/23/2018) Today I learned a definition of the direct image of a D_Y module given a closed embedding $Y \hookrightarrow X$. Namely, given a D_Y module M, the definition is $f_\cdot(f^\cdot D_X \otimes_{f^\cdot D_X^Y} (M \otimes_{D_Y} det(T_Y X))$, where the extra determinant bundle follows from the fact that the definitions we are going for is "distributions supported on Y," so for example given a point we need $x\partial = -\partial$ since $x\partial\delta = -\delta$.

(6/25/2018) Today I learned the functor of points view of a scheme. In particular, you can define a scheme the old way with a structure sheaf and gluability, or you can define a scheme as the full subcategory of sheaves X on affine schemes with the covering of *open immersions* (which is just a functor from affine schemes to sets satisfying certain gluability axioms) such that X can be covered by open immersions $U_i \to X$ with $\coprod U_i \to X$ an epimorphism. This can be shown by constructing the topological space associated to the functor X, say, Y, by viewing the colimit of $Spec(k)$'s mapping into X, and then can be topologized with the open immersions, and then can be given the exact structure sheaf you want, and then you can show X is naturally isomorphic to

$Sch_k(-, Y)$ where Y is the scheme you constructed.

(6/27/2018) Today I learned that you can have chain complexes which are acyclic (i.e. their homologies are zero) such that the complex isn't homotopic to the zero complex. This can be done by chasing the definitions and considering the exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$ and noting that any chain homotopy of that chain complex to itself must be a map $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$, which shows id isn't chain homotopic to the zero map. I also learned a sketch of one reason we like holonomic D modules. Often times, this notion implies coherence, and the notion of holonomic is preserved under the push forward and pullback functors that are "natural" to D modules, which contrasts to the function case, where $k[t, t^{-1}]$ isn't a coherent $k[t]$ module.

(6/28/2018) Today I learned there is a complex for which the tensor hom adjunction holds in the category of the chain complexes. In particular, you can work backwards and note that if you expect the tensor hom adjunction to hold for some mysetry chain complex $M := \underline{Hom}(X, Y)$, given chain complexes X, Y, since you already know what the tensor product of chain complexes is you can feed in $Hom(\mathbb{Z}[j], M)$, which by direct analysis/abelian group reasons *has* to be $Ab(\mathbb{Z}, M^{-j}) = M^{-j}$, you can compute it on the other side and show that it has to be $M^j = Hom(X \otimes \mathbb{Z}[-j], Y)$. Analyzing this, it becomes clear what differentials must be. Namely, if we have the chain complex X shifted further to the left and we would like to get morphisms to the right, you can get morphisms of X shifted less far to the left by precomposing (levelwise) with the differential map of the chain complex X!

(6/29/2018) Today I found a mistake in yesterday's calculations. That was fun. Namely, it's not the case that $Ab(\mathbb{Z}, M^0) = Ch_{\mathbb{Z}}(... \to 0 \to \mathbb{Z} \to 0 \to ..., M^\bullet)$. In fact, you can check the first square to determine that there is a restriction on which abelian group maps can be morphisms from $... \to 0 \to \mathbb{Z} \to 0 \to ....$ Namely, 1 must map to something in the kernel of the differential. Instead, it turns out that $Ab(\mathbb{Z}, M^0) = Ch_{\mathbb{Z}}(... \to 0 \to \mathbb{Z} \to \mathbb{Z} \to 0 \to ..., M^\bullet)$. Using this, you can compute the tensor hom adjunction to get a funny looking pair of morphisms. But it turns out that one of the morphisms is "free", say ψ, and once you have a module Hom on each level (chosen arbitrarily!) choosing signs carefully, the other element of the pair is $\pm(d\psi + \psi d)$ will give you a morphism of chain complexes!

May 2018

(5/1/2018) Today I learned that if D is a squarefree integer which is divisible by three or more primes, then the class number of $\mathbb{Q}[\sqrt{D}]$ must be even. This is because by assumption D must have two odd primes dividing it, say p and q, and you can go through all the various quadratic subfields to show that the only prime of $\mathbb{Q}[\sqrt{D}]$ which could possibly ramify in $\mathbb{Q}[\sqrt{r}, \sqrt{D}]$ over $\mathbb{Q}[\sqrt{D}]$ with $r \in \{p, q, pq\}$ is 2 since we can't have total ramification over $\mathbb{Q}$ because you can show a specific subfield that blocks it. But then you can go through all the various modular cases of D, p, q to show that there's some $r \in \{p, q, pq\}$ where primes lying over 2 don't ramify in $\mathbb{Q}[\sqrt{r}, \sqrt{D}]$.

(5/2/2018) Today I learned that given a graded module over a graded ring, you can do the similar sort of Proj construction and that makes the module into a quasicoherent sheaf. I also learned that another way to describe this construction at the stalks is, for each stalk, to localize the prime of the module and then take the zero graded part. I also learned a trick in computing with norms that says an irreducible representation of a group G restricted to an index two subgroup of H must either remain irreducible or split into two irreducible components of the same dimension.

(5/3/2018) Today I learned about degenerations. The picture that was associated to it was a family of smooth projective varieties parametrized by a punctured disc, and how monodromy sometimes prevents us from smoothly filling that disk. However, algebraic geometry provides a way to fill the disk with a punctured torus. I also learned what a Berkovich space is–essentially it's the space of norms of the function field of a point that extend the norm of the field already given, and what the Berkovich circle looks like–essentially it's many copies of the Bruhat Tits tree.

(5/4/2018) Today I learned a reduction in a problem I'm trying to solve. Namely, the problem is that if the completion of $F := \mathbb{Q}[\zeta + \zeta^{-1}]$ at a prime lying over 7 isn't $\mathbb{Q}_7[\sqrt[3]{7}]$ then the completed field $\mathbb{Q}_7[\sqrt[3]{7}, \zeta + \zeta^{-1}]$ has a 48th root of unity, equivalently, has an unramified extension of $\mathbb{Q}_7$. I learned today that if you can show you have any extension of the form $\mathbb{Q}_7[\sqrt[3]{u}]$ for some $u \in \mathbb{Z}_7$ that isn't already a cube you can show the residue field actually does change.

(5/6/2018) Today I learned how to actually show that problem. Namely you can show that no prime above 7 ramifies in $\mathbb{Q}[\sqrt[3]{7}, \zeta_7 + \zeta_7^{-1}]/\mathbb{Q}[\sqrt[3]{7}]$ because you can argue that if it did, the dimension of $\mathbb{Q}[\sqrt[3]{7}, \zeta_7 + \zeta_7^{-1}]$ over $\mathbb{Q}$ completed at some prime above 7 has to be 9 for ramification reasons which in turn says the composite of the two fields is larger than either field. This gives you the inequality needed to apply the above.

(5/7/2018) Today I learned that the class group of $\mathbb{P}^1_k \times \mathbb{P}^1_k$ is $\mathbb{Z}^2$. This is because you can use the excision exact sequence to show that the divisors $[\{infty\} \times \mathbb{P}^1_k]$ and its opposite must generate the group, and then you can restrict any integer combination to either factor $\mathbb{P}^1$ to show that no integer combination of those two divisors can be principal, since that would $O(n)$ was principal on $\mathbb{P}^1$.

(5/8/2018) Today I learned what a toric variety is and a way to build a ton of them that connects to the theory of polytopes. Namely, a toric variety is a variety V over $\mathbb{C}$ which contains an algebraic torus as a Zariski open subset such that the action of the torus T extends to an action $T \times V \to V$. Given any polytope, you can check what integer points are on the polytope and you can also check which points are on each ray on a fixed face for each face. This determines a specific subring of $\mathbb{C}[x_1^{\pm 1}, ..., x_n^{\pm 1}]$ which turns out to always give an open embedding into any "subface" of that face. This gives you the gluing data to construct varities!

(5/9/2018) Today I learned why the above model of toric varieties is a good model to generate many toric degenerations. This is because given a polyhedron that is closed under translation in the upward vertical direction, this says that you always have the "last" variable, say t, in every affine open set determined by the faces of the polyhedron. You can show that away from zero is essentially the same thing as also requiring the polyhedron be closed under translation in the downward vertical direction, so in particular you can show that away from zero the general fiber is just the toric variety determined by the projected polyhedron, while this doesn't in general work for zero.

(5/10/2018) Today I learned a theorem which states that given a toric variety associated to some polyhedron, the toric prime divisors (i.e. the prime divisors invariant under the torus action) can be recovered by looking at the facets of the associated polyhedron. I also worked through an example which says that the triangle with vertices at the origin, e_1, e_2 gives projective space as a toric variety.

(5/11/2018) Today I learned about the focus focus singularity, which involves gluing $\mathbb{R}^2$ with the closed ray from the origin pointing downward to $\mathbb{R}^2$ with the closed ray from the origin pointing upward on the $x \neq 0$ parts by gluing (x, y) on the first to the same point on the second if $x < 0$, else $(x, x + y)$. This makes an affine manifold and relates to the Gross Siebert degeneration model. This is because in the two different charts if you do the naive toric degenerations you don't get the same on either chart, so instead you give some kind of canonical gluing and then you get the degeneration to the union of axes, which can't be expressed via normal toric varieties.

(5/13/2018) Today I learned what parallel transport is! Specifically, given a section of a vector bundle and a starting point on that vector bundle the way you transport it parallel along a given path is you take a chart of the vector bundle in that path and then in "affine" space you simply slide the same vector along the path (which explains the "parallel" part) to the ending of your path. If your path starts and ends at the same point, you can get a different vector, which is the phenomenon of *monodromy*, I'm pretty sure. Also in the focus focus singularity parallel transport takes $e_1, e_2 \to e_1, e_1 + e_2$.

(5/14/2018) Today I learned that the model in the Gross Siebert model of degenerations models so that the general fiber is not necessarily toric. For example, this captues the general hypersurface degenerating to the union of coordinate hyperplanes, where note that the central fiber is still toric.

(5/15/2018) Today I learned the full (well, almost all) of the example as to what is different about the non toric gluing applied to the old toric degenerations. Essentially what's happening is that the old way that one may naively get a degeneration from $\mathbb{R}^2$ depends on the choice of chart you use for the affine singularity. Specifically the choice is not well defined up to isomorphism of degenerations, each of which has a predetermined embedding of the special fiber into the total space. To rectify this, Gross and Siebert put vectors on each of the pieces and then rectify the problem of two different isomorphisms of the toric strata by gluing the isomorphisms together after applying an automorphism, so to speak.

(5/16/2018) Today I learned a generalization of that example of last time. Essentially you can give the $\mathbb{R}^2$ manifold with an affine singularity as a polyhedral structure with one vertex vertically above the origin and one vertically below with the standard charts. Then the "isomorphisms suggested by toric geometry" for a given vertex are given by the result of moving to the left hand chart to the right hand chart via parallel transport through the vertex, noting that the charts are chosen so that each vertex is on *exactly* one vertex. This generally gives a local model for codimension one polyhedra with vertices on it with this same idea after you localize on a tropical manifold.

(5/17/2018) Today I learned in the Gross Siebert construction of degenerations that the relations obtained via the procedure on affine singular manifolds are attempts to rectify the fact that monomial multiplication is not well defined on a singular affine manifold. Furthermore, I learned what a main idea of mirror symmetry is–namely, the idea is that for each Calabi Yau space X there is some sort of mirror space for which the algebraic geometric constructions on X become symplectic constructions on the mirror space.

(5/18/2018) Today I learned that in order to construct the above version of degenerations on singular affine (well, probably better, tropical manifolds) there may be some problems with compatibility of maps. Specifically, imagining gluing each quadrant together while cutting off/regluing via a nonidentity affine transformation the ray $(-\infty, -1]$ if you "go around" the various gluing morphisms you reach a compatibility problem. The solution is at the "other half" of the ray $(-\infty, 0]$ to also change the gluing function, and furthermore, even if the other ray isn't part of the original polyhedral decomposition, add a ray (which is more generally in higher dimensions called a *wall*.)

(5/22/2018) Today I learned that the valuation map from a local field K^* to $\mathbb{Z}$ gives an isomorphism on the second homology groups of the profinite completion of $\mathbb{Z}$ acting on either. This can be shown by showing that the cohomology of the units are trivial which induces isomorphisms on the long exact sequence, and you can show that the cohomology of the units are trivial by realizing the quotients as isomorphic to the additive field (as Galois modules) and then use a lemma which allows you to show cohomology is zero from a quotient.

(5/24/2018) Today I learned that given a field extension L/K of local fields then the "inv" map on L (regarded via a domain change by Res) is n times the "inv" map on K. This is because you can decompose the invariance map to a diagram involving the valuation (which changes the map by a multiple of the ramification) and then later in the composition that if $\phi \in Hom(Gal(K), \mathbb{Q}/\mathbb{Z})$ note that wherever ϕ takes 1, the associated restriction is f times that.

(5/28/2018) Today I learned that the functional equation for the Riemann Zeta function really comes from a *completed Zeta function* $Z(s) := \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ for which the functional equation reads $Z(s) = Z(1-s)$. From this, you can understand why the trivial zeroes of the Riemann ζ function are where they are–the trivial zeroes come from the fact that the Γ function has a pole at the negative integers. This also shows why all zeroes must occur in the critical strip.

(5/29/2018) Today I learned that the endomorphism ring of any elliptic curve (defined to be the maps from the elliptic curve to itself which send the identity to itself, which turn out to be group homomorphisms) must be either the integers or an *order* of an imaginary number field or a quaternion algebra. Furthermore, due to the existence of the Frobenius, if your elliptic curve is over a finite field (or better, regarded over its algebraic closure) then the endomorphism ring can't just be the integers. This theorem stems from the fact that each isogeny has a dual isogeny which puts an anti involution structure on the endomorphism ring of the curve.

(5/30/2018) Today I learned that any prime $4p$ which can be represented as a quadratic form has the property that the discriminant of that quadratic form is a square modulo $4p$. This can be shown by literally completing the square on the quadratic form if the quadratic form is of full rank. I also learned an analogue of the exponential function in the function field analogy, namely for $\mathbb{F}_q$ the power series $\sum_{i=0}^{\infty} \frac{x^{q^i}}{Q_i}$ where Q_i is the product of all irreducible polynomials of degree n.

(5/31/2018) Today I learned the idea of the proof that says that if the generalized Riemann Hypothesis is true, then given any proper subgroup of the group $\mathbb{Z}/m\mathbb{Z}$ there is an element less than $klog(log(m))$, where k is some constant which turns out can be taken to be 2, such that that element isn't in the subgroup. To show this you note that H is contained in the kernel of some nontrivial character and then prove the statement for a nontrivial character. You then argue that for each time there is an element in the kernel, the first few terms of the L function associated to that character is equal to the first few terms of the Riemann ζ function (which can be made more rigorous with integral transform arguments). You then use some asymptotic arguments using the fact that ζ has a pole at one, whereas any other L function doesn't.

April 2018

(4/1/2018) Today I learned that the polynomial x^4+3x+3 is irreducible over the field $K := \mathbb{Q}(\sqrt{21})$. This is because if $\mathfrak{p}$ is a prime of O_K lying over 2 we can use Galois theory to show that x^4+3x+3 cannot split as the product of an irreducible cubic and a linear factor, since the Galois group of this polynomial over $\mathbb{Q}$ must be of order 4 or 8. Furthermore, assuming it factors as the product of two quadratics we can use Gauss' lemma for O_K (which just so happens to be a UFD). Modding out by $\mathfrak{p}$ we would conclude that x^4+3x+3 is irreducible.

(4/2/2018) Today I learned about the general proof that each ideal group is a class group of some algebraic number field. First off, I learned that this can be reduced to assuming the field in question has roots of unity of order "the index of the ideal class group defined for some modulus $\mathfrak{m}$ in $I^{\mathfrak{m}}$." This can be done by showing that any cyclic extension, through the norm map, if the "upstairs" extension has a class group then the "norm" of that ideal also has a class group.

(4/3/2018) Today I learned how to show that ideal groups operate like rational functions. Namely, if you have two ideal groups and they agree on some modulus, that they must agree on the greatest common divisor (which corresponds to the "union") of the two moduli. Essentially you can use the fact that if $\mathfrak{n}|\mathfrak{m}$ then restriction is unique–i.e. any congruence subgroup defined modulo $\mathfrak{n}$ that restricts to some congruence subgroup H modulo $\mathfrak{m}$ must be $H\iota(K_{\mathfrak{n},1}$ to define what the congruence subgroup should be. Then you can construct a unit subject to certain congruence conditions to show that what the congruence subgroup defined on the greatest common divisor should be, is truly the congruence subgroup which restricts to what we need it to restrict.

(4/4/2018) Today I learned more hardcore into the proof of how the existence of a class field for any ideal group is proven. In particular, after using the norm map described two minutes ago you can reduce it to showing that the fact when your your ideal group has exponent n in the group of ideals besides some modulus the ideal group is defined on. From there you can pick the set of ideals S which are on the modulus as well as any prime ideals which divide p, ∞, or any element of a set of generators in the class group. Then the class field of $\iota(K_{\mathfrak{m},1})K^n$ is K adjoining the S units.

(4/5/2018) Today I finished the proof of the existence theorem, which basically reduces to showing that the index $[K^* : K^n K_{\mathfrak{m},1}] = n^{|S|}$ where S is a finite set of prime ideals with some appropriate powers. This can be done by first showing that $[K^* : K^n K_{\mathfrak{m}}] = n$ and then showing that $K^n K_{\mathfrak{m}}/K^n K_{\mathfrak{m},1}$ can be realized as an appropriate unit group. Then you can locally compute the index by explicitly computing H^1 with the trivial action and the Herbrand quotient.

(4/6/2018) Today I learned the basics of why there is a filtration on $Sym^r(F)$ for any quasicoherent sheaves F, F', F'' which can be written as $Sym^r(F) = G_0 \supset G_1 \supset ... \supset G_{r+1} = 0$ so that $G_i/G_{i+1} = Sym^i F' \otimes Sym^{r-i} F''$ if there is a short exact sequence $0 \to F' \to F \to F'' \to 0$. This is essentially because you can write any element of $Sym^r(A^{n+m})$ as a linear combination of basis elements which are of the form "the first n basis elements come first and then the next m come last". This isomorphism is independent of any basis choices, so it gives a canonical map to the direct sum $\oplus_{i=0}^r Sym^i F' \otimes Sym^{r-i} F''$ which in turn, by letting G^n be the summands from n to r, provides your filtration.

(4/7/2018) Today I learned about *gonality* of a curve C, which is the minimal degree of a rational map $C \to \mathbb{P}^1$. I learned that a curve is rational if and only if this degree is one. I also learned this measure is not a good measure for higher dimensional spaces because we would like $\mathbb{P}_1 \times \mathbb{P}_1$ to be rational too, so we make a new definition, called the irrationality index. Also I learned about the transfer morphism, which is a morphism induced by the corestriction map $Hom(H, \mathbb{Q}/\mathbb{Z}) \to Hom(G, \mathbb{Q}/\mathbb{Z})$ (on which G, a finite group, acts trivially) by identifying G and H with their double duals.

(4/8/2018) Today I learned the idea behind the fact that all rational points $P = (x_0, y_0)$ of finite order on an elliptic curve given by $y^2 = x^3 + ax^2 + bx + c$ with integer a, b, c are integer points themselves, i.e. $x_0, y_0 \in \mathbb{Z}$. Essentially you show that no prime can divide the denominator. To do this, you change coordinates $(x, y) \to (t, s) = (\frac{x}{y}, \frac{1}{y})$ which induces a homomorphism from the set of rational points on the elliptic curve for which the order of the x coordinate is less than some integer $2v$, say $C(p^v)$, and show you have an induced injective homomorphism $C(p^v)/C(p^{3v}) \to \mathbb{Z}/p^2\mathbb{Z}$. From there you use a descent method showing that if a point in $C(p^v)$ had finite order then by looking at its image you can see that that point is also in $C(p^{3v})$ for all v, i.e. it must be the point at infinity.

(4/9/2018) Today I learned that given a coherent sheaf O_X and two coherent sheves $F.G$ that the Hom sheaf between them is also a coherent sheaf. This is because for any quasicoherent sheaves you can localize to the point where they are both sheaves of the form $\widetilde{M}$, in which case the Hom sheaf as a module is isomorphic to just the module maps between the underlying M's. This uses coherence, which I learned is necessary, and shows that the functors $Hom(F, -)$ and $Hom(-, G)$ are left exact functors.

(4/11/2018) Today I learned what a zero dimensional Gorenstein ring is, at least over $\mathbb{C}$. Essentially what it is is a quotient of $\mathbb{C}[x_0, ..., x_n]$ by a homogeneous ideal (where n and later d are implicitly given in the definition) for which the quotient is a graded $\mathbb{C}$ vector space $R_0 \oplus ... \oplus R_d$ such that there is a perfect pairing of vector spaces $R_i \times R_{d-i} \to R_d$ (meaning that the map $R_i \to Hom(R_{d-i}, R_d)$ is an isomorphism). These can be used to parametrize something called a secant variety.

(4/12/2018) Today I learned the proof of Chebotarev's density theorem, which says that given any Galois extension of algebraic number fields E/K and any $\sigma \in G := Gal(E/K)$, the Dirichlet density of the set of primes of K with a prime lying above it whose Frobenius is conjugate to σ is $\frac{c}{|G|}$ where there are c conjugates of σ. This improves upon the Frobenius density theorem, which only gives the similar result but for the density of primes conjugate to some *generator* of $\langle\sigma\rangle$. The proof uses the existence of a class field to argue that in the abelian extension E/E^σ the set of primes of E^σ having Artin map equal to σ has Dirichlet density exactly $\frac{1}{|\sigma|}$ since this corresponds to exactly one coset of the class group. Using the factorization in nonnormal extensions from a larger Galois extension lemma, you can show this implies that up to a set of Dirichlet density zero there are $[C_G(\sigma) : \langle\sigma\rangle]$'s worth of primes of $E^{\langle\sigma\rangle}$ for each prime of K, which shows the theorem.

(4/13/2018) Today I learned why it was that, given a congruence subgroup H of an algebraic number field K defined on some modulus $\mathfrak{m}$ then if E/K is cyclic then the if group $H_E := \{\mathfrak{U} \in I^{\mathfrak{m}_E : N_{E/K}(\mathfrak{U}) \in H}$, which is necessarily a congruence subgroup because $N_{E/K}(E_{\mathfrak{m},1}) \subseteq K_{\mathfrak{m},1}$, has a class field L then H also has a class field. The key to proving this is to show that L/K is still an abelian extension. To show it is Galois, note that for any automorphism $\tau : L \hookrightarrow \mathbb{C}$ fixing L we have that $\tau N_{L/E} \tau^{-1} = N_{\tau(L)/\tau(E)}$ because conjugation induces an automorphism of Galois groups and further that $\tau(E) = E$, so we have that $N_{\tau(L)/E} I^{\mathfrak{m}}_{\tau(L)} = \tau(N_{L/E}(I^{\mathfrak{m}}_E) \subseteq \tau(H_E) = H_E = ker(\phi_{L/E})$ which says that the primes which split completely in $\tau(L)$ are in L, i.e. $L \subseteq \tau(L)$.

(4/14/2018) Today I learned a really cool proof that if F is a finite rank locally free sheaf and G is a quasicoherent sheaf on a scheme X then we have an isomorphism $Hom(F, G) \cong F^\vee \otimes G$. The map is given by taking $\psi : F \to G$, choosing a basis (and thus identifying $F \cong A^n$ for some ring A and some $n \in \mathbb{N}$ you can view the map ψ as $\sum_i e_i^\vee \otimes \psi(e_i)$, and then you have to show that in order to say this defines a map of schemes that this is independent of base change. But you can explicitly compute the inverse which involves the sum of three indices, but it turns out that a pair of them correspond to the identity so that we only end up summing over one, even after base change, and get the same answer coming out!

(4/15/2018) Today I learned a bunch of applications to the "geometric Nakayama's lemma". Namely, on any quasicompact regular curve there is a sheaf called the torsion sheaf which locally quotients to a free sheaf. I also learned that given any finite morphism of schemes $\pi : X \to Y$, you can define the *degree* of the morphism at a point $y \in Y$ to be the rank of the stalk of the preimage sheaf as a sheaf on Y, which coincides with the standard definition.

(4/16/2018) Today I learned about the vector bundles $O(n)$ that you can put on $\mathbb{P}^n_k$ for a field k. It's trivial on the standard open cover, and the transition functions from the places where i doesn't vanish to the places where j doesn't is multiplication by $(\frac{x_j}{x_i})^n$. I also proved that on $\mathbb{P}^1_k$, these are the only possible invertible sheaves. This stems from the fact that they must be trivial on the two standard open sets which you can prove from the structure theorem for modules over a PID. Then you can show the transition function must take 1 to a unit of a ring isomorphic to $k[z]_z$, which are simply scalars of powers of z.

(4/17/2018) Today I literally learned that matrix multiplication was defined that way to be composition of functions. Like, I can't believe I didn't think of that earlier, but yeah, that's why it's defined the way it is.

(4/18/2018) Today I learned that on a Noetherian normal scheme the map of pairs of a line bundle and a nonzero rational section on that line bundle modulo isomorphism to its divisor group is a well defined injective group homomorphism. It's a homomorphism because the order of the product of two things becomes a sum of the orders, and it's injective because if there's any section s with no poles and no zeroes, you can use the fact that the scheme is normal to argue that s and $1/s$ must both be global sections, and from there you can use the isomorphism "multiplciation by s": $O_X \to$your line bundle to show the isomorphism.

(4/20/2018) Today I learned that for any line bundle L on an integral normal scheme X and a nonzero rational section on the line bundle, say s, there is a basis of the topological space of affine open sets U such that each U is affine and for each U there is an isomorphism between $O_X(div(s))|_U \cong O_U$. This is essentially because after you take an affine open set $Spec(A)$ of your favorite point then you can argue that your nonzero rational section must actually be a nonzero element of $Spec(A_g)$ for some $g \in A$. After tracing through the definitions one can then see that $O_X(Spec(A)) = g^p/aA$ after choosing some identification $L(Spec(A)) \to A$ whose associated map of sheaves ends $s \to a/g^p$.

(4/21/2018) Today I learned that any line bundle of the form $O(D)$ for a divisor D on a Noetherian factorial scheme (i.e. a scheme where all stalks are UFDs) is actually locally principal. To see this, note that it suffices to show that each point has an open set around which the bundle is locally principal, so a lot of points are already covered by the fact that any point not in any of the irreducible components in the support of D is already covered. For the points in D, note that it suffices to show it for an irreducible divisor because "localizing" gets us away from the other points and $O(n[Y]) = \otimes_{i=1}^n O([Y])$. Then fixing a random p on one of the irreducible components, you can use that the prime ideal associated to any irreducible codimension one subset Y must be principal since it's a codimension one prime ideal in a UFD. You can argue then that any generator can't have a zero at any other codimension one subset containing p by working in the local ring, and then localize away from every other zero and pole.

(4/22/2018) Today I worked through a diagram which essentially says that locally principal divisors are also the group of line bundles with designated section, with isomorphisms that carry the designated section to the designated section, and that once you mod out by the principal bundles on the other side you get the designated rational section. This says the Picard group injects into the class group. I also learned that it is unknown whether F_2, the free group on two generators, can be recovered by its finite quotients. That is, is there a group G which isn't F_2 for which the finite groups G surjects onto are the same that F_2 does.

(4/23/2018) Today I learned why on a hypersurface of degree d larger than one no hyperplane can be cut out by a single equation. This is because the excision exact sequence for class groups shows that the hyperplane must have class group $\mathbb{Z}/d\mathbb{Z}$ and because the map from the class group of $\mathbb{P}_k^n$ is surjective, it must be generated by the hyperplane class. However, if the hyperplane was cut out by a single equation in the affine scheme, it would be trivial in this group.

(4/25/2018) Today I learned about the existence of a local Artin map, which is defined at a prime $\mathfrak{p}$ of an algebraic number field K by taking an element of the field $K_{\mathfrak{p}}^\times$ and approximating it by a modulus $\mathfrak{m}$ such that $\mathfrak{p}^a\mathfrak{m}$ has the reciprocity law holding and then taking that approximation and killing off any primes in $\mathfrak{m}\mathfrak{p}$ and then applying the regular Artin map. I found out that because the kernel of this local Artin map for an abelian extension L/K is given by $N_{\mathfrak{p}}(L_\beta^\times)$ where $\beta|\mathfrak{p}$, this, and the fact that all primes with "local conductor" 1 must be unramified, implies that any prime which ramifies must divide the conductor $\mathfrak{f}(L/K)$.

(4/26/2018) Today I learned what an affine manifold is, which is just a manifold whose transition functions are affine transformations. I learned about a theorem of Bieberbach which says that any Euclidean manifold (i.e. whose transition functions are affine with zero translation) must be covered by some torus, so they in particular have Euler characteristic zero.

(4/27/2018) Today I learned that the class group of the spectrum of an integrally closed domain A is the only obstruction to A being a unique factorization domain. Namely, if A is an integrally closed domain with zero class group, it is a UFD. To prove this, one can write show that each codimension 1 prime ideal is principal, and by an application of Krull's theorem, this shows that all irreducible elements are prime and thus gives unique factorization. To show any codimension one prime ideal is principal, pick one, and by assumption that the class group is 0, we can write it as $div(f)$ for some f which a priori is in the field of fractions but after applying algebraic Hartog's lemma is in A. Then (f) is inside the prime and actually is a prime ideal, since if $fj = gh$ for some $g, h, j \in A$ then without loss of generality g vanishes at our prime in question which in turn implies that $div(g/f) \geq 0$ so $f|g$.

(4/28/2018) Today I learned about local conductors and their relation to ramification. In particular, you can show that if every unit of a completed extension $L_\beta/K_\mathfrak{p}$ is a norm (where $\mathfrak{p}$ is a prime of K with β lying over it) then $\mathfrak{p}$ cannot ramify in L. What happens is that the inertia degree is essentially taking up as much as it possibly can which forces the ramification to take the minimal value it possibly can, i.e. 1. You can use this to show that every ramified prime has to divide the conductor because you can use the relations of various congruence subgroups in an ideal class group to show that any unit in the local field has local Artin map evaluate to zero and that the kernel of the Artin map is the unit group.

(4/30/2018) Today I learned about the Hilbert class field of an algebraic number field, which is the maximal abelian unramified extension of that number field. It's also the class field of the "global" function, and moreover using the fact that a certain morphism, called the transfer morphism, is always trivial when the domain and codomain are $G \to G/G'$, you can show that every ideal in the ground field becomes principal in the Hilbert class field!

March 2018

(3/1/2018) Today I learned how to compute for moduli $\mathfrak{m}$ that the index $a(\mathfrak{m}) := [K^* : N(L^*)K_{\mathfrak{m},1}] = \prod_{\mathfrak{p}|\mathfrak{m}} e_\mathfrak{p} f_\mathfrak{p}$ for a cyclic extension L/K. Essentially the first thing one does is separates the problem using a "Chinese remainder theorem" like tool which gives an isomorphism of groups that shows if $\mathfrak{m}, \mathfrak{n}$ are relatively prime moduli, then $a(\mathfrak{m}\mathfrak{n}) = a(\mathfrak{m})a(\mathfrak{n})$. This basically rests on the fact that $N(L_{\mathfrak{m},1}) \subseteq K_{\mathfrak{m},1}$. The infinite case is easy, and the finite prime case can be reduced with an isomorphism to the group of units. From there, you can use the fact for a prime power $\mathfrak{p}^n$ with $\mathfrak{q}$ a prime of L lying over $\mathfrak{p}$, if n is large enough so that *exp* and *log* are mutually inverse isomorphisms from the additive group $\mathfrak{p}^n$ to the set of units congruent to 1 mod $\mathfrak{p}^n$ to show that any element there is a $[L_\mathfrak{q} : K_\mathfrak{p}]^{th}$ power, i.e. if n is large enough we have reduced the problem to computing the index of scalars modulo just norms, and *this* is a homology group, so the homology can be computed with homological methods (and the normal basis theorem!)

(3/2/2018) Today I learned the proof of the Hasse Norm Theorem, which says that any element of a cyclic extension L/K is a norm if and only if it is a local norm for every prime of K. The proof that a norm is a local norm of, say, $\mathfrak{p}$ with $\mathfrak{q}$ lying over it follows quickly from taking coset representatives of $G(\mathfrak{q})$. On the other hand, assuming that an element α is locally a norm, you can first show that it's the norm of some ideal of L. Then you can find an element γ for which $\alpha N_{L/K}(\gamma)^{-1} \in K_{\mathfrak{m},1}$, which we can appeal to the proof of the fundamental inequality for cyclic extensions which shows us that this implies that $\alpha N_{L/K}(\gamma)^{-1}$, and thus α, actually is the norm of some element. I also learned that this theorem is not true for noncyclic extensions.

(3/3/2018) Today I learned that it is unknown, but conjectured, that for every $n \in \mathbb{N}$ and every prime p the field $\mathbb{Q}(\mu_{p^n})$ has class number one. As evidence for this, I learned that Iwasawa proved for all p the class number of the field is prime to p. I also learned that the odd negative values of the zeta function relate to the field $\mathbb{Q}(\mu_p) \cap \mathbb{R}$ for a prime p. I also learned what the cyclotomic character of a Galois group G_F is, it's just a map $G_F \to \mathbb{Z}_p$ which encodes all of the actions of a given $\sigma \in G_F$ on all the roots of unity. Additionally, I learned what a Tate twist is, namely, given any Galois representation V we can tensor it with the above representation to get the twist $V(1)$. We can also recursively define $V(m) := V(m-1)(1)$ and get the positive Tate twist for all positive

m. Finally, you can use the dual representation of the cyclotomic character (which I'm not too sure what it is right now) to get $V(-1)$.

(3/4/2018) Today I learned another statement of the theory of Iwasawa, which in particular says that if you have any $\mathbb{Z}_p$ extension then there are positive integers μ, λ and an integer ν such that the p torsion of the class group of the n^{th} layer has order $p^{\mu n + \lambda p^n + \nu}$ and that conjecturally for the cyclotomic extension we always have that $\lambda = 0$.

(3/5/2018) Today I learned why if $\mathbb{Q}_\infty/\mathbb{Q}$ is the cyclotomic $\mathbb{Z}_p$ extension, then each prime $q \in \mathbb{Q}$ has only finitely many primes in $\mathbb{Q}_\infty$ lying above it. For $q = p$ this follows because there is literally only one prime. For $q \neq p$, viewing $\mathbb{Q}_\infty$ as a tower of subfields $\mathbb{Q}_0 \subseteq \mathbb{Q}_1 \subseteq ...$, we first claim that there is a field $\mathbb{Q}_i$ for which the primes don't split completely upon passage to $\mathbb{Q}_{i+1}$. If this were not true, then we would have splitting completely forever. Picking $\mathfrak{q}_1$ lying above q and $\mathfrak{q}_i$ lying above $\mathfrak{q}_{i-1}$ recursively, we then obtain that $(\mathbb{Q}_i)_{\mathfrak{q}_i} \cong \mathbb{Q}_q$, since there is no residue field degree or ramification in the extension. However, note that $\mathbb{Q}_q$ has only finitely many roots of unity by the log map, and thus, this can't happen forever since $\mathbb{Q}[\zeta_{p^n}]$ as $n \to \infty$ contains infinitely many roots of unity. Thus there is an i which don't split at the next stage. Let $\mathfrak{q}_i$ be a prime lying above $q\mathbb{Z}$ in that field. Then if it split completely again, say at stage j, this would be a contradiction since in the finite extension $Gal(\mathbb{Q}_j/\mathbb{Q}_i)$, the decomposition group of a prime $\mathfrak{q}_j|\mathfrak{q}_i$, say $D(\mathfrak{q}_j)$, must have associated fixed field larger than $\mathbb{Q}_i$ itself. But this would imply that $\mathbb{Q}_{i+1} \subseteq \mathbb{Q}_i^{D(\mathfrak{q}_j)}$, and since the prime $\mathfrak{q}_i$ splits completely in the larger field, it would split completely in $\mathbb{Q}_{i+1}$. Thus splitting stops, and by prime decomposition of abelian extensions, once splitting stops, it stops for good.

(3/6/2018) Today I learned why, with the usual Iwasawa setup, there is an equivalence for a torsion $\Lambda[[\Gamma]]$ module X to have finite Γ fixed points, covariance X_Γ, and having the characteristic polynomial of the ideal not have a 0 at 0. What happens is that you can take either the fixed point functor or the covariance functor, which are left and right exact respectively, and this will shortly give you some of the implications. The fact that finite Γ covariance yields that T isn't in the characteristic ideal is obtained by taking the covariance of both sides and noting that one of the factors is $\mathbb{Z}_p$, so there are no elements of finite order that aren't already in the kernel. This also involves recognizing covariance as the same thing as that which is annihilated by T, the variable in the power series ring.

(3/7/2018) Today I learned where all of the extra units are going in the math of reciprocity laws. Specifically, if L/K is an abelian extension then we know the reciprocity law holds for some modulus $\mathfrak{m}$ of K containing all of the primes that ramify to sufficiently high powers. Specifically, the weirdness is that if we have a new modulus $\mathfrak{n}$ which raises those primes to even higher powers, we have the curious equality $\iota(K_{\mathfrak{m},1}N(I_L^{\mathfrak{m}}) = \iota(K_{\mathfrak{n},1}N(I_L^{\mathfrak{m}})$ which seems like it should depend on the exponents. But I figured out what's really happening here. We showed that for any prime which is unramified and any prime lying over it (in a cyclic Galois extension, at least), each element is a local norm. For the ramified primes, you can raise to high enough powers so that they're d^{th} roots, and for a proof of the Hasse Norm Theorem, you can show that any element that is locally a norm everywhere is actually the norm of some ideal. Thus you're just turning up all the ramified primes until they become norms of ideals by raising the exponents of $\mathfrak{m}$.

(3/8/2018) Today I learned some of the theory for why, given a cyclic extension L/K of algebraic number fields, if the fundamental equality holds for a given modulus then the reciprocity law holds for that modulus $\mathfrak{m}$. This is done (at least in Janusz's book) by showing that the kernel of the Artin map when restricted to $I_K^{\mathfrak{m}}$ is contained inside the group of norms and one-y units. This proves the claim because the kernel of the Artin map has index $[L : K]$ (since, regardless of the modulus, the Artin map is still surjective and so the 1st isomorphism theorem gives this), and assuming the fundamental equality holds, the groups are equal.

(3/9/2018) Today I learned how you can prove part of the Artin Reciprocity Theorem for a cyclic extension L/K. As in yesterday, if you assume that you have a prime power $\mathfrak{p}^a$ in the kernel of the Artin map, you can use a little bit of elementary number theory to find a primitive root of unity θ_m for which $\mathbb{Q}[\theta_m] \cap L = \mathbb{Q}$. From there you can argue that $K[\theta_m] \cap L = K$ for degree reasons, and there you can argue that your prime $\mathfrak{p}$ is the norm of the fixed field F of the product of the Artin maps of the two intermediate fields applied to $\mathfrak{p}$. From there, you can also argue that L is a subfield of $F[\theta_m]$, and since you know the reciprocity law holds for $F[\theta_m]/F$, you can then translate down to the extension L/K using the Galois sliding lemma/how it applies to Artin maps.

(3/10/2018) Today I learned how given any abelian extension L/K you can prove the Artin reciprocity theorem for L/K assuming you have it for cyclic extensions. Namely, write $Gal(L/K)$ as the product of cyclic extensions, and find each field fixed by all but one factor of this product. Then these Galois groups are cyclic, and you can find a modulus for which the reciprocity law holds here. Letting $\mathfrak{m}$ be the product of all these moduli, we can see that for any $\alpha \in K_{\mathfrak{m},1}$, then the restriction of $\phi_{L/K}(\alpha)$ to these cyclic extensions are trivial, but this implies $\phi_{L/K}(\alpha)$ is trivial itself.

(3/11/2018) Today I learned that the Kronecker-Weber theorem doesn't hold for any quadratic extension. Specifically, there exists a quadratic extension of $\mathbb{Q}[\sqrt{D}] \neq \mathbb{Q}$ which isn't Galois over $\mathbb{Q}$, which in particular says it can't be contained in a cyclotomic extension. This is because we may immediately assume that D is a squarefree integer, and if D is an integer that isn't -1, then $\mathbb{Q}[\sqrt[4]{D}]$ does the trick since it isn't Galois over $\mathbb{Q}$. (For the case where $D > 0$ this follows since $i \notin \mathbb{Q}[\sqrt[4]{D}]$ and for the complex case you can show the group has to be the Klein four group but this would imply that $(\sqrt[4]{D} + i\sqrt[4]{D})(-\sqrt[4]{D} - i\sqrt[4]{D}) \in \mathbb{Q}$ since it's fixed by every element of the Galois group. You can do similar tricks for $\mathbb{Q}[i]$ with the extension $\mathbb{Q}[\sqrt{1+i}]$.

(3/12/2018) Today I learned the fact, and an application of the fact, that $\sum_{i=0}^{n}(-1)^i$n choose i= 0. This can be proven by induction and works exactly like you'd think it works. Furthermore, this essentially implies that if you sum the primitive $n = p_1...p_s$ roots of unity for a squarefree n, then you get $(-1)^s$, which you can also show by induction based off this. Because of these two facts, you can argue that 1 is in the $\mathbb{Z}$ span of the primitive n^{th} roots of unity and from there you can fancy up that argument a little more to argue that any n^{th} root of unity is in the $\mathbb{Z}$ span of the primitive n^{th} roots of unity. In fancier terms, this is saying that for any squarefree n, any primitive n^{th} root of unity gives an *integral basis* for the n^{th} cyclotomic field. Furthermore, since $1 \notin \mathbb{Z}i + (-i)\mathbb{Z}$, I learned that the squarefree hypothesis is necessary.

(3/13/2018) Today I learned what the class group of an abelian extension of an algebraic number field L/K is! Specifically, for each modulus $\mathfrak{m}$ you can define a *congruence subgroup* as a group that could be thought of as a "group of the ray class group", namely, some H which fits in $\iota(K_{\mathfrak{m},1}) \subseteq H \subseteq I^{\mathfrak{m}}$. Furthermore, if $\mathfrak{n}|\mathfrak{m}$, then $I^{\mathfrak{m}} \subseteq I^{\mathfrak{n}}$ and so we can define the *restriction* of a congruence subgroup to be the intersection $H \cap I^{\mathfrak{n}}$. It turns out that if there is an element restricting to a congruence subgroup, it is unique, and furthermore if there is a congruence subgroup for two moduli $\mathfrak{m}_1, \mathfrak{m}_2$ then there is a congruence subgroup for the greatest common divisor of the two. This says that declaring two congruence subgroups equivalent if if they restrict to some modulus to the same congruence subgroup is an equivalence relation, and furthermore the *class group* for L/K is the ideal group given by the kernel of the Artin map on any modulus for which the reciprocity law holds.

(3/14/2018) Today I learned that any abelian *tamely ramified* extension $L/\mathbb{Q}$ of a number field (that is, an extension for which $p \not| e(\mathfrak{p}/p)$ for any prime $\mathfrak{p} \subseteq L$ lying over p) is contained in a cyclotomic extension $\mathbb{Q}[\theta_m]$ where θ_m is a primitive root of unity *and m is squarefree.* This is obtained by using the Kronecker Weber theorem to obtain that $L \subseteq \mathbb{Q}[\theta_n]$ for some $n \in \mathbb{N}$ not necessarily squarefree. Then you can argue that there is a natural map from the inertia group of a prime lying over p of $Gal(\mathbb{Q}[\theta_n]/\mathbb{Q})$ to the inertia group of a prime lying over p of $Gal(K/\mathbb{Q})$ given by restriction. This in turn implies that by assumption $p \not|$ the order of the inertia group, we see that any element of $Gal(\mathbb{Q}[\theta_{p^a}]/\mathbb{Q}[\theta_p])$ fixes L, which says that the associated product of all such groups, indexed over the primes $p|n$, fixes L, so $L \subseteq \mathbb{Q}[\theta_m]$ for m the product of primes dividing n.

(3/16/2018) Today I learned a criterion to determine whether a given O_X module (where X is a scheme) is a quasicoherent sheaf. Namely, a module F is a quasicoherent sheaf if and only if the map $F(Spec(A))_f \to F(Spec(A_f))$ induced by mapping to an A_f module is an isomorphism for all such maps. The reason this is true is that one can locally define an isomorphism of $F|_{Spec(A)}$ by merely defining the map of schemes $F|_{Spec(A)} \to F(\widetilde{Spec(A)})$ for each affine open $Spec(A)$, and then noting that the maps glue since for each pair of affine open sets their intersection is the union of sets which are distinguished affine open sets for both rings.

(3/17/2018) Today I learned (or reviewed, I'm honestly not sure) why given a map of sheaves $\phi : F \to G$ that the kernel of the stalk is the stalk of the cokernel. This essentially follows from the fact that you can literally view the kernel of a map as a "subsheaf" of F, and from there you can argue that the stalk at a point can be "included" into F, which makes a natural injection to the kernel of the map $F_p \to G_p$, which is the map defined on stalks by taking an $f \in F(U)$ to $\phi(U)(f)$. The surjectivity part essentially comes from the fact that if your stalk is zero then there's some open set on which you're zero.

(3/18/2018) Today I learned the fact that you can check the exactness of a complex of sheaves at the level of stalks. The reason for this is that it turns out in any abelian category if $A \to B \to C$ is a complex, then $A \to im(A)$ is an epimorphism, which implies that you get a map $im(A) \to ker(B \to C)$. And there you use the fact that a map is an isomorphism if and only if the induced map on each stalk is an isomorphism, which to prove injectivity on each open set you can use sheaf properties and then to verify surjectivity you actually use injectivity to verify that gluability is possible to get surjectivity on each open set.

(3/19/2018) Today I learned that just because the tensor product presheaf is locally a sheaf that it doesn't imply that the tensor product is actually a sheaf, as the separated presheaf of "functions which have a logarithm" shows. I also learned that locally free sheaves are a strictly larger class than vector bundles–i.e. locally free sheaves don't necessarily determine a vector bundle. To see this, note that the injection of $k[t]$ modules $k[t] \to tk[t]$ obtained by multiplication by t yields an isomorphism of $k[t]$ modules. However, if we try to interpret this as sections, at $t = 0$ we only have the zero function but for $t \neq 0$ we have a whole lines worth of functions, so this can't be a vector bundle.

(3/20/2018) Today I learned that any section on any quasicoherent sheaf F defined on the set where a fixed section s of a line bundle L doesn't vanish can be realized as the quotient by s of a global section. What happens is that locally you can naively define the map by just taking the transition function of the line bundle to the ring and divide by s and then multiply your section of F by this element of the ring. This is well defined because the transition functions are just multiplication by some unit in the ring $u_{\alpha\beta}$ so it essentially uses the fact that $g_\alpha/s_\alpha = g_\alpha u_{\alpha\beta}/s_\alpha u_{\alpha\beta} = g_\beta/s_\beta$.

(3/21/2018) Today I learned *Shapiro's Lemma*, which says that if A is an H module and H is a subgroup of a subgroup G then for all $n \in \mathbb{N}, H^n(G, M_H^G(A)) = H^n(H, A)$. This almost all comes from an isomorphism of abelian groups $Hom_{\mathbb{Z}G}(P, Hom_{\mathbb{Z}H}(\mathbb{Z}G, A)) \to Hom_{\mathbb{Z}H}(P, A)$ given by $f \to (p \to f(p)(1))$.

(3/22/2018) Today I learned a simple lemma which says given two extensions of algebraic number fields $L, K/F$ where K/F is Galois then ramification at a prime $\mathfrak{p} \subseteq L$ in the composite KL occurs only if ramification at $\mathfrak{p} \cap F$ in K, which can be proven simply by noting that restriction map injects the inertia group of a prime into the inertia group of the prime lying below it. Two consequences of this help in the proof of Iwasawa's theorem–namely, if $K \subseteq K_1 \subseteq ...$ and L_i is the maximal abelian unramified $p-$extension of K_i, then $L_iK_{i+1} \subseteq L_{i+1}$, so in particular $L_i \subseteq L_{i+1}$. You can also show that eventually $L_i \cap K_{i+1} = K_i$ which implies an isomorphism of Galois groups and furthermore that you can realize $Gal(\bigcup_i L_i/K_\infty)$ as a colimit of the groups $Gal(L_i/K_i)$.

(3/23/2018) Today I learned that given a map of abelian groups $A \to A'$ where A, A' are G, G' modules for some groups G, G' respectively given a map $G' \to G$ you can talk about whether the morphisms are *compatible*, namely, that if one realizes A as a G' under the map $G' \to G$ whether the map $A \to A'$ is a G' module morphism. If the morphism is compatible, then in particular not only does one obtain a map $Hom(G^n, A) \to Hom(G'^n, A')$ (by precomposing with the map $G' \to Gn$ times and post composing with $A \to A'$) but furthermore this map commutes with the differential map, and so in particular it induces a map on homology!

(3/25/2018) Today I learned the overview of the proof of a theorem of Iwasawa which says that for a $\mathbb{Z}_p$ extension K_∞/K that there are constants $\mu, \lambda \geq 0, \nu$ such that the order of the $p-$primary part of the class group at the n^{th} field extension is $p^{\lambda p + \mu p^n + \nu}$. This is essentially because you can use class field theory to relate this class group to the maximal unramified abelian $p-$extension of K_n and you can take the composite of all of these fields, say L, to obtain an extension Galois over not only K_∞ (which happens to have an abelian Galois group), but also K. From this, you can argue that the abelianness of both groups makes $X := Gal(L/K_\infty)$ into a $\Gamma \cong \mathbb{Z}_p$ module. You can also show that this module quotients with a well behaved quotient to $A_n^{(p)}$, the p primary part of the class group, and that you can compute the order using this and homological algebra for sufficiently large n.

(3/26/2018) Today I learned the overview of the proof of a cool lemma which says that given any psuedoisomorphism of Λ modules $Y \to E$, where E has a direct summand as in the structure theorem of finitely generated Λ modules and assuming $Y/\frac{\Phi_{p^n}(1+T)}{T}Y$ is finite, then there is a constant p^c such that eventually for n large enough the order of $Y/\frac{\Phi_{p^n}(1+T)}{T}Y$ is p^c times the order of $E/\frac{\Phi_{p^n}(1+T)}{T}E$, which can be computed explicitly. The reason this is true is because one can show that the ranks of the various kernels and cokernels of the maps on the n^{th} level are bounded and strictly increasing or decreasing using basic spectral theory and basic set theoretic arguments which implies that those ranks must stabilize!

(3/27/2018) Today I learned that any continuous bijection of profinite groups is automatically a group homeomorphism (the fact, although not the proof yet). I also learned that in the proof of the Iwasawa growth formula that if you take K_∞/K as a $\mathbb{Z}_p$ extension and L_∞/K as the maximal abelian unramified $p-$extension then it turns out that $Gal(L/K)$ is the semidirect product of the normal subgroup $Gal(L/K_\infty)$ times Γ, which can be identified with any choice of inertia subgroup. Using this, one can show that if L_n is the maximal abelian unramified extension of K_n, the unique subfield of K_∞ whose Galois group is Γ^{p^n} then $Gal(L_0/K_0)$ is a certain quotient of X (*not* just of $Gal(L/K)$) and moreover $Gal(L_n/K_n)$ is just a bit thinner quotient.

(3/28/2018) Today I learned that you can use the fact that conjugating an element x in a subgroup by an element of $\gamma_0^j \in I_1$ can be realized as an element as the product of a commutator and x. This allows you to take the quotient you need in the proof of the Iwasawa proof.

(3/29/2018) Today I learned that if K_n/K is an cyclic p^n extension and L_n is the maximal abelian unramified extension of K_n then L_n is actually Galois over K. This is true because if M is the Galois closure of L over K where $L = K(\theta_1)$ for some θ_1 with K Galois conjugates $\theta_2, ..., \theta_n$ then you have an injection $Gal(M/K) \to \prod_{i=1}^n Gal(K(\theta_i)/K_n)$ (which, by the way, you can show $K_n \subseteq K(\theta_i)$ is by the Galoisness of K_n) and you can show by conjugation that $Gal(K(\theta_i)/K_n) \cong Gal(K(\theta_1)/K_n)$ and that any prime of K_n can't ramify.

(3/30/2018) Today I learned a fact which I thought was obvious but wasn't as obvious as I had hoped. Namely, I learned that for all finite extensions of algebraic number fields L/K there exists some n where if m has prime factors all prime factors larger than n, then $L \cap K(\zeta_m) = K$. This is true when $K = \mathbb{Q}$ by choosing n larger than all of the primes that ramify in L. This result can be translated to K by simply computing the group fixing $L \cap K(\zeta_m)$.

February 2018

(2/1/2018) Today I learned a whole bunch about gerrymandering, and different problems involving gerrymandering and ways to detect gerrymandering. For example, you can measure the *efficiency gap*, which specifically is an arithmetic way to measure the number of wasted votes for each party. I also learned that you can use a random walk to determine if a district is fair, and about a proposal system where those seeking to be elected "choose" their districts by people voting by dividing their votes up.

(2/5/2018) Today I learned that you can argue very easily from the fact that the stalk of the tensor product is the tensor product of the stalks (which you can basically argue directly), and the fact that the sheafification of a presheaf has stalks agreeing with the original presheaf, that taking the product with a locally free sheaf is exact. This is also because exactness of a sequence of sheaves can be checked locally. I also learned the beginnings of an isomorphism that says for O_X-modules F, G, H where H is locally free, $Hom(F \otimes H^\vee, G) \cong Hom(F, H \otimes G)$. Specifically I proved this for when F, G, H are actually just modules and H is free, which essentially this statement becomes the statement that $Hom(F, G^n) \cong Hom(F^n, G)$, which is becuase maps $F^n \to G$ are determined by how they operate on $\{0\} \times ... \times \{0\} \times F \times \{0\} \times ... \times \{0\}$.

(2/6/2018) Today I literally spent most of my way learning why $Hom(F \otimes H^\vee, G) \cong Hom(F, H \otimes G)$ is true (with the notation as in yesterday). It turns out that you can define the morphism in the exact way you would do it in the free case (which is not an obvious morphism by any means) and show that coordinates on vectors change in the same way that the dual coordinates change when you change coordinates and that's why the dual is necessary. Also I learned that you can pull back a locally free sheaf given a map of ringed spaces $\pi : X \to Y$ by using the gluing construction with transition functions given by making a diagram involving the old transition functions commute.

(2/7/2018) Today I learned that with the above construction of pullback of locally free sheaves, that Pic can be viewed as a functor from locally ringed spaces to abelian groups. Basically because the transition functions of the pullback of a locally free sheaf is essentially designed to make it so that a morphism $\pi : X \to Y$ gives a map on each locally open set level given by the induced map of rings, it gives a map of sets. I have also mostly proven it's an abelian group homomorphism, which basically involves meticulously defining everything and making sure that $\pi^*G_1 \otimes \pi^*G_2$ satisfies the gluing construction.

(2/8/2018) Today I learned finally why $Pic(\pi) : Pic(Y) \to Pic(X)$ gives a homomorphism of abelian groups. Basically you just argue that there are a bunch of unique maps making certain diagrams commute, and that the tensor product of the transition functions given by $\pi^*G_1 \otimes \pi^*G_2$ are the unique maps making the diagrams commute by how they operated on their own. I also learned the formal definition of the sheaf of sections on a locally free sheaf of rank n over a ringed space X, which is a vector of functions in $O_X^n(U_\alpha)$ for an open cover $\{U_\alpha\}_\alpha$ where the β coordinate can be obtained by doing the transition function of the locally free sheaf $\beta\alpha$ to the α coordinate.

(2/9/2018) Today I learned that you can think of fractional ideals of an algebraic number field (and moreover any Dedikind Domain) as an invertible sheaf on the spectrum of the ring of algebraic integers! In particular, you can define a sheaf structure for a fractional ideal I by declaring that on the open sets $D(f)$ where I_f is generated by one element, the sheaf is the ring $(O_K)_f$ and defining transition functions to be multiplication by the units by two different generators. This interpretation makes it clear that principal ideals are an equivalent notion to the invertible sheaf O_X, the identity in the Picard group. It even turns out that these are isomorphic groups!

(2/11/2018) Today I learned that a profinite group (which a priori was defined as the inverse limit of partially ordered finite groups subject to the standard "cocycle condition" on the maps between them) is also equivalently a totally disconnected Hausdorff compact topological group. In particular, today I proved that if you have a profinite group you can actually view that group as the inverse limit of finite groups where the maps from your profinite group to your "quotients" are all surjective (which follows by basically replacing the group with the image of the map) and so you can intrinsically view a profinite group as being built by a collection of normal subgroups of that group whose intersection is trivial.

(2/12/2018) Today I learned what an Iwasawa algebra was and a few basic facts about them. The Iwasawa algebra of a profinite group Λ is defined as the colimit of the group rings $\mathbb{Z}_p[\Lambda/\Lambda_i]$ where Λ_i ranges through a collection of normal subgroups whose intersection is trivial. I learned that with this construction if $\Lambda \cong \mathbb{Z}_p$ then its Iwasawa algebra is $\mathbb{Z}_p[[T]]$, the power series ring, and through the fact that $\mathbb{Z}_p[[T]]$ has division with remainder in some sense (which I am currently proving) we can get nice properties about this algebra.

(2/13/2018) Today I worked through an implication for a $\mathbb{Z}_p[[T]]$ module to act continuously on a finite abelian $p-$group A. Specifically, $T^nA = 0$ for some n. The reason that this makes sense (and why we want to define an action to be continuous with respect to formal power series) is because for each $a \in A$, we know there is some $b \in A$ such that $b = (1+T+T^2+...)a$. We would like to say by continuity that $b = \lim_{n\to\infty}(1 + T + ... + T^n)a$, but of course, there is no limit in a discrete topological space (at least, the obvious one is defined only for a metric space). However, one can show that the set of elements taking a to b is open using a topological basis argument, which says that there is some N with $n > N$ implying that $(1 + T + ... + T^n)$ is in the open set, which says that T^n eventually can't do anything to a except take it to 0.

(2/14/2018) Today I learned about the specific proof where given a F_∞/F, where F is an algebraic number field and $\Gamma := Gal(F_\infty/F) \cong \mathbb{Z}_p$ and given a γ_0 which does not fix the subgroup associated to $p\mathbb{Z}_p$, then the action of T defined by $T = \gamma_0 - id$ is topologically nilpotent, that is, for all elements of $a \in F_\infty, T^n a = 0$ for $n >> 0$. Basically you can first show by working through the definition that it suffices to show for some n that $T^n a = 0$. Originally I thought to show via a Chinese remainder theorem argument that expanding $(\gamma_0 - id)^{p^m}$ we would have coefficients all divisible by p^m. This was sad when I found out $p||\binom{p^m}{p^{m-1}}$. But then I realized that continuity still gives you that $\gamma_0^{p^m} = id$ for large enough so even though multiplication by T^{p^m} may not immediately yield 0, it does reduce the order of a, and then you can just keep applying the procedure to get 0.

(2/16/2018) Today I learned what might be called the fundamental theorem of finitely generated modules over the ring $\Lambda := \mathbb{Z}_p[[T]]$. This says that any finitely generated Λ module has a map to a direct sum of Λ's and Λ's modulo a distinguished polynomial. The way you can do show this, at least when you have no $\mathbb{Z}_p$ torsion and no free Λ components (I haven't read the proof in the general case yet) is to recognize that due to the fact you conveniently assumed there were no free Λ components, you can find a polynomial which kills your module. Then you can use the Weierstrass preparation theorem and, noting that you have no $\mathbb{Z}_p$ torsion, get a distinguished polynomial to be what kills your module. Then you can tensor with $\mathbb{Q}_p$ and then use the regular finitely generated modules over a PID theorem, along with a little lattice theory, to get your desired map.

(2/17/2018) Today I learned that the first cohomology of a cyclic Galois group $G := Gal(L/K) = \langle\sigma\rangle$ acting on $L^\times$ and acting on the set of prime ideals of O_L relatively prime to any given modulus is trivial. For the latter, you can argue that given any ideal whose norm is trivial (since in cohomology of cyclic groups acting on ideals, the map is the norm map) you must have the primes over any fixed prime raised to powers sum to zero, so you can argue that because of that you can construct an element $\mathfrak{U}$ with $\mathfrak{U}\sigma(\mathfrak{U})$ hitting those primes. Namely, for example, if your factorization was $\mathfrak{q}_1^2\mathfrak{q}_2^{-1}\mathfrak{q}_3^{-1}$, then $(id/\sigma)(\mathfrak{q}_1^2\mathfrak{q}_2)$ does the trick.

(2/19/2018) Today I learned about why a "continuous" Nakayama's lemma holds for pro-p groups X. Namely, this says that $X/(p,T)X = 0$ if and only if $X = 0$. This holds true because one can realize that if $X = (p,T)X$, then the $p-$primary group $A := Hom_{cont}(X, \mathbb{Q}_p/\mathbb{Z}_p)$, which has a natural Γ action on it by acting on the inside of the coordinates, has no $p-$torsion. Then you can use a lemma which states that A must be zero, and then through a semi-basic argument you can argue that any nonzero X has a nonzero map $X \to \mathbb{Q}_p/\mathbb{Z}_p$ by just using the quotient maps given.

(2/20/2018) Today I learned why the Kronecker-Weber Theorem is true, assuming the Artin reciprocity law holds. I also learned what the Artin reciprocity law says, which says in particular any abelian extension L/K has a modulus $\mathfrak{m}$ of K for which the reciprocity law holds. Using this we can find a modulus over $\mathbb{Q}$ for which the reciprocity law holds, which implies that you can find a modulus of the form $(m)\infty$ for which the reciprocity law holds. But then in the field extension $\mathbb{Q}(\theta_m)$ (oh, for fun, θ_m's are the primitive m^{th} roots of unity now), this tells you that the kernel of the Artin map is $\mathbb{Q}_{\mathfrak{m},1}$. You can use this to argue that $N(I_L^{\mathfrak{m}}) \subseteq ker(\phi_{L/K})$, which tells you that, up to a finite set, the primes of $\mathfrak{Q}$ which split completely in $\mathbb{Q}(\theta_m)$ are primes that split completely in L, which gives the containment $L \subseteq \mathbb{Q}(\theta_m)$!

(2/21/2018) Today I learned a deeper insight as to why the fundamental equality holds for cyclic extensions. Basically what happens is that you can do some computations and realize that you can get the order of the ray class group modulo the norms by splitting it into three separate problems–two involving the scalars modulo norms and one involving a Herbrandt quotient. I also learned why when we take the homomorphisms of a projective resolution of, say, $\mathbb{Z}$, to an abelian group A, we don't mod out by the image for the 0^{th} homology group–this would always be trivial since $Hom(-, A)$ is exact!

(2/22/2018) Today I learned about the completed group ring of a profinite group over $\mathbb{Z}_p$. There are a few ways to define it, but the way I like to define it is that if our profinite group G is given by $lim(... \to G_1 \to G_0)$ then the completed group ring is given as the limit of the groups $\mathbb{Z}/p^j\mathbb{Z}[G_i]$ where we have a grids worth of maps. Moreover, you can show that the completed group ring of Λ (a Galois group defined to be isomorphic to $\mathbb{Z}_p$ is non canonically isomorphic to the ring of formal power series $\mathbb{Z}_p[[T]]$. The way you show this isomorphism is by choosing a topological generator γ and on each level of our ring of formal power series, identifying it with $\gamma - id$. You can show this is a surjection using topological ideas that I learned, namely, that any homeomorphism with dense image from a compact set to a Hausdorff space is surjective.

(2/23/2018) Today I learned an applicaiton of the structure theorem for Λ modules. Namely, I proved that if the characteristic ideal pf a Λ module X (i.e. the product of the polynomials/powers of p which appear in the decomposition) has a power of the formal power series variable T inside, then the largest quotient of X for which Λ, a group topologically generated by some γ, say, acts trivially is an infinite group. This is essentially because you can prove by your structure theorem that you have a map $X \to \mathbb{Z}_p$ which has a finite cokernel, and therefore the image is infinite. But the image, of course, is a quotient of X! I also learned about *Kummer Theory*, the basics of which argues that any Galois extension with Galois group subgroup $\mathbb{Z}/n\mathbb{Z}$ over a field which has all n^{th} roots of unity is obtained by adjoining an n^{th} root of your field. This is really weird because the proof almost goes through homology.

(2/24/2018) Today I learned even more weird things about homology, but probably the best thing I got out of today was why the Fundamental Equality for cyclic extensions might be hoped to be true. In particular, if you were told to prove the fundamental equality for cyclic extensions L/K for a fixed modulus $\mathfrak{m}$ then noting that you're considering the group $I_K^{\mathfrak{m}}/N(I_L^{\mathfrak{m}})\iota(K_{\mathfrak{m},1})$ you might first note that you could recognize that as the quotient of a homology group $K^\times/N(L^\times)$, and you also have a map $K^\times/N(L^\times)K_{\mathfrak{m},1} \to I_K^{\mathfrak{m}}/N(I_L^{\mathfrak{m}})\iota(K_{\mathfrak{m},1})$ and you can actually extend these to a bicomplex which you can do homological algebra!

(2/26/2018) Today I learned a retrospectively pretty obvious principle. Namely, through the identification that $\mathbb{Z}_p[[T]]$ is isomorphic to the completed group ring $\mathbb{Z}_p[[\Gamma]]$ (uniquely, too, after picking a topological generator to map T to), you can identify the Γ fixed points of a Λ module as the same thing as the points annihilated by T when regarding the module as a $\mathbb{Z}_p[[T]]$ module! This helps with the structure theory. In particular, you can show that with the help of the structure theorem of Λ modules that a Λ torsion module X has an infinite amount of fixed points X^Γ if and only if the characteristic ideal does not contain the element T. This is because when modding out by any other irreducible ideal, T is a unit and therefore can't be annihilating anything, and conversely, you can show if you do have T then the kernel of "all other" maps must be infinite but multiplying this set by T yields a finite image, thus an infinite subset annihilated by T, i.e. fixed by Γ.

(2/27/2018) Today I learned a few statements which come from class field theory. One of which says that if F is a fixed algebraic extension of $\mathbb{Q}$ and p is a fixed prime number, then if K denotes the maximal p extension for which all primes of F lying over p are unrammified, then the Artin map canonically identifies the p torsion part of the class group with $Gal(K/F)$. Moreover, I learned that you can define a certain $\mathbb{Z}_p$ module of local units of the ring at each prime of F lying above p, and that furthermore the (local) Artin map gives an isomorphism of a certain quotient of this unit group with $Gal(M/K)$, where M is the maximal extension for which all primes except for the ones lying above p don't ramify.

(2/28/2018) Today I learned that the only prime that can possibly ramify in a $\mathbb{Z}_p$ extension F_∞/F, where F is an algebraic number field, is a prime of F lying above p. This is because any other prime is either infinite or finite lying above a different prime. In the first case, the inertia group corresponding to that prime is clearly finite, and in the second, the prime tamely ramifies, which implies that the inertia group is finite. However, the only finite subgroup that $\mathbb{Z}_p$ has is zero.

January 2018

(1/2/2018) Today I learned a huge chunk of why if you have a representable functor to Groups, say represented by some Y, then the multiplication, identity element, and inverse maps with respect to making Y a group scheme are uniquely determined by the fact that we require them to behave with the groups $F(X)$ for all schemes X. Essentially, what this statement is saying is that there is a natural transformation from the groups to $Hom(-, Y)$ and the only multiplication that could possibly work is the one that turns this natural transformation, evaluated at each scheme, into a group homomorphism. I specifically proved that this "multiplication" map $Y \times Y \to Y$ is well defined and associative, which essentially stems from the fact that you can group homomorphism-ly show that stems from the fact the multiplication of $F(Y)$ is associative.

(1/3/2018) Today I learned a bit more about group schemes. In particular, I learned that given any morphism of group schemes $G \to H$, then the kernel of the associated map $Hom(X, G) \to Hom(X, H)$ is a functor and moreover, if this functor is representable by some object G_0 then there is a canonical $G_0 \to G$ which is the kernel of the map $G \to H$. This is essentially because you can argue that any morphism which when postcomposed with the morphism of group schemes is the zero map must factor through G_0 and (loosely speaking) this implies you hope you can find a canonical injection $G_0 \hookrightarrow G$. And it turns out you can, using the natural isomorphism and applying it to the identity $G_0 \to G_0$.

(1/4/2018) Today I finally learned an application of the Yoneda like idea which says that the map of functor from schemes to set $X \to Hom(-, X)$ is a full subcategory. Basically, the idea behind group objects is that you have a multiplication map. In the old days, you would say multiplication *of* something, say $m(a, b)$. But nowadays, we're fancy and we think of these things like maps! So we think of them as $m \circ (a, b)$. But the nice thing is we can just define the map m (or any other map we want) based on how it behaves on all other maps $W \to X$. Thus we can talk about group homomorphisms acting on k valued points for example! I also learned what an abelian variety was–a geometrically integral projective algebraic group, and I learned that the operation of an abelian variety is actually abelian! This stems from the ridigity lemma, which says that if a function from a proper geometrically irreducible $k-$scheme with a k valued point times an irreducible domain is constant on some point for each factor then the function itself is constant. This can be used to show that in an abelian variety, inversion is a group homomorphism, which characterizes abelian groups.

(1/5/2018) Today I learned one easy way to view writing rational maps from projective schemes. Essentially, if you're mapping to something that sits inside of $\mathbb{P}^m$, you can simply specify $m + 1$ polynomials in the fraction field that aren't all zero such that the $m + 1$ equations satisfy all of the equations that your target projective scheme satisfies. This is because in viewing maps $\mathbb{P}^n \to \mathbb{P}^m$, we can use the fact that a map of graded rings gives a map on projective space minus the places where all the target polynomials vanish, which gives a rational map if all the target polynomials are nonzero. Also, it gives a clear way to distinguish and canonically give rational maps–two maps are equivalent (not necessarily on the same domain) if you can multiply a scalar multiple of one to get the other. This is nice because you can use this to get your rational map in a standard form for $\mathbb{P}^n_k$–in particular, you can specify that the polynomials are uniquely determined by the fact they are all coprime. Then you can see what the maximal domain of definition is too–the places where not all of the polynomials vanish.

(1/8/2018) Today I learned about the variety $x^2 + y^2 = pz^2$ for odd prime p. In particular, I learned that this variety is not isomorphic to $\mathbb{P}^1$ over $\mathbb{Q}$ when $p \equiv 3 mod 4$. This is because any variety with any rational point can be multiplied and divided so that we may assume the solution $[x, y, z]$ are all linearly independent integers with one of them nonzero. However, the integer pz^2 will have an odd number of factors of p, a prime congruent to three modulo 4. This says that we can't express pz^2 as a sum of integer squares.

(1/11/2018) Today I learned that there is an equivalence of categories between isomorphism classes of curves with maps being dominant maps between them over a perfect base field k and the category of finite field extensions over k. Essentially, what's happening is that any nonconstant map of curves must be surjective, since you can argue that the map between the curves must be proper and irreducible to a one dimensional scheme, and so in particular you obtain that you can an injection of function fields going the other way, and you can use transcendence theory to show that this injection gives a finite extension, since both of the curves have dimension one. What's also happening is that given any curve, you can define a rational map of curves that isn't constant (and thus is dominant - noting that all "curves" are assumed to be irreducible) by just specifying the map on coordinates.

(1/12/2018) Today I learned about *hyperelliptic curves*, which are specifically curves of the form $y^2 = f(x) \subseteq \mathbb{A}^2$ where $f(x)$ doesn't have any repeated roots as a polynomial. I first learned that you equivalently by the Jacobson criterion ask that the (affine) curve be smooth. Also I learned that the points at infinity (the points in $\mathbb{P}^2 \setminus \mathbb{A}^2$ are not smooth. Specifically in the degree four case, I learned that you can realize this affine hyperelliptic curve as an affine part of another curve in $\mathbb{P}^3$ which has two additional points. I also learned about ramification of rational functions and specifically that a rational function ϕ isn't rammified if and only if it has $deg(\phi)$ distinct preimages at each point.

(1/13/2018) Today I learned about the Frobenius morphism of elliptic curves and a few things about inseparable extensions. First off, if you're working over a field of characteristic $p > 0$ and q is some positive power of p, then the Frobenius map is a map from a (projective irreducible) curve to the curve cut out by the q^{th} power of the equations cutting out your curve, and it's given by the map of function fields in the other direction by raising each coordinate to the q^{th} power. In particular, after moving to the equivalence of categories to fields and injections sense of things, this is the map corresponding to $K(C)^q \hookrightarrow K(C)$ (which you can prove because we are always background assuming K is perfect). I also proved that the degree of the q^{th} power Frobenius map is actually q, which you can prove by using the fact that an intermediate extension of a separable/purely inseparable extension is separable/purely inseparable too. I also learned that, while not the definition Dummit and Foote uses, you can define a purely inseparable extension to be an extension L/K where if $p := char(K)$ then every element of L can be raised to some power to get inside L.

(1/15/2018) Today I learned about differentials and some facts about them. In particular, you can low level define differentials on an irreducible smooth space as the vector space over the function field of df's for each f in the function field subject to the rules that constants derive to zero, and the standard addition and product rules. On smooth curves, this caueses a one dimensional tangent space, and moreover, you can determine whether or not the derivative is zero by whether or not the function field is separable over adjoining the element to the ground field, which is pretty neat. You can also extend this to show that a map of smooth curves is a separable map if and only if the induced pullback on differential forms (which pulls back both functions and differentials) is a nonzero map.

(1/16/2018) Today I learned about why there are no holomorphic differential forms on all of $\mathbb{P}^1$. Essentially you can take a uniformizer, say t, at a point and argue that dt vanishes at no point in the affine plane containing it, while on the other hand you can show it has a pole of order 2 at the point at infinty. Then you can use the fact that dt is a basis for the space of differential forms and that the degree of the divisor of any function is zero to show that the degree of any differential form is -2 if it isn't zero.

(1/17/2018) Today I learned the Riemann-Roch theorem, which says that given a smooth curve C and a canonical divisor E on the curve, then for any divisor $D \in Div(C)$, we have that $l(D) - l(E - D) = deg(D) - 1 + g$ for some constant g, called the genus of the curve! I don't know why this is true (it's assumed for the elliptic curves book) but I'm happy to hit this milestone.

(1/18/2018) Today I learned an application of the Riemann Roch theorem, which is a theorem due to Hurwicz. Specifically, given a nonconstant separable map of smooth curves $\psi : C_1 \to C_2$ with genus g_1 and g_2 respectively, one has the "covering space like" formula $2g_1 - 2 \geq 2g_2 - 2 + \sum_{P \in C_1} e(P) - 1$ where $e(P)$ refers to rammification, where $=$ occurs if and only if the characteristic of the underlying field doesn't divide the rammification index at any point. This is basically because you can compute things term by term using a nonzero differential form and the fact that there is a canonical divisor. I also learned a more general fundamental theorem of Galois theory–specifically, that the subfields correspond to *closed* subgroups of the Galois group, and the finite dimensional extensions correspond to open ones.

(1/19/2017) Today I learned that any curve of genus zero is isomorphic to $\mathbb{P}^1$, and equivalently that a curve has genus zero if and only if there is a divisor on the curve of the form $P - Q$ where $P \neq Q$ and $P - Q = div(f)$ for some f. Basically, what happens is that you can use the assignment $(x, y) \to (f, 1)$ to argue that you very well could have considered P as the "zero" point. You can then use rammification to argue that because f is a uniformizer at P (since it vanishes at exactly order one) you can use the rammification formula to determine that the map above is degree one, and thus an isomorphism! Furthermore, I learned that you can extend this argument a little bit for genus one curves and argue that there is a canonical operation on any genus one curve with a fixed point P_0 on the curve given by a point P being associated with the divisor class in $Pic^0(C)$ of $P - P_0$.

(1/20/2017) Today I learned some okay stuff about the equations of elliptic curves. In particular, there are objects called the discriminant of the elliptic curve and the *j-invariant* of the elliptic curves. It turns out wading through the computations that over an algebraically closed field, the j-invariant of two elliptic curves agree if and only if they are isomorphic, and moreover there is a elliptic curve having j invariant j for every element in the algebraic closure. Also, I learned that because of the number of conjugacy classes of D_{2n} for n even, except for the four representations induced by the commutator, all of the remaining representations must be two dimensional.

(1/21/2017) Today I learned the group operation on an elliptic curve over an algebraically closes field k. Given two non infinite points on the curve, one can construct a unique line between them, interpreting the unique line between them at the tangent line if the two points are the same. Using Bezout's Theorem, one can argue that the line must intersect some unique point. You can use explicit equations to determine what this curve actually is, and then you can use it to show that any even function is a polynomial of just one variable—in the Weirstrass form, just x. Moreover I learned that the same Bezout argument can show that the nonsingular points on any genus one curve form a group when written in the Weirstrass form.

(1/22/2017) Today I learned about isogenies between elliptic curves. Specifically, isogenies are identity preserving morphisms of curves. Since these are morphisms of curves, these are either the zero morphism (which maps every point to zero) or it is a surjective map, so isogenies have degrees and separability, etc. Moreover, it turns out that because we have the translation option on elliptic curves, each point has the same number of preimages in an isogeny, given by the separability degree of the induced map of function fields, and similarly each point in the domain has the same rammification. Moreover, it turns out that all isogenies are group homomorphisms, and that these isogenies form a (probably not commutative) ring of characteristic zero with no zero divisors!

(1/23/2017) Today I learned about the existence of what I call "good uniformizers" and how they are useful. Specifically, I proved a lemma (which is really just a not too hard application of the Chinese Remainder Theorem) which says that given any list of distinct primes $\mathfrak{p}_1, ..., \mathfrak{p}_n$ of an algebraic number field (or any Dedikind ring) K, there exists some $f \in K^\times$ (or in the associated field of fractions) which is a uniformizer of $\mathfrak{p}_1$ and isn't in any of the other $\mathfrak{p}_i$. This is helpful because it can be used to prove that the class group is the same if you forget the existence of finitely many primes, and it is also helpful to resolve an ambigouous definition of $K_\mathfrak{m}$ for a modulus $\mathfrak{m}$. In particular one can show that if $K_\mathfrak{m} := \{\frac{a}{b} \in K^\times : \exists c, d s.t. \frac{a}{b} = \frac{c}{d}$ and cO_K, dO_K have no prime factors in common with $\mathfrak{m}_0\}$, then $K_\mathfrak{m} = \{\alpha \in K^\times : \alpha O_K$ has no prime factors in common with $\mathfrak{m}_0\}$.

(1/24/2017) Today I learned that you can give the functor $Hom(-, D)$ a set of *derived functors*, which are obtained on an element A by taking a projective resolution of A and then taking the functor applied to both sides and then taking cohomology of both sides. You can show that this is independent of projective resolution and furthermore you can argue that by the way you've defined it that short exact sequences yield long exact sequences involving the functors Ext, which explain why the right derived functors "complete" the not necessarily exact sequence of the image of the functor.

(1/27/2018) Today I learned the whole proof of Dirichlet's Theorem on infinite progressions, as well as some kind of conceptual way of realizing why it's true. The idea is that you can imagine the arithmetic progressions with constant distance m as the group $(\mathbb{Z}/m\mathbb{Z})^\times$ and noticing that this group is in particular isomorphic to $\mathbb{Q}_\mathfrak{m}/\mathbb{Q}_{\mathfrak{m},1}$. You would hope that the primes in these cosets are "evenly distributed," and you can use the Frobenius density theorem to argue that they actually are evenly distributed, and moreover you can do a computation with L functions to show that they each have nonzero density! This in turn implies that are infinitely many in each coset.

(1/28/2018) Today I learned the proof of the Frobenius density theorem. Basically, you can use an induction step and argue that the primes lying over a fixed prime whose Frobenius is inside the subgroup $\langle\sigma\rangle$ can be determined by noting that if E is defined as the fixed field of $\langle\sigma\rangle$ then it's the number of primes of the ground field that have a prime of relative degree one over the prime (after a fudge factor accounting for the fact you could have multiple primes of relative degree one above your prime), and use the fact that the set of primes of relative degree one over the ground field has Dirichlet density one, along with some group theory, to split up your primes and compute the density of those Galois group elements which conjugate to an actual generator of σ, not to some arbitrary element in $\langle\sigma\rangle$.

(1/30/2018) Today I learned a bunch of stuff about reducing polynomials $f(x)$ modulo p and how these relate to factoring the prime p in the ring of adjoining a single root. One of the cooler things I learned was that you can have polynomials that are reducible modulo every prime p, but not reducible over $\mathbb{Q}$. In particular, the polynomial x^4+1 is irreducible because it is the minimal polynomial for the primitive roots of unity, but note in particular that the Galois group given by adjoining a primitive fifth root of unity is not cyclic. This is a problem because reducing $\mathbb{Q}[\psi_5]$ by a prime $\mathfrak{p}|p$ for any prime that isn't two will give give you a necssarily cyclic Galois group of $\mathbb{Z}[\psi_5]/\mathfrak{p}$ (which is the splitting field of x^4+1) that is generated by one element. Thus we must have had the Galois group has order strictly less than four, so the polynomial f factors! And of course, for the prime 2 we have $(x^2+1)^2 = x^4+1$.

(1/31/2018) Today I learned a boatload of things about free sheaves and free modules. One of the first things I proved was that the Hom sheaf between locally free O_X modules was also a locally free O_X module, which basically stems from the fact that once you get an open set on which the sheaf is locally free, the restrictions are locally free too, so you can define a morphism by essentially saying where a basis in the sections on that open set go to. Similarly, you can use this to show that the tensor product of two locally free O_X modules of rank m and rank n respectively give you a locally free sheaf of rank mn, and this gives you a law of composition on *invertible* locally free O_X modules that I learned will turn it into an abelian group. (Although I haven't verified associativity or invertibility, I can guess why O_X is the identity and the group is abelian.)

December 2017

(12/1/2017) Today I learned that given any finite set of primes Σ, there are only finitely many real characters which have conductor which are divisible by primes only in Σ. This stems from the fact that if you have a map from $(\mathbb{Z}/p^n\mathbb{Z})^\times$, it is determined by where it sends the generator (since the group itself is cyclic), and it either sends the generator to 1, which implies it has conductor 1, or it sends the generator to -1, and for p an odd prime, this implies that the morphism factors through $(\mathbb{Z}/p^n\mathbb{Z})^\times \to (\mathbb{Z}/p\mathbb{Z})^\times$. This is then extended by the Chinese remainder theorem and the fact that there are 1 or 2 real characters of order $f, 4f, 8f$ where f is an odd squarefree number.

(12/3/2017) Today I learned that the only two ways to extend a group G with a normal subgroup of order $\mathbb{Z}/p^n\mathbb{Z}$, say $\langle a\rangle$ whose quotient is $\mathbb{Z}/2\mathbb{Z} = \langle g\rangle$ is to have the dihedral group of order $2n$ or the cyclic group of order $2n$. This is a pretty elementary proof–you know that $gag = a^s$ and conjugating that by g again you get the equation that $a^{s^2-1} = 1$, which in particular says that

$s = \pm 1$ since $(\mathbb{Z}/p^n\mathbb{Z})^\times$ is a cyclic group so in particular there are exactly two elements of order dividing 2.

(12/4/2017) Today I learned that any closed additive subgroup of the $p-$adic rationals is of the form $p^n\mathbb{Z}_p$ for some n. I also learned that there is a bijection between characters of irreducible representations of the Galois group of fields over $\mathbb{Q}$ which only rammify in a certain finite list and dirichlet characters whose conductors are divisible by only primes on that list. I also learned about the $p-$adic logarithm, which you can basically define as a power series and use to show that the group of units of $\mathbb{Z}_p$ is merely a cyclic group times $\mathbb{Z}_p$.

(12/5/2017) Today I learned about a cool example involving the natural projection map $\mathbb{P}^n_A \to Spec(A)$, where A is some ring. The fundamental theorem of elimination theory says that that map is closed, so given a closed subset it's the solution set of a bunch of homogeneous polynomials. To sort of visualize this, consider the set $V(y-z, xy-z) \subseteq \mathbb{P}^n_A$ where $A = k[x]$. Then you'll notice that in order for there to be a solution, $y = xy$, and if $y = 0$ then so does z, so there is a nontrivial solution to this if and only if $x = 1$, and this is the image under the map.

(12/6/2017) Today I learned that given any two manifolds, if their ring of functions to $\mathbb{R}$ are isomorphic as $\mathbb{R}$ algebras, then the original manifolds were isomorphic too. This also can be viewed as saying that in differential topology, everything is algebro-geometrically affine. Moreover, I learned that for any two schemes, if their coherent sheaves are isomorphic, then the schemes are isomorphic themselves.

(12/7/2017) Today I cleared up a misconception I had about whether $Hom(M, N) \cong Hom(N^\times, M^\times)$. Unfortunatley, it's not, as one can see by considering both modules to be $\mathbb{Q}$ over $\mathbb{Z}$. Then the homomorphisms as modules are many, however, the only $\mathbb{Z}$ linear maps $\mathbb{Q} \to \mathbb{Z}$ send $1 \to 0$ since if $1 \to m$ then $\frac{1}{2m} \to \frac{1}{2}$ otherwise.

(12/8/2017) Today I learned what a division of a finite group element σ is–it's defined to be the set of elements which are conjugate to σ^m where m is a number relatively prime to the order of σ. I also learned a way to count the number of elements in the divisor–for each power relatively prime to the order n of σ, of which there are $\phi(n)$ of them, we need to count the order of the orbit of conjugacy action, which is $\frac{|G|}{|C_G(\sigma)|}$ since for m which are relatively prime to n, $C_G(\sigma) = C_G(\sigma^m)$. But you count each element not necessarily once, but really you count each one by the order of the conjugacy action on the group H generated by σ, which is $N_G(H)/C_G(\sigma)$. So dividing out by the order of that group, we get that each division has $\frac{|G|}{|N_G(H)|}$ elements in it.

(12/9/2017) Today I learned that it's not necsesarily true that the restriction of a proper morphism is affine. For example, the map $\mathbb{P}_\mathbb{C} \to Spec(\mathbb{C})$ is affine (it's trivially finite type and we showed it was separable by basically arguing the intersection of the two obvious affine open sets is still affine, although I don't remember why it's separable). But anyway, I learned this because I also learned that every proper affine morphism is finite, and every finite morphism is affine proper.

(12/10/2017) Today I learned an application of Noether normalization and the going down theorem–a proof of the Krull Height Theorem! The first thing I proved was that you can use isomorphism nonsense to argue that it suffices to prove that any minimal prime ideal containing some fixed element f has codimension at most 1 (instead of $f_1, ..., f_r$ having codimension at most r) and then you can argue that Noether Normalization gives you a finite morphism from your irreducible k variety $Spec(A) \to \mathbb{A}^m_k$ and then argue that any minimal prime maps to (f_0) for some irreducible $f_0|f$ in $k[x_1, ..., x_m]$ and then use finiteness to pull the chain up. You can then use that codimension is the difference of dimension to prove the codimension claim.

(12/13/2017) Today I learned Dirichlet's Theorem on primes in arithmetic progressions! This is a theorem which says that any arithmetic progression starting at any number relatively prime to a fixed m will have infinitely many primes. This essentially stems because you can argue that the primes of the modulus $(m)\infty$ correspond to the primes in Arithmetic progressions, and then you can use the density theorem and a computation with L functions associated to nonidentity characters to show that each of these cosets have Dirichlet density nonzero.

(12/14/2017) Today I learned why my original "proof" of fiber dimension was wrong. Basically the dimension of any set is not the dimension of its closure (consider the fiber of the identity $Spec(k[x]) \to Spec(k[x])$. But I did learn a nifty formula that shows dimension is what you think it is for a point, and it follows from realizing that by base changing over some point $Spec(K(q))$ makes you a $K(q)$ variety, which means that dimension is the transcendence degree which in particular implies that the dimension of the fiber is the difference of the dimension of the total space minus the dimension of the point.

(12/15/2017) Today I finally got the idea of a representable functor. Basically, a functor $C \to Set$ is representative if it is naturally isomorphic to $Hom(-, X)$ for some object in the category. It's particularly helpful to argue that products exist to just argue that the product $Hom(-, X) \times Hom(-, Y)$ is representable, and this is because there is a bijection of natural transformations $Hom(-, X) \to Hom(-, Y)$ and maps $X \to Y$, i.e. every map is given by a pullback.

(12/16/2017) Today I learned a slew of fun things about smoothness and tangent spaces. In particular, since we're heavily defining schemes in terms of the functions on them, you can define the cotangent space of a local ring A at the maximal ideal $\mathfrak{m}/\mathfrak{m}^2$, which makes sense, I learned, because the dual of the $A/\mathfrak{m}$ vector space $\mathfrak{m}/\mathfrak{m}^2$ can be viewed as a *derivation*, also called a tangent vector, which takes functions in and returns a value in the base field (which is one way you can view a vector on the tangent plane). I also learned that you have a sort of Krull's principal height theorem in the sense that for any local ring A the tangent space of $A/(f)$ is the subspace of the tangent space of A cut out by how f acts. I also learned that the tangent space behaves well with respect to the intersection of two subschemes, but it doesn't necessarily behave well with respect to the union of two subschemes–in particular, the tangent space of the parabola $y = x^2$ and the tangent space of the x axis $y = 0$ in $\mathbb{A}^2_k$ both are kx, however, their union gives a dimension two space.

(12/17/2017) Today I learned a few facts about the ring R of matrices with entries in a division algebra. In particular, the identity matrix is the only primitive central idemopotent, which is defined to be any idempotent (which, by definition, do not include zero) which cannot be written as the sum of two primitive elements in the center of the ring. I also learned that any simple R module is isomorphic to Re_1 which you can prove using formal manipulations.

(12/18/2017) Today I learned a really neat thing. Specifically, if X is a k-scheme I learned that there exists a bijection between maps of k schemes $Speck[\epsilon]/(\epsilon)^2 \to X$ and the data of a point $p \in X$ with residue field (which, not so related, by dimension reasons must be closed) and a tangent vector at that point. The really cool thing in constructing that is trying to construct a map $Speck[\epsilon]/(\epsilon)^2 \to X$ you basically can quickly realize with the point you can map to the point p, say $SpecA/(p)^2 \hookrightarrow X$. The harder part is trying to construct the ring map $A/p^2 \to k[\epsilon]/(\epsilon)^2$. What you originally think is that you can take a function on A and plot to its value at p (i.e. modding it out by p and obtaining some k value) and then add the tangent vector evaluated at the function minus its value, and then multiplied by ϵ. But at first glance, this may not appear to be a map of rings. However, for two $f_1, f_2 \in A$, you can add $(f_1 - \overline{f_1})(f_2 - \overline{f_2})$ to the tangent vector to show that you still have multiplicativity!

(12/19/2017) Today I learned a new notion of smoothness, this time literally called smoothness. We say a pure dimension d finite type k scheme is smooth if you can cover it with affine open sets such that for each affine open set in the cover, which must be of the form $Spec(k[x_1, ..., x_n]/(f_1, ..., f_n))$, the Jacobian of the f's has corank d, which means that the image as as large as it can be. This can more easily be checked on just closed points to show every point is smooth, since the set where matrices have a certain corank is given by the vanishing and nonvanishing of certain polynomial equations, and locally closed subsets of $\mathbb{A}_k^n$ always have a closed point in them. Also, at least in the case of an algebraically closed field, smoothness is the same as being regular at each point. This can also be upgraded with some fun theorems.

(12/20/2017) Today I learned that, by definition, regular rings are Noetherian. That is just a definition thing but it tripped me up for a while. Anyway, you can use this fact to show that if you assume the theorem which states that any regular local ring of finite dimension is an integral domain, that an ideal of a regular ring which cuts out a regular ring is actually a regular embedding. You can do this by arguing that by dimension reasons of the linear spaces, there is a basis of elements which are in the kernel of the associated map of cotangent spaces. Then you can argue that this actually cuts out the entire ring because of dimension reasons, and you can argue that these form a regular sequence by an inductive hypothesis by using the theorem and arguing that you got a regular local ring of dimension one less, so you can use induction (with base case also used by the theorem).

(12/21/2017) Today I learned what the tangent space in affine space and projective space is for a k valued point is. Basically, the idea behind a k valued point is that you can translate it, and then you can exploit everything you can do regarding evaluating partial derivatives while staying in the field k. I proved that no matter how you naively define the tangent space of projective space (including picking an open cover and putting the affine tangent space in) you always get the same tangent space, a lot of which follows from the fact that $deg(f)f = \sum_{i=1}^n \frac{\partial f}{\partial x_i} x_i$ for homogeneous polynomials f. This can also be used to show in fields which the characteristic doesn't divide $deg(f)$, you can check that the Jacobians vanishing for a hypersurface implies the point is actually at the hypersurface!

(12/22/2017) Today I learned about a theorem which says that $\mathbb{A}_k^n$ is regular, which is basically proven by a proposition which says that if a local ring B is regular of finite dimension, then after localizing any prime ideal of $B[x]$ above the maximal ideal of B, that's also a local ring. This proposition basically stems from the fact that you can specifically compute each fiber and show that from the fiber, you can determine the actual fiber and use dimension theory, specifically using the fact that if $\mathfrak{p} \in Spec(B)$, then $\mathfrak{p}B[x]$ is a prime ideal in $B[x]$. Using this proposition and a fancy isomorphism (well, it's not so fancy) this argument proves that $\mathbb{A}^n$ is regular over any regular base ring.

(12/26/2017) Today I learned the proof of Bertini's Theorem, which says that for any given subvariety X, there is a dense open subset of algebraically closed k hyperplanes such that no component of X is contained in H and there are no singular points in $H \cap X$. You can argue this by essentially showing that in the product space of $\mathbb{P}_k^n$ with its dual, the set of "pairs" where there is a hyperplane with a point in $\mathbb{P}_k^n$ with a "problem", is a closed set whose dimension is bounded by one less than the total dimension, implying that the set of bad points is also a closed set of small dimension, which uses the fundamental theorem of elimination theory.

(12/27/2017) Today I learned the proofs of a ton of equivalences of discrete valuation rings. Specifically, discrete valuation rings are regular, local, Noetherian rings of dimension one, but you can also show that the only nonzero ideals are the ideals that are powers of the maximal ideal. I also learned how poles are specifically defined–they are the codimension one points, and showed using Noetherianness that an element of the ring can only have a finite number of poles and zeroes, and I learned that e^{-x^2} shows that the ring of global sections of real smooth functions at zero is not Noetherian because it violates a subcase of the Artin Reis lemma, which says that in a Noetherian local ring, there is no element whose powers series is trivial that isn't zero, i.e. $\cap_i \mathfrak{m}^i = \{0\}$.

(12/30/2017) Today I learned a bunch of extensions of Bertini's theorem. Namely, that if you have any locally closed embedding into projective space, there is an dense open set in dual projective space which doesn't contain any component of that set and the intersection is smooth at each point in the locally closed embedding. This essentially follows because you can follow the proof in the exact same proof that Vakil writes for the smooth case, and the dimensions still hold because the codimension is still large in the set of points which don't satisfy the criterion, so the dimension is small. You can also adapt this proof in the exact same way to adapt this to the intersection of n general hyperplanes.

(12/31/2017) Today I got a better insight as to how group schemes work. In particular, I learned that group schemes should be interpreted as letting the ring maps in reverse give "coordinates." This is how you can define GL_n, for example. Specifically, you can define the GL_n over a ring A as a group scheme by calling it $Spec(A[x_{ij}, y]/(yd-1)$ where d is the determinant polynomial. Then if you want to put values of an A algebra B, you simply look at the coordinates going in the other direction. With this logic, you can also argue why morphisms of group schemes give maps of A valued points (or even a general scheme X valued points)!

November 2017

(11/1/2017) Today I learned about the definition of *proper* morphism. This by definition is a separated, finite type, universally closed morphism and I learned that it's closed under things like base change, composition, etc. I also learned that a map of $\overline{k}$ is determined by the map on the closed points. This is because (at least the way I proved it) you can talk about the locus of points where the two functions agree, and you can argue that it contains the disjoint union of the residue fields of the domain (since there is a unique map of residue fields over $\overline{k}$, since both residue fields are $\overline{k}$ algebras), which is the reduction of the domain. This is helpful because varieties are reduced, so the then the locus where they agree as maps is the domain itself.

(11/2/2017) Today I learned why it was that the graph morphism of a rational map is well defined. Basically, you can argue it down to showing that any rational map is the same as if you had defined it on the largest possible domain of the map, and then from there you can argue that if you can get the closed embedding to agree after embedding into just the domain times the codomain (instead of the whole space times the codomain) then the closed embeddings remain the same. Then you can argue that because your rational maps are defined over all the associated points of your scheme, you have that the schematic closures of both are the same in the domain times the codomain.

(11/3/2017) Today I learned the only functions on a proper, connected, reduced scheme over an algebraically closed field k that are global functions are the constant ones. This is because you can argue that a global function is equivalent to a map to the affine line (which is equivalent to just specifying where in the ring of global sections you send x) and you can argue that by properness the composite of that map to the projective line is closed and since its image is contained in the affine line and must be connected, it only maps to one point. Then you can show that the map to affine space factors through the the point which has a ring of functions a reduced scheme with one prime ideal, i.e. a field. Then you can show that because you had an algebraically closed field, the residue field was just the algebraically closed field k itself. Then you can go back to the question, where did x go? Because it factors through k so it had to be an element of k.

(11/5/2017) Today I learned the approximation theorem for inequivalent norms of an algebraic number field. This theorem says that given any collection of n norms on a given algebraic number field K which are pairwise inequivalent, and some n elements of K, say, $\beta_1, ..., \beta_n$, then given an $\epsilon > 0$ you can find an element which approximates them all at the same time, meaning that $|\alpha - \beta_i|_i < \epsilon$ for all the norms $|.|_i$. That's crazy. But it essentially stems from the fact that for any pair of inequivalent norms, there exists some element γ such that one of the norms on γ is larger than 1, and the other is smaller than 1. The rest of it is using clever algebra tricks to extend this lemma to the fact that for each i you can get an element where the ith norm is larger than one and the rest are smaller, and then use more algebra tricks with these elements to get the desired approximation. It's a really weird result.

(11/6/2017) Today I learned that for any irreducible affine k-variety X, the dimension of the space X equals the transcendence degree of the fraction field of X over k, say, n. This is essentially because you can argue *Noether's Normalization Lemma*, which says that there is always a finite morphism from $X \to \mathbb{A}_k^n$. Since finite morphisms preserve dimension, this lemma entirely reduces the problem to showing that $\mathbb{A}_k^n$ actually does have dimension n. This can be done by induction, noting that the $n = 0$ and $n = 1$ cases are trivial. But then for the inductive step you can pick the first nonzero prime ideal appearing in a chain and you can figure out whatever irreducible nonzero function is in there (there has to be one) actually gives the nonzero ideal, and mod out to see the transcendence degree, and thus the dimension of the modded out space, is $n - 1$, and so your dimension of $\mathbb{A}_k^n$ is n.

(11/7/2017) Today I learned that there is a ring that is finite dimensional but has ideals that are generated by an arbitrarily high number of elements. In fact, the ring is $\mathbb{Z}[x]$! To see this, consider the ideal $I_n = (2^n, 2^{n-1}x, ..., x^n)$, and let $\mathfrak{m} := (2, x)$. Then you can show that $I_n/\mathfrak{m}I_n$ is an n dimensional $\mathbb{F}_2 \cong \mathbb{Z}[x]/\mathfrak{m}$ vector space, so it must be generated by n elements! Woah.

(11/8/2017) Today I learned one of the proofs that the Ray Class group is finite. This uses the fact that the ray class group of a modulus is the group generated by prime ideals that don't divide the modulus modulo the elements which are one modulo each prime power of the modulus. From there, you can break up the order into two pieces–the first index is the order of the group over the restricted principal ideals allowed, which you can show actually is the (finite) class group, and then through a modulus "Chinese Remainder Theorem" you can argue that the second part is finite as well, since the CRT says you can simply test it at each prime.

(11/10/2017) Today I learned a filtration on the Galois group of the finite Galois extension of a complete field L/K. In particular, if π is a uniformizer of L, then by computing norms we know that for each $\sigma \in G := Gal(L/K), \alpha := \frac{\sigma\pi}{\pi}$ is a unit, by computing norms and noting that any element of G doesn't affect the norm. You can take one from α and see where it ends up, and let G_u be the subgroup of G that sends it into $\mathfrak{p}^u$. You can show in particular that G_1 is the Sylow p subgroup of G_0, which is the Galois group of L over the maximal unrammified extension of K. This is by showing that all of the p^{th} powers of G_u are in G_{u+1} when $u \geq 1$, which you can do by a fraction trick noting that you can write $\frac{\sigma^p\pi}{\pi} = \frac{\sigma^p\pi}{\sigma^{p-1}\pi}...\frac{\sigma\pi}{\pi}$ and then noting that each of those contributes a factor of p. You can then show G_1 is the Sylow p subgroup of G_0 by arguing that any if it wasn't, there would be an element $\tau \in G_0 \setminus G_1$ with $\tau^p \in G_1$ and then show using a similar fraction trick that this can't happen.

(11/11/2017) Today I learned when Dirichlet series converge and what they are. Dirichlet series are functions of the form $f(s) = \sum_{n=0}^{\infty} \frac{a_n}{n^s}$ where $s, a_n \in \mathbb{C}$. I showed in particular that if a_n is $O(n^b)$, then you can show if you cut the ray stemming from any angle smaller than $\frac{\pi}{2}$ from it and go a little to the right of b you get uniform convergence in the series. Basically this stems from integral tricks and clever rewritings. I also learned today that you can reduce proving statements about codimension of an irreducible closed subscheme to proving them when the underlying topological space is an affine k scheme. Essentially, you argue first that you can replace the underlying space with its reduction, which (more than just intuitively) doesn't change dimension because it doesn't change the topological space. Then you work in the irreducible component so now you have an integral scheme so it doesn't matter what open set you compute dimension in. Then you compute it in any open set containing it, and you cleverly pick the one that contains a maximizing chain of

Y.

(11/12/2017) Today I learned why it was that the codimension of an irreducible closed subset of an affine finite type k scheme adds the way it should add. Essentially, you can do some reductions to first reduce to the affine case (which I talked about earlier) and then you can further use an inductive argument to show that you can work with hypersurfaces, which are minimal irreducible closed subsets that aren't the entire space itself. Then you can use the *Going Down Theorem*, which says that given any integral extension of rings $B \hookrightarrow A$ and two prime ideals of B, say $q \subseteq q'$ such that there exists a $p' \in Spec(A)$ for which $p' \cap B = q'$, then there's a $p \in Spec(A)$ with $p \cap B = q$ and $p \subseteq p'$. This theorem I also learned the proof of, and it's essentially an extension of an argument that the primes on a Dedikind ring are transitive, since you use the going up theorem in an integral closure of B in the normal closure of $K(A)$ after reducing to the case where A is an integral domain.

(11/13/2017) Today I learned that if you have a minimal prime $\mathfrak{p} \subseteq A \otimes K$, where K/F is a finite field extension and A is an F algebra, it is not necessarily the case that if $\mathfrak{q} := A \cap \mathfrak{p}, A/\mathfrak{q} \otimes K \cong (A \otimes K)/\mathfrak{p}$. This is in particular because the right hand side is always an integral domain, whereas the left hand side may not be, for example, in the case where $F = \mathbb{R}$ and $K = A = \mathbb{C}$ and thus the minimal prime $\mathfrak{q} = 0$.

(11/14/2017) Today I learned an analytic number theory trick about Dirichlet series $\sum_{n=1}^{\infty} \frac{a_n}{n_s}$. Specifically, I learned that if it just so happens that $\frac{s(x)}{x} \to z_0$ as $x \to \infty$, where $s(x) = \sum_{n \leq x} a_n$, then $\lim_{s \to 1}(s-1)\sum_{n=1}^{\infty} \frac{a_n}{n_s} = z_0$ for s in a ray where the function is defined. This is proven by reducing to the case where $a_n = 1$ (i.e. the Riemann zeta function) by using clever summing tricks, and then the case of the Riemann function is proving by showing that there are analytic functions which approximate the Riemann Zeta function everywhere but a countable collection of real points whose intersection is just the point 1, which in particular says that the limit actually does exist if you multiply the Riemann zeta by $(s-1)$ to get rid of the pole problem there. Then you approximate it using the same box method you might do in calculus. This is helpful because it turns a two dimensional problem into a one dimensional limit problem.

(11/15/2017) Today I learned about intersections in projective space relating to dimension. In particular, today I proved that if you have a hypersurface in $\mathbb{P}_k^n$ and any closed surface of dimension larger than zero, there is a point which intersects. This doesn't work in affine space (picture two parallel lines/planes). The way you prove this is to show that any closed subset of projective space can be written in the form $Proj(k[x_0, ..., x_n]/I)$ for some homogeneous ideal I. Using this information, we can pull back both the hypersurface and the other closed subset back to the affine space in one dimension up, where we can use affine cones. Then we can use dimension theory, and in particular, Krull's height theorem, to tell us that the codimension of the function cutting out the hypersurface can be at mostone in the affine cone of the other closed subset. This in particular says that we have to have more than just the trivial point of intersection, the origin!

(11/16/2017) Today I learned more stuff about intersections on projective spaces. In particular, you can show that given any closed subset $X \subseteq \mathbb{P}_k^n$ of dimension r and showing that you can intersect $r+1$ hyperplanes of k to miss X if k is an infinite field. This is because given any polynomial in an infinite field that itself isn't zero, you can show that there's a value you can plug in where none of the values are zero to show that you can tilt the polynomials by a scalar factor to construct a hyperplane missing all of the generic points of X. You can also use this lemma to argue that the intersection of r hyperplanes can be chosen to be a finite number of points.

(11/17/2017) Today I learned that if you are given a field chose characteristic does not divide a given integer n, then the representation of S_n given by "coordinates that sum to zero" is an irreducible representation. This is because if you are given a vector in any representation that does not have all the coordinates the same and is nonzero (which must happen given that the characteristic doesn't divide n) then you can take $v - (1,2)v$ and divide out by the (necessarily nonzero) first coordinate to show that $e_1 - e_2$ is in the representation. Then you can use permutations to show that $e_i - e_{i+1}$ is in your representation for all i.

(11/19/2017) Today I learned shit about the Riemann zeta function in general function fields. In particular, I used the trick which says that if you can compute the one sided limit $\lim_{x\to\infty} \frac{s(x)}{x}$ where $s(x) = \sum_{n\leq x} a_n$ then the pole of your Dirichlet series is that exact same value. Then you can show that the sums are corresponding to certain special ideals in a given modulus, which you moreover can show are up to roots of unity represented by elements in a lattice. Then you can integrate this region to figure out what the pole of the general zeta function of a modulus of a given algebraic number field is.

(11/20/2017) Today I actually learned how to pick that canonical point of the lattice is picked. In particular, you can show the number of principal ideals you need to count can be muliplied by a $K_{\mathfrak{m},1}$ preserving unit, which equivalently scales the point on the lattice down to all but possibly one basis vector chosen, namely, the vector $(1,...,1,2,...,2)$ having coordinates strictly less than one.

(11/21/2017) Today I learned why there are infinitely many primes. Hah. It's a proof that generalizes though to a lot of other things though. Here's the proof. The idea is that the Riemann zeta function $\zeta(s) = \Pi_p(1-\frac{1}{p^s})^{-1}$ and you can use logarithms on ζ (or any zeta function, or any function which can be realized as a product over a set where all the elements greater than 2) to see that $-log(\zeta(s)) = \sum_p \frac{1}{p^s} + g(s)$ where g is bounded near one. Then you can use another algebra trick to show if there were finitely many primes, you could show that $-log(s-1)$ was bounded near one!

(11/22/2017) Today I learned how to reduce the problem of showing that the base change of any k scheme over an algebraic field extension, or the base change of a finite type k scheme over any field extension, preserves pure dimensionality, to assuming that your scheme is an affine integral scheme. Essentially you base change to an irreducible component. The thing that was originally tripping me up was that I thought we needed to argue pure dimensionality at the reduction step, but the irreducible component $\overline{\mathfrak{p}}$ is also an irreducible component of any affine open set containing it. I also learned why it was that if one is given two equidimensional irreducible subvarities of affine d space of codimensions m and n respectively, then their intersection has codimension at most $m+n$. This is because you can reduce the problem to showing it for the case where each of your subvarities (a priori a locally closed subset) embeds specifically into an affine open subset of $Spec(k[x_1,...,x_d])$. You can then use the magic diagram to realize the intersection can also be viewed as the product of the two subvarities intersected with the diagonal, and the affine assumption allows you to specifically compute what the dimension of that variety must be.

(11/23/2017) Today I learned that you can localize any irreducible component of some closed subset of $\mathbb{A}^n_k$, say, $Spec(k[x_1,...,x_n]/I) = Spec(k[x_1,...,x_n]/(f_1,...,f_n)$ such that there is a unique minimal prime ideal leftover containing $(f_1,...,f_n) =: I$ and so thus dimension doesn't change. You can do this by noting the fact that there are finitely many irreducible components of the closed subset, and you can use prime avoidance to find a function that vanishes on one of the primes but none of the others. Then multiplying this for every irreducible component that isn't your favorite one, you obtain a function f you can localize by to make this happen.

(11/25/2017) Today I learned that the fiber of a map of locally Noetherian schemes can never be too low. In particular, if you are given a map π and $\pi(p) = q$, then $codim_X p \leq codim_Y q + codim_{\pi^{-1}(q)} p$. This is essentially because you can argue that you can cut out p locally in its local ring by using equations cutting out q in its local ring, pulling them back, and combining them with equations cutting out p in $\pi_{-1}(q)$. Then we use the fact which says that dimension of the

Noetherian ring is bounded by how many equations cut it out, and the fact that codimension is the dimesnion of a point is the dimension of the corresponding local ring, to show the inequality. I also learned that the "points" of scheme theory are meant to represent basically every geometric object, which is an observation I maybe should have made earlier but didn't.

(11/26/2017) Today I learned the official (well, another official) definition of an elliptic curve over a field k–it's a curve of genus one with a specific marked point, which I guess with the group action becomes the identity point. You can define a group structure using line bundles. I learned that there is a theorem that says the L valued points of an elliptic curve, where L/k is finite, form a finite rank $\mathbb{Z}$ module with a finite amount of torsion, and moreover that this isn't true when the field extension is infinite, since $\mathbb{C}/\mathbb{Q}$ is and the $\mathbb{C}$ valued points make the elliptic curve topologically isomorphic as groups to a torus, which does not have finite torsion. The paper I am reading is investigating when finite torsion can happen in infinite extensions, which uses Iwasawa theory and $\mathbb{Z}_p$ *extensions*, that is, Galois extensions of a given base field whose Galois group is isomorphic to $\mathbb{Z}_p$.

(11/27/2017) Today I learned about why the Riemann Zeta function doesn't converge for any complex number s with real part larger than one. This is because you can first look at the series $\sum_{n=1}^{\infty} \frac{1}{n^s}$. and notice that that converges absolutely, since $|\sum_{n=1}^{\infty} \frac{1}{n^s}| \leq \sum_{n=1}^{\infty} \frac{1}{n^{Re(s)}} < \infty$ by the triangle inequality, and when the sum converges absolutely, the product does as well, and in particular no factor can be zero when the product converges absolutely to a noninfinite number by the continuity of the logarithm. I also learned that for any character $\chi : X \to \mathbb{C}$ of a group (which I relearned was just the trace of a given representation), for any $g \in G$ which maps to a $S^1 \subseteq \mathbb{C}$ (which is true for any g of finite order), then $\chi(g^{-1}) = \overline{\chi(g)}$. This is because you can diagonalize the matrix because you can show that the minimal polynomial of the matrix of g divides $x^{dim} - 1$ in $\mathbb{C}$ and thus is separable. Then you can diagonalize it and cleverly compute in that basis.

(11/28/2017) Today I learned that you can put any quadratic field into a cyclotomic field, in particular, you can show it's contianed in $\mathbb{Q}(\zeta_{|D|})$ where D is the discriminant of the field. This uses Gauss sums, which you can do some manipulations involving Dirichlet characters of cyclic groups to reduce the problem to constructing a character of the disciminant $|D|$, which you can break into cases based on what power of 2 divides $|D|$.

(11/29/2017) Today I learned the reduction of a theorem proven in the Park City series I believe was originally proved by Greenberg. The original theorem says for an elliptic curve there is a bound B that for all abelian extensions of $\mathbb{Q}$, say K, which only have primes that rammify in a finite given finite set of primes Σ then the rank of the group of the $K-$ valued points of E is bounded by B. I learned that given any elliptic curve and any algebraic number field L, you can tensor the L valued points with $\mathbb{C}$ to get all of the possible representations at a multiplicity depending only on the representation itself (and in particular, not on the particular field extension L). Thus for each L, if we are able to show only a finite number of these representations have multiplicity nonzero, we can argue that the rank will always be finite.

(11/30/2017) Today I learned a pretty cool fact about L series associated to characters χ. What I showed in particular is that there is a unique character of conductor $f, 4f, 8f$ for any squarefree odd f and this shows that the conductor we used to get a given quadratic field inside a cyclotomic field was canonical. You can use this fact to prove that $\prod_{\mathfrak{p}|p}(1-N(\mathfrak{p}^{-s})^{-1}) = (1-N(p)^{-s})^{-1}$ for any complex s, which you can use to compute Dirichlet series from Dirichlet series of the character by multiplying all these primes! This in turn gives you one way to compute the L series of the character in terms of the class number. But you can also compute this another, more straightforward way that does not involve the class number. This provides a formula for the class number, at least of quadratic field extensions!

October 2017

(10/1/2017) Today I learned so much about the blow up of 2 dimensional affine k space at a point. First of all, I learned what it is–it's essentially taking the point $(0,0)$ and deciding we need to allow a bunch of lines to pass through it instead of it just being the point $(0,0)$ anymore. Then, I learned what the fiber over each closed point $p \in \mathbb{P}^1_k$ is–it's just a one dimensional affine $K(p)$ space! Normally to compute a fiber product over a nonaffine scheme you have to break the nonaffine scheme into a union of affine open schemes and figure out gluing. But of course, if you cleverly pick all but one of your affine opens to not have the point you're taking the fiber over, you can help yourself out. This makes it easy to compute. I also learned the idea of the blow up. You'd want points (p,l) where p is a point on the line l, a line passing through the origin.

(10/2/2017) Today I learned that the fiber of the blow up map $Bl_{(0,0)} \to \mathbb{A}^2_k$ at the point $(0,0)$ is an effective Cartier divisor! I also learned that the fact that you have an isomorphism of global sections of a map $X \to Spec(k)$ for X a scheme definitely does not imply that the map is an isomorphism. This helped in my work, and the map $\mathbb{P}^1_k$ shows this.

(10/3/2017) Today I learned that the existance of any nontrivial subfield of an algebraic number field immediately implies its discriminant over $\mathbb{Q}$ is not squarefree. To see this, note that if you have nontrivial inclusions $\mathbb{Q} \subseteq K \subseteq F$ then you can use the primitive element theorem to write $K = \mathbb{Q}(\alpha), F = K(\beta)$ for appropriate $\alpha \in K, \beta in F$. Then $\{\alpha^i\beta^j\}$ as i,j vary appropriately form a basis of $F/\mathbb{Q}$ which you can use block matrix multiplication to show that the determinant must be a power of the discriminant of F over K.

(10/4/2017) Today I learned that the product of two irreducible integral finite type k schemes, where k is an algebraically closed field, is itself an irreducible integral finite type k scheme. Essentially this follows because you can prove a lemma about a scheme being irreducible if and only if it's irreducible on an open cover where any two sets intersect to nonempty sets, and then you can argue that on an open cover $A \otimes_k B$ the product is an integral domain because you can mod out one of the rings by a maximal ideal not containing your nonzero elements (which gives k by Nullstellensatz). I also learned about the Segre embedding of projective space $\mathbb{P}^n_A \times_A \mathbb{P}^n_A \hookrightarrow \mathbb{P}^{mn+n+m}_A$ which is basically "multiply all the coordinates together and then keep them in a matrix sort of thing.

(10/6/2017) Today I learned that the product of two projective spaces, say $Proj(S)$ and $Proj(T)$, is the projectivization of another ring, specifically, it's $Proj(\oplus_0^\infty S_n \otimes T_n)$. This essentially comes from an isomorphism that for each $(f,g) \in S \times T$, $((\oplus_0^\infty S_n \otimes T_n)_{f \otimes g})_0 \cong (S_f)_0 \otimes (T_g)_0$, so locally everywhere the ring is the product of the two projective spaces. The informal way I think about this isomorphism is that if you had a function $s/f^n \otimes t/g^m$ then it's basically the same thing as $(sf^m \otimes tg^n)/(f \otimes g)^{n+m}$.

(10/17/2017) I think I lost all my "What I Learned's" in Germany somehow. This is a bummer because I did write them. Oh well. It's not like I need to be writing this every day. Anyway, today I learned about the decomposition subgroup. It's the fact that the Galois group acts transitively on prime ideals of a larger field lying over a smaller one, and so you can define the stabilizer and deduce some important properties. I also learned that in any "Eisenstein" polynomial for a prime p and root α, you can argue that you can apply Kummer's lemma. This follows because you can show that $p \not| [O_F : \mathbb{Z}[\alpha]]$ by showing that $p | \alpha^{n-1}$ where $n := deg(\alpha)$ and then use this to show that $p^n | a_0^{n-1}$.

(10/18/2017) Today I learned that if you have a finite extension of algebraic number fields L/K and you pick a prime $\mathfrak{p}$ of the ring of algebraic integers and a prime $\shortparallel$ lying over it with decomposition subgroup D, then you get a map $Gal(L^D/K) \to Gal(k_\mathfrak{q}/k_\mathfrak{p})$ which is just realizing that the action of a Galois group element fixing $\mathfrak{q}$ also is an automorphism of the quotient fields.

I learned this map is surjective, which comes from a black box that says that $k_{\mathfrak{p}} = k_{\mathfrak{q}^D}$, where $k_{\mathfrak{q}^D} = \mathfrak{q} \cap L^D$, giving us that we can lift any root of a generator of $Gal(k_{\mathfrak{q}}/k_{\mathfrak{p}})$ to a root of some universal minimal polynomial, and then the result follows since the action on the Galois group is transitive on roots of a minimal polynomial.

(10/19/2017) Today I learned that in any algebraic number field K and an algebraic extension L/K with primitive element θ and minimal polynomial $f(x) \in F[x]$, then you can factor $f(x) = f_1(x)...f_n(x) \in K_{\mathfrak{p}}[x]$ and that will give you each the primes that your ideal can extend to in L, and moreover that $K_{\mathfrak{p}} \otimes_K L$ is the direct sum of these.

(10/20/2017) Today I learned through a spectacular diagram chase that $(X \times Y) \times (X \times Y) \cong (X \times X) \times (Y \times Y)$. Boy today was a long day (NSF!)

(10/22/2017) Today I learned a new proof that if the discriminants of two algebraic number fields have no common factor and the fields are linearly disjoint Galois extensions then the ring of integers simply scales multiplicatively, meaning that if K, L are the two field extensions then $O_{KL} = O_K O_L$. This is essentially because you can show that for both fields, the discriminant of the field times any coefficient on an element of O_{KL} (written in the field basis of the product of bases), and you can use the adjoint matrix and the fact that Galois elements take integral elements to integral elements to show this.

(10/23/2017) Today I learned the full inner workings of the magic diagram. Basically I wrote out the whole proof of what, at the level of sets, reduces to "You don't get any new information of the product if you specify what the point is that the two elements in the product are equal." One of the main tools I used today was a lemma which said that you can show that two maps to a product are equal by simply showing that their projections are both equal. It's pretty obvious, but worth staying explicitly.

(10/24/2017) Today I learned a bunch of random shit that comes with the basics of separability. Like I learned the fact that the composition of two separated/quasiseparated maps is separated/quasiseparated, and that's because you can show the diagonal map of the composite is the composite of the first diagonal map with the map $X \times_Y X \to X \times_Z X$ which turns out to be a projection map because of the argument of the magic square, which in particular is the base change of the second diagonal morphism. In particular this gives that the diagonal map is the composite of two closed embeddings/quasicompact morphisms, so thus the diagonal map is a closed embedding/a quasicompact morphism. Similarly I learned that the product of two separated schemes are separated.

(10/27/2017) Today I learned a lot of number theory. Specifically, I learned why given a prime β lying over another prime $\mathfrak{p}$ in an algebraic number field extension L/K, the decomposition subgroup $G(\beta)$ acts as the Galois group of the completion field $L_\beta/K_{\mathfrak{p}}$. In particicular, it first fixes the field L_β because you can show that any element in $G(\beta)$ not only fixes L_β, but it also fixes the ideal β/β^2, so it sends a uniformizer to a uniformizer (where the action on the completion is given by realizing it as embedded in $K_{\mathfrak{p}} \otimes L$). You can also argue that $G(\beta)$ fixes only $K_{\mathfrak{p}}$ by associating it to the fixed points of the whole tensor product $K_{\mathfrak{p}} \otimes L$, which you can show by an "equalizer/average " function $\frac{1}{|G|}\sum_{\sigma \in G} \sigma$ is just $K_{\mathfrak{p}} \otimes 1$.

(10/28/2017) Today I learned all about completions, and hwo they can make dealing with prime ideals so much easier. In particular, I learned that you can realize the Galois group fixing a prime $\mathfrak{q}$ lying over $\mathfrak{p} \subseteq L$ of a field extension of algebraic number fields K/L, the *decomposition subgroup*, by passing to the completions $K_{\mathfrak{q}}/L_{\mathfrak{p}}$ and then using Hensel's lemma to purely split up the extension into a part that doesn't rammify (corresponding to the extension of having a root of unity, where the root of unity it is is determined by what's happening in the finite field extension and Hensel's lemma) and then a totally rammified part (because you can argue through some dimension counting) that all you can do above the root of unity is rammify.

(10/29/2017) Today I continued my awesome journey on the path of learning why completions are God's work made for making prime ideals split into their rammify parts, their "inertia field increasing" parts, and their "splitting into many primes part". In particular, I learned that if you have an extension where the decomposition group associated to a prime is normal, then you can use Galois theory to show that you can find an increasing chain of fields (specifically, the ground field, the field fixed by the whole decomposition group, the field fixed by the inertia subgroup, and the top field) such that, going up in order, the prime only splits into all its pieces, with no rammification or inertia degree, then in the next field extension all the primes remain inert, and then in the next one any rammification occurs. I also learned about the *Artin map* for abelian extensions, which takes the group of fractional ideals of a field extension L/K whose Galois group is abelian to the Galois group via, on primes, $\mathfrak{q} \to (\mathfrak{q}, L/K)$. I also learned that this map is surjective, but do not yet have a proof for this fact.

(10/30/2017) Today I learned how the Artin map interacts when you slide up an abelian extension L/K to the abelian extension EL/E. It turns out that you can determine the new Artin map by simply applying the norm map and taking the old one. This is essentially because if you take the norm of a prime of L, you get the prime ideal lying over it to the relative degree power, and you can also trace through the definition of the new prime (and how the Frobenius map acts on the new finite field extension) to see that's also exactly what happens applying the Artin map to the new prime. I also learned (first, actually) the domain of the slid up function can be taken to be the free group generated by the ideals that don't divide one of the "bad" primes that we didn't allow in the original Artin map. This is because you can pass to a completion and show the contrapositive. Namely, if a prime doesn't rammify upon passage to E, then the original extension had to be adjoining an appropriate root of unity. Then you can show the composite extension also was adjoining a root of unity.

(10/31/2017) Today I learned a spooky "primitive element" like theorem for finite extensions of $\mathbb{Q}_p$ that I think you can pretty reasonably extend to finite extensions of any complete field. The theorem actually states that the ring of integers of $F/\mathbb{Q}_p$, a finite extension, is just $\mathbb{Z}_p[\alpha]$ for some α. The way you find this is that you use Hensel's lemma to lift up a primitive root of unity of highest order you can, and then first note that ring of integers of $F/\mathbb{Q}_p$ is generated by the uniformizer and that root of unity by a power series argument. Then you can show ring of integers of $F/\mathbb{Q}_p$ is generated by the sum of the uniformizer and the root of unity (over $\mathbb{Z}_p$ because the sum of the normalizer and the root of unity raised to the order of the residue field to the n^{th} power tends to the root of unity as $n \to \infty$ and the $\mathbb{Z}_p$ module is closed since it's a finitely generated module over a PID and all norms are equivalent on it.

September 2017

(9/3/2017) Today I learned about the two rulings on the quadric surface $Spec(k[w,x,y,z]/(wz - yx))$. Another way to imagine this is that the quadric surface X is parametrized by $\mathbb{P}^1 \times \mathbb{P}^1$. I've showed that each point on the quadric surface has a surface from each line passing through it, which you can essentially do by doing the algebra you might think to do originally and then make it carry over scheme theoretically using the $(x-a, y-b)$ means $x = a, y = b$ idea. Today I also learned that you can make Simpsons characters in LaTeX after downloading a style file. Here's Homer. But sadly, this doesn't work on the computers at school, so I had to comment it out. But all you have to do is download a file, usepackageSimpsons with a backslash and then type backslash Homer!

(9/4/2017) Today I learned that through each point there exists a *unique* line passing through it from each family of lines on the quadric. Essentially the "moral" I learned from this was to treat it like it's regular old algebra, where x, y, z etc. represent numbers, and manipulate it formally, algebraically. Then the next step to solving a problem is to figure out how you can apply the formal algebra in the setting where not all points are the traditional points.

(9/5/2017) Today I learned a few things. One of the things I learned was that you can determine whether a set with the right number of elements is a basis if and only if its discriminant is nonzero. This is helpful in the interpretation of the discriminant as a volume form, and is not too hard to prove. The reason is that you can use a change of basis formula to get that the discriminant of any basis is just a square multiple away from any other basis. At least over separable fields, you can use the primitive element theorem and the specific determinantion on the particular basis of powers of one element to conclude it's not zero, by separability. Awww yeah baby. Also I solved a problem regarding the weights of projective space, in particicular, showing that if x_m, y_n have weight $m, n \in \mathbb{N}$ respectively, then $Proj(k[x_m, y_n]) \cong \mathbb{P}^1_k$. Also I learned what Feynman Diagrams were in zero dimensional quantum field theory and how they can be used to compute certain integrals that come up in the subject.

(9/6/2017) Today I learned what the Affine Cone of a graded ring is. Well, they actually call it the affine cone of $Proj$ of the graded ring, but it should better be called $Proj(S_*)$, where S_* is a graded ring. What it is is just $Spec(S_*)$, but what's more important is that if S_0 is a field, then the irrelevant ideal of S_*, S_+ is just a point, which we can call the origin, and we can note that if we take out the origin we can map our space naturally to $Proj(S_*)$. You can view a picture of this and note that $Speck[x, y, z]/(x^2 + y^2 - z^2)$ gives the exact cone shape you're looking for.

(9/7/2017) Today I learned what the scheme theoretic image is. It's defined in the following way– we're identifying a closed subscheme of Y, where $X \to Y$ is our map, and we're going to identify it by the ideal sheaf, which I've called the "ideal picker" before. Anyway, we have talked about how we can intersect closed subschemes by adding the ideal sheaves and this process actually does give an ideal sheaf, so you can ask for the maximal sheaf such that the inclusion combined with the pullback map is the zero map. Then I worked on a pathological example. If you embed the infinite disjoint union of points with increasing fuzz to the affine line, it turns out the scheme theoretic image is the entire line (no polynomial function besides zero vanishes at every point with fuzz) but set theoretically it's just the origin point. But apparently in "good" cases the topological image closure is the scheme theoretic image's topological space.

(9/8/2017) Today I learned a few times when the scheme theoretic image can be interpreted affine locally. So for example, in the case where the domain is reduced, the ideal sheaf picker of our sheaf theoretic image at every affine level must be inside the kernel of the associated ring map. So in particular if it turns out that associated ideal picker that picks every kernel of the associated ring maps gives you a closed embedding (means that the localization condition holds) then that has to be the closed subscheme associated to it. First I learned that in two pretty standard conditions, when the domain is reduced you have the localization property of the ideal picker held. Also, if your morphism is affine, or more generally quasicompact, you can show that the ideal localization property you need to obtain a closed subscheme holds. Finally, I worked through an example to see where the proof fails, countably many points with fuzz increasingly large.

(9/10/2017) Today for a small part of the day I learned that the image sheaf of a reduced scheme is reduced. But really, this should make some sort of sense because reducedness means that every ring is reduced. But at least on the affine local level, the functions on the image sheaf are isomorphically the functions on the domain anyway, which are reduced. For most of the day I spent showing that in two nice cases of a map of schemes $\pi : X \to Y$ that can be factored as a closed embedding and then an open embedding can also be factored the other way around. (The two nice cases being either π quasicompact or X being reduced). This stems because you can construct an isomorphism between the closed image sheaf of the closed embedding and the restriction subsheaf of π to the open set you're embedding, and the fact that X is isomorphic to its image as a sheaf in these cases.

(9/11/2017) Today I learned what a locally principal closed embedding is–quite literally, it's an embedding that's locally isomorphic to something cut out by a single equation. Hyper surfaces in $\mathbb{P}^n_k$ are good examples of this. I also learned a few equivalent definitions of the reduced scheme of a given closed set $X \hookrightarrow Y$, and why they're equivalent. You can define it as the smallest closed subscheme whose topological space contains X, or you can define it affine locally (which you can show are equivalent because the affine local set must contain all of the non nilpotent elements to contain X and does in our case) and you can also show it's the image of a particular map of schemes $\rho : W \to Y$ where W is the disjoint union of all points of X where on each point the functions are just the residue field.

(9/12/2017) Today I learned the idea of an $\mathbb{E}_{\text{inf}}$ ring spectra–essentially it's a ring structure in the homotopized version of algebra, meaning that it's a ring but every time equality holds it really holds "up to homotopy." I also learned that in a spectral sequence associated to a double complex, the first map is required to be the rightward map and the next one is required to be the upward map and from this you can derive a natural map that goes three up and two left.

(9/13/2017) Today I learned the natural way to construct the second page of a given spectral sequence morphism associated to a double complex. It's actually just diagram chasing, but the part that really tripped me is that your element has to be d_{up} of some element. However, it turns out that the particular element need not be in $im(d_1)$, since you don't know that element itself is actually in E_1.

(9/14/2017) Today I learned why the following map is an exact sequence $0 \to E_2^{0,1} \to H^1(E^*) \to E_2^{1,0} \to E_2^{0,2} \to H^2(E^*)$ is exact for a double complex, where every map is induced by the vector space except for the differential map $E_2^{0,1} \to E_2^{2,0}$. I also proved that it was well defined all as maps. It was pretty freaking hard. I also learned a new way to show why you can factor ideals of Dedikind rings into prime ideals, using the fact that you can actually show that $II^{-1} = R$. It relies on a nice lemma which says that for any proper ideal $J \subseteq R$, there exists a $\lambda \in K(R) \setminus R$ for which $\lambda J \subseteq R$.

(9/15/2017) Today I learned the proof of the five lemma using spectral sequences. The five lemma says that if you have two short exact sequences and maps with commuting squares between the entries such that all but the middle are isomorphisms, then the middle one must be as well. (This can actually be weakened to the leftmost map merely only being a surjection and the rightmost map being only an injection). The reason for this is that if you compute the spectral sequence using the rightward orientation, a bunch of zeroes pop out due to exactness. Thus the cohomology of the total complex must also be zero. But then computing it the right hand way, you can show that what remains at the infinity page at important spots is the kernel and cokernel of the middle map.

(9/17/2017) Today I learned one application of Spectral Sequences–they show that any rearrangement of any regular sequence of a finitely generated A module M, where A is a Noetherian local ring with maximal ideal $\mathfrak{m}$, is in fact a regular sequence upon rearrangement! You can prove this by arguing that you can swap any two elements $\{x, y\}$ in any finitely generated module M which in that order form a regular sequence, which follows because the properties of local rings essentially fit right into the properties of being a spectral sesquence. You compute one of the orientations to see it's zero everywhere you care about (in particular, in 0 and 1 cohomology) and then you do it the other way to see that $x\{m : ym = 0\} = \{m : ym = 0\}$ which since $y \in \mathfrak{m}$ and

Nakayama's lemma implies that $\{m : ym = 0\} = 0$ and similarly you can read off from the spectral sequence that y isn't a zero divisor of M.

(9/18/2017) Today I learned, courtesy of a Math Stackexchange post by Tom Oldfield, why it was that given "the equality of two ideals when localized away from a prime ideal implies that the two ideals are equal after localizing just one function," you could argue that a closed embedding being a regular embedding at a point is an open condition. This is because if you have a point and a regular sequence there, you can first find an open set where the kernel of an ideal is given by some functions which are a regular sequence at that one point. Then you translate "being a regular embedding at a point" into a statement about the kernel of a few maps being zero, which are ideals, and then argue that the localization can happen for each of the finitely many ideals for a function associated to them. Then you can just localize via the product of all the funcitons to show that in that neighborhood the sequence forms a regular sequence.

(9/19/2017) Today I learned that in a Dedikind ring, a ring being a UFD is equivalent to the ring being a PID. This actually seems reasonably obvious in hindsight (as does most of the math I do ever) viewing the interpretation of a Dedikind ring as being a Noetherian, integrally closed, integral domain with Krull height one–by factoring reasons you argue you can show this for prime ideals and then for any nonzero prime ideal you simply take any nonzero element in the prime ideal, factor it, take the element of the factorization that's in the prime ideal, and use Krull height ness to get that the prime ideal must be generated by that irreducible element itself. I also argued that you can localize far enough in Noetherian rings so that if an element is a nonzero divisor at every local ring, then it is a nonzero divisor on some open set of the spectrum of the ring.

(9/20/2017) Today I learned that in a locally finite type k scheme, a point is a closed point if and only if its residue field is a finite extension of k. The "only if" essentially comes from the Nullsellensatz, and the only if part comes from the fact that if you're given any possible affine set containing the point, then you can show that the ring modulo the prime ideal embeds into the residue field. Then, because by assumption the residue field is a finite extension of k, you have an integral domain embedding into a finite extension of k. Thus it's a field! (And a closed point, since it's maximal in every open set).

(9/21/2017) Today I learned a neat little thing which says that given a fractional ideal I of the ring of integers R in an algebraic number field K, there exists a nonzero *integer* m (whereas before we were only guaranteed a nonzero element of the ring) such that $mI \subseteq R$. The reason for this is that every element of R is integral over $\mathbb{Z}$, so if we take some $\alpha \in R$ element where $\alpha I \subseteq R$, then write the minimal polynomial and then take the nonzero integer on the coefficient and write it as the product of α times an element in the ring. Then you know that that integer times I, then, is the same thing as an $R\alpha$ element times I, which must then be in R.

(9/22/2017) Today I learned the main piece for why the product of schemes $X \times_Z Y$ exists, which is proving that it exists when X and Z are affine. Essentially, what you have to do is write Y as the union of affine open sets and then argue that you can use the equivalence of categories between affine schemes and rings (and the fact that the tensor product is the coproduct in Ring) to argue that these local products exist. Then you can use the fact that the intersection of any two of these affine open sets embeds into the affine open set, and then the product there exists (by a special case argument) and then you argue that you can glue all these products together (although I haven't gotten why the cocycle condition holds yet) and then argue locally that you get a unique map going to that glued scheme in the right setup for products.

(9/24/2017) Today I learned that if one is given a complete field K which is complete with respect to a nonarchimedian valuation and the associated prime ideal $\mathfrak{p}$ doesn't rammify in a finite extension L/K of fields with valuations, then there's only one way that can occur–if $L = K(\psi)$ for a certain primitive root of unity ψ. This essentially follows because you can take the quotient of

L modulo $L\mathfrak{p}$ and then because ramification doesn't occur (and moreover there's a unique prime of the ring L) then your quotient is a field, which has a certain primitive root of unity. You use Hensel's lemma to lift up that root of unity to a primitive root of unity $\psi \in L$, and then you use Nakayama's lemma to show that $K(\psi)$ really is L, and not just a subset.

(9/25/2017) Today I learned why the fucking morphisms glue to show that the product of schemes exist. Essentially, you want to show the cocycle condition holds for affine open schemes glued by their associated intersections. So what you do to show that the two maps you need to show are equal actually are equal is to argue that all three of the restrictions to the triple intersection could have been a possible triple intersection. Then you just argue that both of the maps you need to agree could serve as a certain product map, and then since there's a unique one of those you show they're equal. I also learned what it meant for a functor to be representable today–it means that the functor is naturally isomorphic to the contravariant functor from schemes to sets $h_X(Y \to Z) = Hom(Z, X) \to Hom(Y, X)$ via pullback. This is sort of a more sophisticated way to prove the product exists–you can define a product of functors (since morphisms are in set, these exist) so now you just have to show it's representable.

(9/26/2017) Today I learned these two weird tricks to compute any tensor product $A \otimes_B C$ of any two B algebras A, C. Basically, it stems from the not too easy but not too hard fact that $A/I^e \cong A \otimes_B B/I$ where if $\phi : B \to A$ is our algebra map and $I \subseteq B$ is some ideal than I^e is the ideal generated by $\phi(I)$. This is true, for what I've been able to come with currently, just directly. You can see where the A lies and if you map $A \to A \otimes_B B/I$ you can use the map $A \otimes_B B/I \to A/I^e$ to show that any element in the kernel was also in I^e to start with. Also the "adding variables" trick, which is easier and just notes that if $\phi : B \to A$ then $B[x] \otimes_B A \cong A[x]$. You can now express any ring map as the domain ring, plus a bunch of variables and then all the relations and use this trick combined with a clever tensor rewriting to get the tensor product of any two rings!

(9/27/2017) Today I adventured into the wild and crazy world of a few properties of k schemes and maps of them that, if the property holds for base change over a certain field extension l, then it had to hold for the original map. The main thing I proved today was that if l/k was a finite extension, then the above applies to the property of the associated ring being a normal integral domain. Also I learned that even though the tensor product might naturally have a ring structure to it, the induced maps only need to come from bilinear maps, which was incredibly helpful in proving that!

(9/28/2017) Today I learned a bunch of shit related to the class group. For example, one of the things I learned is that there's an application of Minkowski's theorem which gives a *Minkowski bound* such that for all elements in the class group, there is an integral element whose norm is bounded by that Minkowski bound. This makes it much, much easier to compute class groups, especially in small field extensions such as $\mathbb{Q}[\sqrt{d}]$. Actually, today I specifically computed that when $d = 5, 19$ then the group is trivial, and $\mathbb{Z}/2\mathbb{Z}$ respectively.

(9/29/2017) Today I learned how to take the preimage as a scheme, properly. The first thing I learned was that topologically, if you have a map $\pi : X \to Y$ then you have an isomorphism $\pi^{-1}(\{y\} \cong X \times_Y p$ (again, of topological spaces) which leads you to define the definition of the inverse image of a map of a map of schemes $\pi : X \to Y$ to be $X \times_Y p$, where the map $p \hookrightarrow$ is given locally by the map $B \to B_p/p$ for each open $Spec(B) \subseteq Y$.

August 2017

(8/2/2017) Today I learned a neat trick in the world of completing fields with absolute values on them. Essentially you can realize the Cauchy sequence $\{a_i\}$ in the completion as a limit of the constant sequences, i.e. $\lim\{\{a_1, a_1, ...\}, \{a_2, a_2, ...\}, \{a_3, a_3, ...\}, ...\}$ and use this to show that every extension (and in particular, the identity) can be extended to a *unique* map. This is used to show that the completion of a field is unique up to isomorphism.

(8/3/2017) Today I learned that you can specify a map from affine $A-$space to affine $A-$space by saying where functions go *or* by saying where points go, it turns out it's the same thing! The reason is, if I give you the map $(x, y) \to (p(x, y), q(x, y))$, say, and you at first interpret that as a pullback map of functions $\pi^\# : A[x, y] \to A[x, y]$, then $(\pi^\#)^{-1}(x - a, y - b) = \{r(x, y) : \pi^\#(r) \in (x - a, y - b)\} = \{r(x, y) : r(p(x, y), q(x, y)) \in (x - a, y - b)\} =$ "The set of points where if you plug in $x = a$ and $y = b$ zero comes out" $= \{r : r(p(a, b), q(a, b)) = 0\} = (x - p(a, b), y - q(a, b))$. This makes me really happy!

Also I learned about the concept of a locally closed set, and how if you have a constructable set then it's actually just the union of disjoint locally closed sets. I am also barking at the door of Hilbert's Nullstellensatz!

(8/4/2017) Today I learned the proof of Nullstellensatz, assuming that we know Chevalley's Theorem, which states that given any finite type morphism between constructable schemes, the image of any constructable set (and in particular, the space itself) is a constructable set. Combine this with the fact that the point that's just the generic point of $Spec(k[x])$ isn't a constructable point, and you can argue that any field generated finitely as an algebra is also generated finitely as a module, since otherwise you can embed $k[x]$ into the field and show that the image is the generic point!

(8/5/2017) Today I learned how the Nullstellensatz implies a "weak" form of the Nullstellensatz, which basically says that if $\mathfrak{m}$ is a maximal ideal of $k[x_1, ..., x_n]$ for some algebraically closed field k, then $\mathfrak{m} = (x_1 - a_1, ..., x_n - a_n)$ for some $a_i \in k$. This is because you can reduce to the one variable case by intersecting with $\mathfrak{m} \cap k[x_i]$ and then arguing that if you have finite generation as a moudule, then $\{1, x_i, x_i^2, ...\}$ solves some polynomial equation. In particular that polynomial equation lifted up is in $\mathfrak{m}$. But you can factor it as a product of $(x - \beta_j)$ for some roots, and since $\mathfrak{m}$ is in particular prime one of those roots is in the prime ideal.

(8/6/2017) Today I learned the proof of the Grothendieck Freeness Lemma (and how to spell Grothendieck.) The Grothendieck Freeness Lemma says that if B is a Noetherian integral domain and A is a finitely generated B algebra, then for any finitely generated A module M there exists some nonzero $f \in B$ such that M_f is a free B_f module. The proof comes from the fact that this is true for $A = B$ (essentially because if $A \cong B/I$ for some nonzero ideal I, then you can localize by a nonzero element to make the localization of A zero, which makes it free by definition). Then you can argue that it suffices to show that A is satisfies the theorem implies $A[T]$ does shows our theorem is true and then use some sexy category theory to write M as a direct sum of finite A modules.

(8/7/2017) Today I learned that if you have a field K that is complete with some respect to a non-archimedian valuation and R is the ring associated to this valuation, and the ring just so happens to be a DVR, then if you take the integral closure of that ring R in a finite dimensional separable extension, that ring is a DVR also, and there's a unique way to extend that valuation to the field extension (which essentially follows because you can show that the unique prime ideal of R rammifies totally, which you can prove using the Chinese remainder theorem and the fact that idempotent elements lift.) I also learned that you can reduce the problem of showing that $Spec(A) \to Spec(B)$, a finite type morphism of Noetherian schemes with $Spec(B)$ irreducible, has a nonempty open set $V \subseteq B$ that is either entirely contained in the image or not at all touching the image can be reduced to the case where B is an integral domain. This is essentially because *as topological spaces* $Spec(B) \cong Spec(B/I)$, where I is an ideal contained in all prime ideals of B.

(8/8/2017) Today I proved that it suffices to reduce the problem of yesterday can actually be reduced to the affine case. Essentially, you can use the compactness of both of your schemes to argue that you can cover your target scheme in a finite number of affine schemes such that the preimage of each one is the finite union of some affine schemes where the induced map gives a finitely generated algebra. Then assuming it works in the affine case, you get a finite number of open $V_{ij} \subseteq Y$, if $\pi : X \to Y$ is your morphism. If any of them are in the image of the smaller subset then they're in the image of the larger subset (obviously) and if the image of the smaller subset (i.e. the nice affine scheme we had discussed earlier) avoids the associated V_{ij} for all of them, just take the intersection of all of them, which is still open and nonempty since the finite intersection of dense open subsets is still a dense open subset.

(8/9/2017) Today I spent the entire day showing that Chevalley's Theorem implies that the surjectivity of any map between affine schemes on closed points implies that the map is surjective on all points. One thing I did was reduce this to the case where the thing you're quotienting out by was a prime ideal, i.e. your variety is actually an integral domain. Another thing I did was spend a lot of the day using maximal ideals/closed points to try and prove that a prime ideal is the intersection of all maximal ideals containing it in $k[x_1, ..., x_n]$. This makes sense–if you imagine the functions on the curve $(x - y)$, it's exactly the stuff that's valid functions on all of the closed points $(x - a, y - a)$. Hopefully tomorrow.

(8/10/2017) Today I learned about a stronger form of the Nullstellensatz dealing with Jacobson rings–i.e. rings where each prime ideal is the intersection of all maximal ideals containing it. This, combined with Chevalley's Theorem, implies that the surjectivity of a map on a $k-$ variety on closed points gives the surjectivity of the entire map, for if the image contained all the closed points, the compliment, also a constructible set, contains none. But if it contains any point, you can use the fact that your ring is a Jacobson ring to show that the set of maximal ideals are dense, and thus the open set must contain a closed point.

(8/11/2017) Today I learned the proof of Chevalley's Theorem! Essentially you can argue that a corollary of Grothendieck's Freeness Lemma says that given any map of schemes to an irreducible Noetherian scheme Y there exists a nonempty open set U such that either the image contains U or is entirely disjoint from it. You can break these into two cases and make an algorithm iterating finding this open set, which can't go on forever, lest you have a decreasing chain of closed ideals. Therefore it eventually must stop, which you can show must give U being the entire image.

(8/12/2017) Today I learned the proof of the Fundamental Theorem of Elimination Theory, which says that the natural morphism $\mathbb{P}^n_A \to Spec(A)$ (given by the fact that $\mathbb{P}^n_A$ is an $A-$scheme) is a closed map. This doesn't sound so sexy, until you unpack what it says algebraically. This essentially says that if you're given some polynomials in $A[x_0, ..., x_n]$, you can write polynomials in A itself to tell you when homogeneous polynomials in $A[x_0, ..., x_n]$ have a nontrivial solution for substituting the x_i's for elements of A. The essential idea is to associate whether a point $\mathfrak{p} \in Spec(A)$ is in the vanishing set by looking back at the quotient field $\kappa(\mathfrak{p})[x_0, ..., x_n]$) and then using some linear algebra to show that $Proj(\kappa(\mathfrak{p})[x_0, ..., x_n]))$ has a point in the vanishing set if and only if a certain set of linear maps aren't surjective–i.e. a certain larger set of determinants are zero.

(8/13/2017) Today I learned about closed embeddings. I learned a lot of things about them! First of all, a closed embedding is an affine morphism such that the induced morphism of rings on each affine open subset is a surjection. This isn't immediately obvious, but having this on all affine open sets says that the map actually identifies the domain with a closed subset of the codomain. Moreover, I learned a bunch of easier properties, like the fact that it's a finite (and thus finite type) morphism and the property of being a closed embedding is affine local on the target. Also I learned that there's a necessary and sufficient condition for an *ideal sheaf* to give you an actual closed embedding. What needs to happen is that you have to have an ideal sheaf on each affine open subset and for any $B \to B_f$, the ideal chosen for B_f must be the f localization for the ideal chosen or B. Then you can build a closed subscheme out of it!

(8/14/2017) Today I learned a pretty fact about prime ideals and their relation to the finite intersection of ideals of a ring. Specifically, if A is *any* ring, $\mathfrak{p}$ is a prime ideal of A, and $J_i \subseteq A$ are a finite collection of ideals in A, then $\mathfrak{p} \supset \cap_i J_i$, then in fact $\mathfrak{p} \supset J_j$ for some j. That's pretty cool. The proof works by induction, with the base case trivial and can be reduced to the case of $i \in \{1, 2\}$ through clever use of parenthesis. Assume swapping if necessary that $J_1 \not\subseteq \mathfrak{p}$. Then there exists an $f \in J_1$ but $f \notin \mathfrak{p}$. But for all $g \in J_2, fg \in \cap J_i \subseteq \mathfrak{p}$. Thus $g \in \mathfrak{p}$ for all $g \in J_2$. Really pretty.

(8/15/2017) Today I learned about the power of love. Ha ha ha. Just kidding. I learned something more valuable than that–what it means to take the closed subsheme of projective space! It's not too hard to define because you just give a set of homogeneous polynomial equations of some degree and it determines a closed subscheme and on the different gluing patches using the standard open cover you can show that the vanishing sets agree since the different homogeneous polynomials only differ by some unit in the corresponding rings.

(8/16/2017) Today I learned about the proof that $I = (wz - xy, x^2 - wy, y^2 - xz)$ is a prime ideal of $k[w, x, y, z]$. Essentially you can use all three of those equations to take a polynomial and write it as the sum of polynomials that just have a few variables in them, including a polynomial with just x terms and a polynomial with just y terms. You can then argue that $k[w, x, y, z]/I \cong k[a^3, a^2b, ab^2, b^3]$ by showing that if it just so happens that the polynomial $p(w, x, y, z) \in k[w, x, y, z]$ and $p(a^3, a^2b, ab^2, b^3) = 0$ then you can argue that those polynomials can't have terms that "interact" with each other–i.e. each term has unique monomials associated to it that don't appear in any others. This shows that each polynomial is individually zero!

(8/17/2017) Today I learned about the relation of projective space and linear spaces. Essentially if you take an $n + 1$ dimensional k vector space, say $W = (kx_0 + ... + kx_n)^\perp$ for some chosen elements of a basis, then the symmetric construction $Sym(W^\perp) \cong k[x_0, x_1, ..., x_n]$ so then can define projective W space $\mathbb{P}W = Proj(Sym(W^\perp))$. You can use this to prove that if you have an injection $V \hookrightarrow W$ (then you have a surjection $W^\perp \to V^\perp$ as graded rings) then you get a closed embedding $\mathbb{P}V \hookrightarrow \mathbb{P}W$.

(8/18/2017) Today I learned another proof that finite morphisms are closed. Essentially you can reduce the problem to a map of affine schemes $Spec(A) \to Spec(B)$ and then realize that you can write $Spec(A)$ as a projective space! Namely, if you have B in the zero grading and a copy of A in every other grading, you can realize $Spec(A)$ as embeddable into $\mathbb{P}^n_B$ for some N. And then you write the map as a composition of $Spec(A) \hookrightarrow \mathbb{P}^n_B \to Spec(B)$, which we proved the second maps was closed via the fundamental theorem of elimination theory.

(8/21/2017) Today I learned the product formula for all the possible norms over any algebraic number field. Essentially, viewing primes as the equivalence classes of valuations on the field, the product of all of the valuations is one. Essentially this follows by reducing to the rational case, almost. What's going on is that we take any arbitrary field extension and then we write all the possible completions and associate them to all of the different prime ideals or Galois embeddings depending on whether they are archimedian or not. Then we can manipulate each particular extension of a prime ideal so that they multiply to the norm (and this uses the fact that the characteristic polynomials/norms after a certain decomposition decomposes as products) and then multiplying over all possible norms is just the same thing as multiplying the norm of an element over all posisble rational norms, which gets you 1!

(8/30/2017) Today I learned some facts relating to the Bruhat-Tits tree on SL_2, which today mostly was about linear algebra-esque things involving fields that have a discrete valuation on them. I learned that you can normalize any invertible entries in the field using a technique called integral column/row operations, where the main shift is at first to identify where the smallest power of π, an element whose unit multiples and powers generate the field. I also learned that you can diagonalize any invertible matrix in a discretely valuated field, where "diagonalize" loosely means that you can write it as the product k_1dk_2 where the k's are invertible matrices whose entry and whose inverse's entires are also in the associated discrete valuation ring. You can use this to show that given any two lattices, you can find a basis of one that you can multiply by π_i^m for some $m_i, i \in \{1, 2\}$.

(8/31/2017) Today I learned what the actual tree is for a complete field with a discrete valuation K. In particular, the nodes of the tree are just the lattices of K^2 subject to equivalence that the lattice L is also the lattice πL where π is a generator of the unique maximal ideal of the associated DVR. The nodes are then determined by the principal divisor theorem. I also learned the theorem which essentially says that what we've been calling a tree actually is a tree. Essentially this stems from the fact that if you have any path, where edges are if the principal divisor theorem gives that you can find a basis of one of them (e, f) where a basis of the other is just $(\pi e, f)$. Then you can basically argue that any path can be translated by some $GL_2(K)$ element, using completeness, to the path $Id, \nu_1, \nu_2, ...$ where nu_j has first column $(\pi^j, 0)^t$ and second that of the identity.

July 2017

(7/1/2017) Today I learned that you can separate any finite field extension E/F into a separable extension K/F where every element in K is separable over F, and a *purely inseparable* extension E/K where for each element $x \in E$, there exists a power of $p := char(F)$ such that x raised to that power of p is in K. This result is mostly founded on the fact that the set of elements in a field extension separable over a base field is actually a field, and that relates to how many embeddings of a field you could, potentially, have. This means that an extension of degree n is separable if and only if for every embedding of the ground field into some field L, there is an extension field L' such that this embedding of the ground field can be extended n different ways. And you notice in this proof that all you needed was the separability of some generating set, so you see that "field is separable" iff "n extensions" iff "set of generators is separable". This says that you can get a real thing called the separable closure, and then the inseparable part afterward is just writing the polynomial as a power of x^{p^k} (i.e. so its derivative is no longer zero) and then realizing whatever is left has to be separable, i.e. in the ground field.

(7/2/2017) Today I learned two cool number theory things. One of the things was how to factor a prime ideal $p\mathbb{Z}$ in the larger ring $Z[\theta]$, where θ is a primitive m^{th} root of unity for some m. Essentially, you can first argue using Kummer's Theorem that if you have any prime q that doesn't divide m, then if ϕ_m is the minimal polynomial of θ, then after reducing the coefficients of ϕ_m modulo q, you see that each irreducible factor corresponds to having a primitive root of unity in some field extension of $\mathbb{F}_q$. So, letting r be the smallest positive integer such that the field of order q^r has a primitive m^{th} root of unity, that's the splitting field of the polynomial ϕ_m reduced modulo q. Then you can use Kummer's Theorem to write out the structure of the prime ideals (and moreover, if you traced back, you could compute them explicitly). Moreover, given a prime that does divide m, you can use the fact that (say $p^a||m$) $p\mathbb{Z}$ totally rammifies in $Z[\alpha]$ where α is a primitive p^a-th root of unity. Then you can use the above to sort of "composite extension" your prime ideal, and then use the nifty $efg =$ "degree of field extensions" to show that this technique works (I'm being hand-waivey, but my full proof is in my notes and if anyone reads this and is interested in discussing, feel free to email me.

(7/3/2017) Today I learned the proof of quadratic reciprocity! Essentially if p, q are distinct odd primes then p being a square modulo q is essentially translatable to the factorization of $p\mathbb{Z}[\psi]$ where

ψ is a primitive q^{th} root of unity. Then you can use Kummer's Theorem to turn this statement back on itself, involving reducing the polynomial $x^2 \pm q$ and whether or not it has roots. It's a pretty interesting proof!

(7/5/2017) Today I learned the proof of quadratic reciprocity if one of the primes is 2. Essentially the proof runs pretty similarly, except instead of splitting the polynomial $x^2 - q$ for some odd prime q you're splitting the polynomial $x^2 - x + \frac{1 \pm p}{2}$. I also learned that you can construct some kind of natural map on affine schemes given a ring map $B \to A$. The interpretation of this is exciting to me. The main insight I gained today was to say this: Given a ring map $\psi : B \to A$, thinking of elements of the ring as *functions* instead of just elements, and the functions are on the points $Spec()$. Then given a point $\mathfrak{p} \in Spec(A)$, you can ask, what B functions vanish at $\mathfrak{p}$ by defining $f \in B$ "vanishing" to mean that $\psi(f)$ vanishes. Then as two functions don't vanish if and only if their product doesn't vanish, you get a prime ideal where $\mathfrak{p}$ naturally goes! You can extend this idea to argue that you can make this a map of ringed spaces where the pullback on a base is exactly what you might think it is–with the insight above being the key to showing that it's easy to define and well defined and commutes with restriction, etc.

(7/6/2017) Today I learned about *Minkowski's Theorem*, which says that given any bounded convex (meaning closed under midpoint) set $X \subseteq \mathbb{R}^n$ such that $-X = X$ (called *centrally symmetric*) and a full lattice $\mathfrak{L}$ such that $vol(X) \geq 2^n vol(\mathfrak{L})$, where $vol(\mathfrak{L})$ denotes the volume of a fundamental region, then X contains a nonzero point of $\mathfrak{L}$.

Essentially, the reason this works is that if you're given a set T whose $\mathfrak{L}$ translates are invariant, you can integrate over all the translates of $\mathfrak{L}$ (which you can show is actually a finite sum) to show that the volume of T must be less than the fundamental region (again, by just translating each point into the fundamental region). Then you can apply this to the set $\frac{1}{2}X$ showing that there's a point $\frac{x}{2} + \lambda_1 = \frac{x'}{2} + \lambda_2$ with $\lambda_1 \neq \lambda_2$ and you can show that those conditions imply $\lambda_2 - \lambda_1 = \frac{x'-x}{2} \in X$.

(7/7/2017) Today I finished going through the proof of why a morphism of affine schemes is determined by a ring map! Essentially, this observation comes from the fact that any map of locally ringed spaces on the level of structure sheaf sends all the functions (and only the functions) that vanish at some prechosen point in the preimage of a prespecied point to the functions that vanish at the preprescribed point. But we can identify points with all the functions that vanish on them, so you can use this to argue that the map of global sections determines what the map of topological spaces must be. Moreover, you can extend this to show that the category of Rings is equivalent to the category of affine schemes with arrows reversed!

(7/8/2017) Today I worked hard in two problems in algebraic geometry. One was explicitly showing that the projection map from affine $k-$space of dimension $n+1$ projecting onto projective $k-$ space of dimension n actually is a morphism of schemes. And it's pretty hard, but I'm happy about the fact that I'm getting a lot out of this. Most of what I've learned is in the other problem, which I made more headway on. Essentially, there is a "natural bijection" (which, to be honest, I didn't much work through the natural part yet) between maps from a scheme $(X, \mathbb{O}_X)$ to an affine scheme $Spec(A)$ and maps from $A \to \mathbb{O}_X(X)$. This is because it can be shown on the level of affine schemes and then glued!

(7/9/2017) Today I solved the above two problems! The first one, saying that the projection map from affine $k-$space of $n+1$ dimension onto n dimensional projective $k-$space is actually a map of schemes, can be done in affine coordinates and then the reasons that the gluing maps agree come from a nice sexy commutative diagram I drew and took a picture of. The other problem (see yesterday) essentially did just follow as given proper gluing instructions, there really only is *one* map satisfying those gluing conditions. But what's nice about this fact is that we have a ring morphism if $A = \mathbb{O}_X(X)$, the identity! So you get a canonical map. Here are some more canonical maps I learned about today–I learned that $Spec(\mathbb{Z})$ is the final object in Sch (it's a corollary of the last theorem) and that you can create a canonical map from the spec of a stalk at a point to the whole scheme.

(7/10/2017) Today I learned that no compact complex manifold can be embedded into $\mathbb{C}^n$. This is because if such an embedding occurred, then you could take coordinate functions $\mathbb{C}^n \to \mathbb{C}$ and restrict them to the manifold. But these are holomorphic functions that attain maxima, since the forward image of compact sets is compact. I also learned what a linking number is, which is defined as a way to the intersection of two knots in an "oriented intersection number" kind of form.

(7/12/2017) Today I learned that the tautological bundle is the-1^{st} indexed bundle of $\mathbb{C}P^n$, taking its dual you get the first, and then tensoring you can get $\mathbb{Z}$ distinct bundles of $\mathbb{C}P^n$.

(7/14/2017) Today I learned that the class group of the integral closure of any ring over $\mathbb{Z}$ into some finite extension is a finite group! The reason for this is that you can use lattice theory to argue that given any such ring R, there exists a certain constant M such that for any ideal $\mathfrak{U} \subseteq R$, there exists a nonzero $a \in \mathfrak{U}$ such that $N(a) < MN(\mathfrak{U})$. This helps a ton, because you can use this to show that for every fractional ideal in a ring, you can use this fact to find a multiple of that ideal that is in the ring R and has norm less than M, which essentially comes from the fact that $\frac{N(a)}{N(\mathfrak{U})} < M$ and norms play nice with inverses. Then since the norms of ideals are determined by their product of primes, there's only a finite number of primes there can be in your product for the representative with relatively small norm. Thus your class group is finite.

(7/16/2017) One thing I learned today was the specific constant that proved the above thing and one application of looking at the specific constant. It turns out that you use the specific bound on the norms to show that the positive integer generating the discriminant ideal must be larger than one, which in particular implies that some prime ideal rammifies in any nontrivial integral closure of any finite field extension over $\mathbb{Q}$.

(7/17/2017) Today I learned some things about morphisms of projective schemes, including the fact that morphisms of graded rings induce a map on most, but not necessarily all, of the projective space of the two rings. The fact that it's most and not all can be seen with the inclusion map $\mathbb{C}[x,y] \hookrightarrow \mathbb{C}[x,y,z]$. The question is, where do we send the point $[0,0,1] \in \mathbb{P}^2$? Well, the problem here is that every function in (x,y) vanishes at the point $[0,0,1]$ so the only thing $[0,0,1]$ could be mapped to as a point map is something where everything vanishes. But we specifically design projective space so that every point has a function that doesn't vanish at that point!

(7/18/2017) Today I learned at least two things. One of them was the proof that you can basically think of any graded ring S_* as generated in degree one, provided that you only care about the projective space formed by that graded ring. This is essentially because if you have n generators x_i of degree d_i and define $N := nd_1...d_n$ then you can show any monomial of degree dN is in the S_0 algebra generated by S_N by an inductive argument, using a weighted average point to show you can factor out some monomial of degree $d_1...d_n n$ times. I also learned about the geometric intersection number, which is the minimum number of times two curves have to intersect, and that if you apply a Dehn twist among one of the curves k times, then your geometric intersection number is k times the old geometric intersection number squared. $k > 0$.

(7/19/2017) Today I learned that you can use ideas of algebraic geometry to extend something else I learned today–a formula to obtain all pythagorean triples. Since pythagorean triples are essentially rational solutions to $x^2 + y^2 = 1$, we can pick a "start point", say $(1,0)$ and then any other point has a slope associated with it (as in the stereographic projection, almost). This map is invertible so you can show that all pythagorean triples that aren't $(1,0)$ are basically on the affine line $\mathbb{A}^1_{\mathbb{Q}}$. But then you sort of can extend this rational function $Spec\mathbb{Q}[x,y]/(x^2+y^2-1) \to \mathbb{A}^1_{\mathbb{Q}}$ by including it into $\mathbb{P}^1_{\mathbb{Q}}$. This includes the point we left out earlier, and hints at a theorem I will learn probably next year.

(7/20/2017) Today I learned the proof of Dirichlet's Unit Theorem, which says that if the algebraic closure of some field $K/\mathbb{Q}$ has r distinct real embeddings and $2s$ distinct complex embeddings, then the group of units of the ring of algebraic integers in K is a cyclic group times a free group of order $r+s-1$, which corresponds to the fact that the group of units form a full lattice after taking an appropriate logarithm transformation, which you can show by arguing that you can pick units as generators with large "diagonal" entries when the log transformation is applied and very small, but negative, other entries.

(7/21/2017) Today I used the above Dirichlet's Unit theorem to show that there's a very easy method to determine the **fundamental unit**–the unique positive generator of the group of positive units larger than one–of the ring of integers in a quadratic extension $\mathbb{Q}[\sqrt{d}]$ for some squarefree integer d. Essentially, any fundamental unit must solve the equation $x^2 - ax \pm 1$ for some positive integer a, and the first one that works is your fundamental unit!

(7/22/2017) Today I learned that the ring of integers in $\mathbb{Q}[\sqrt{10}], R$, is not a principal ideal domain, and moreover, it is a ring with class group the unique group of order two. This is because the Minkowski bound gives that every element in the class group has some representative contained entirely in the ring and norm less than 4 in this particular case. This then implies that the class group is generated by the prime divisors of $2R$ and $3R$. You can use norms and rammification to show that $2R = \beta^2$ for some beta, and then show that β cannot be principal by arguing that if it were, we could write $2 = a^2 - 10b^2$, which one can see cannot happen by taking both sides modulo 5. Then you can use the element $\theta := 4 + \sqrt{10}$, which just so happens to have norm 6, to argue $R\theta$ is β times some divisor of $3R$. Thus since $3R$ must factor into two distinct prime ideals (by Kummer's Theorem) you can show that one is the inverse of the other, and thus the group is the group $\langle \beta : \beta^2 = 1 \rangle$.

(7/23/2017) Today I learned a way to extend the concept of an element/field being integral over a field! The idea is to note that if you're talking about these things, you have an injection of fields $K \hookrightarrow L$. So at the level of rings, you can talk about whether a *morphism* $\psi : B \to A$ is integral, or whether an element $a \in A$ is integral over it, by requiring that the element a solves a monic polynomial with coefficients in $\psi(B)$. I also learned/proved a million and one thing about restriction maps and inclusion maps, including that the property of being an open embedding is closed under restriction, "gluing," composition, and base change of fiber products. I also proved that the induced map of schemes induced by inclusion is a monomorphism!

(7/25/2017) Today I learned two theorems relating to algebraic geometry and integral homomorphisms/extensions. One of them is called the *Lying Over Theorem*, which says that given any integral extension $B \overset{\phi}{\hookrightarrow} A$ and a prime ideal $\mathfrak{q} \in Spec(B)$, there is an ideal $\mathfrak{p} \in Spec(A)$ that maps to $\mathfrak{q}$ under the associated map of schemes, i.e. the associated map of schemes is surjective. Also I learned about the *Going Up Theorem*, which says that given an integral homomorphism $B \to A$, if you just so happen to have a list of prime ideals $\mathfrak{q}_1 \subseteq \mathfrak{q}_2 \subseteq ... \subseteq \mathfrak{q}_n \subseteq B$ and some possibly smaller length list of prime ideals in $Spec(A)$ hitting the first few elements, then you can complete that list to primes hitting the larger ones! The map $eval_{x,0} : \mathbb{C}[x,y] \to \mathbb{C}[x]$ can show what's going on.

(7/26/2017) Today I learned a bunch of "finiteness" properties of morphisms, which are three definitions of the form a morphism of schemes $\pi : X \to Y$ is a property P of sets if for every affine open set $U \subseteq Y$ the preimage is that property. (This holds for quasicompactness, quasiseparatedness, and affine-ness at least). Oh by the way–I learned what quasiseparatedness is. It's when the intersection of two compact open sets is still compact. This can actually not happen–for example, glue two copies of $Spec(k[x_1, x_2, ...])$ everywhere but at the origins. Then the two affine open sets are compact but the intersection is $D(x_1) \cup D(x_2) \cup ...$ which isn't compact. I also learned about the quasicompact quasiseparatedness condition, which basically says that the scheme can be covered with a finite number of affine open subsets and the intersection of any two is the finite union of affine open sets. Then there's the quasicompact quasiseparatedness LEMMA, which says that given a scheme, then the sheaf on the set of points where a function s vanishes is just the localization of the structure sheaf.

(7/30/2017) Today I learned about the concept of a finite morphism–this is a morphism that is not only affine, but moreover the inverse image of $Spec(B)$ is a finite B algebra, where finite means finitely generated as a B module. This is something that you can semi-easily show is affine local on the target (at least if you know that the property of being an affine morphism is local on the target). I also learned of a way that you can view any A algebra $A \to R$ as projective R space, you essentially define the zero grading to be A and every other level grading to be a copy of R. It seemed kind of dumb, but I suppose it argues that if something is true of a projective scheme, then it's true for any finite R module.

(7/31/2017) Today I struggled through a topological lemma which said that if $X = \cup U_\alpha$, then a set K is closed in X if and only if $X \cap U_\alpha$ is closed in the subspace U_α topology. It took me a little too long, but honestly it's something you struggle through once and then never forget.

June 2017

(6/2/2017) Today I learned an important lemma which I missed in linear algebra. Well, not exactly, but basically I was thrown by the counterexample I will describe. The theorem says that if A, B are finite dimensional F vector spaces with a surjection $T : A \to B$, then $A \cong ker(T) \oplus B$. This can be proven by taking a basis of both $ker(T)$ and B, noting that the number of elements adds up to the dimension of A and then showing they're linearly independent by taking T of it to show that the B part coefficients are zero, and then noting that you picked a basis of the kernel, which is all that's left after you show the B coefficients are zero. This threw me because there is a surjection $\mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ given via multiplication by two but those aren't vector spaces (since if F is a vector space for both $F = \mathbb{Z}/2\mathbb{Z}$ but $1 + 4\mathbb{Z} + 1 + 4\mathbb{Z} \neq 0$. You can use this to show that if $\beta \in R'$ is a prime ideal, then $R/\beta^n \cong R/\beta \oplus ... \oplus \beta^{n-1}/\beta^n$.

(6/3/2017) Today I learned a good chunk of things, mostly at the Temple University Graduate Math Conference. I learned that there's an intersection to algebraic geometry, arithmetic geometry, and dynamical systems, and that this connection is what essentially makes the Mandlebrot set. I also learned that in the same sense that covering space theory and Galois Theory are very connected, you can also translate the language of Galois theory/covering space theory to the language of algebraic geometry, which uses the word etale a lot, but with a little accent over the e which I don't want to figure out how to do because I'm not connected to the internet right now. I also learned that a lot of arguments involving modules over principal ideal domains essentially boil down to "isolate the ideal that stuff in the last coordinate can be. Then it's a principal ideal so pick a principal generator, and then write any element as the module element with last coordinate that principal generator direct sum something with last coordinate zero (or whatever you need).

(6/4/2017) Today I learned about the theory of groups, which is the set of first order sentences they satisfy. Obviously isomorphic groups satisfy the same theories, so you can use theories to distinguish some isomorphic groups. For example, the sentence "There exists x, y such that $xy \neq yx$ distinguishes abelian groups from non-abelian ones. However, it turns out that the first order theory of any free group on $n > 1$ generators has the same first order theory as any free group on $m > 1$ generators. Thus they are not distinguishable from first order theory. Moreover, it turns out that any group whose first order theory is that of a free group is hyperbolic, which is pretty neat, even though I don't know what it means for a group to be hyperbolic yet.

(6/5/2017) Today I learned about bilinear forms on free abelian groups. One of the things I learned was that if you have a subspace that is unimodular (that is, the matrix associated to the form has determinant ± 1) then you can write your space and the bilinear form as a direct sum of the subspace and the space orthogonal to it and the form restricted to those two things. This shows that if you happen to have any set of the correct number of elements in the free abelian group and their bilinear form associated to the set of elements has determinant ± 1 then that set actually forms a basis for your space.

(6/6/2017) Today I learned that any scheme that is irreducible and is also reduced is an integral scheme. This is because a scheme being reducible means that any open subset is also irreducible (and in fact, these are equivalent notions), so you pick some open set U that you want to show $\mathfrak{O}(U)$ is integral. So pick $f, g \in \mathfrak{O}(U)$ and assume $fg = 0$. Then you can show that $U = (V(f) \cap U) \cup (V(g) \cap U)$ (by some abuse of notation here–you're writing the set of points where the function vanishes at the stalk as a vanishing set). These are closed, and therefore one of them must be the entire space. But then since you can restrict that function to any point and get zero, *and* your scheme is reduced (i.e. functions really are determined at their points) then that function must be zero in $\mathfrak{O}(U)$.

(6/7/2017) I finally freaking learned what Poincare Duality is actually saying! It's not just saying there's an isomorphism, it's saying that if you have an oriented closed n manifold X, what you've essentially done is chosen a fundamental class $[X] \in H_n(X)$, i.e. a generator. Now Poincare duality says if I define the map $D : H^k(X) \to H_{n-k}(X)$ via $[\alpha] \to [\alpha] \cap [X]$ where $\cap$ means that for every simplex, α should eat the first $k+1$ vertices and treat it as a k simplex and spit out a number, and then multiply that number by the $n-k$ simplex that the rightmost $n-k+1$ coordinates yield, then the map D gives an isomorphism. You can use this to show that in a closed four manifold the intersection form is unimodular.

(6/8/2017) Today I learned about Kummer's Theorem, which is a theorem which helps factor a prime ideal $\mathfrak{p}$ in a Dedikind ring R with fraction field K with $L/K, S = R - \mathfrak{p}, R'$ the integral closure of R in L with L/K a finite extension and in the special case where $R'_S = R_{\mathfrak{p}}[\theta]$ for some $\theta \in L$. Then if $f(x) \in K[x]$ is the minimal polynomial of θ over K, then the coefficients are in $R_{\mathfrak{p}}$ and so you can reduce them modulo $\mathfrak{p}$ and factor it. Then those factors of f correspond with the same power and the relative degree of each prime ideal is the degree of the polynomial.

(6/9/2017) Today I learned that there's certain properties of schemes, called *affine local* properties, which are properties where if they are true for an affine open set, they are true for any restriction to any distinguished open set, and if they are true for restrictions of some affine scheme A_{f_i} such that the ideal generated by the f_i is the entire ring, then the property holds for $Spec(A)$. This is true due to a lemma which says that you can realize any intersection of two affine schemes (in a larger scheme) as a union of open sets which are distinguished open sets in each scheme.

(6/11/2017) Today I learned that you can write any Notherian scheme as a finite union of irreducible components, and each connected component is merely just a union of some of the irreducible components. The first part comes from the fact that any Notherian topological space can be written as the finite union of closed irreducible sets, none of which contained in any other, and the irreducible components, then, are those sets. And the connected components part comes from the fact that if you have a connected component and you write that connected component as a union of irreducible components then those sets are actually irreducible components of the whole space. At least I'm pretty sure.

(6/12/2017) Today I learned about Kahler manifolds and Hyperkahler manifolds and what they're like. Essentially Kahler manifolds are symplectic, Riemannian manifolds with a complex structure on them with these complex structures that go together and play together nicely. I also learned that a nondegenerate two form ω just means that ω as a map from $V \to V^*$ (where if you take in one vector and have one vector left to take in) is an isomorphism. And a hyperkahler manifold is just a manifold with three almost complex structures which make it a Kahler manifold and those complex structures satisfy some quaternion like relations!

(6/13/2017) Today I learned a bit more about forms (in particular, simply connected manifolds are isomorphic if and only if their intersection forms are the same). Moreover, I translated a hard problem (at least hard for me!) into a problem I made good headway on, where I'm trying to prove that if you can write the spectrum of a ring... well. I put a lot of work into it damnit! It's hard to say what you learned in a problem.

(6/14/2017) Today I learned that my problem yesterday was just reduced to the fact that any integral domain has exactly one closed point. I also learned about connected sum, which is taking two disks and taking an orientation reversing diffeomorphism between them and identifying them. Also the intersection form of the connect sum of two manifolds is the direct sum of the two forms.

(6/15/2017) Today I learned that if you're a locally finite type k-scheme, where k is a field, then a point being a closed point is the same thing as the quotient field being a finite residue field. One direction of this proof uses Hilbert's Nullstellensatz, and the other half of the proof uses the fact that if you have a function in a larger prime ideal and it has k linear combination of the powers of the function are zero in the quotient ring, then whatever coefficient is on the f^0 power must be in that larger prime ideal, since f and the smaller prime ideal are in the larger prime ideal. But then you can do an induction type argument to argue that no linear combination of the f^i is zero. Which is pretty cool.

(6/16/2017) Today I learned that, given a UFD A where 2 is invertible, all of the obstructions to adjoining a square root in your ring being the integral closure of the ring in its field of fractions. Let's say we adjoin a root of D with $D^2 \in A$ but not D. Then if D doesn't have any repeated prime factors, you can take a monic polynomial with coefficients in $B := A[D]$ and multiply it by the "conjugate" (i.e. conjugate all the coefficients) polynomial and you get a polynomial with coefficients in A! Then you can argue that there has to be a degree exactly 2 polynomial which your element solves, and then use the fact that you don't have any prime divisors to conclude that the denominator of the A coefficient on D actually has no prime divisors–i.e. it is a unit.

(6/18/2017) Today I learned something about the interacting composite extensions of rings and how they behave with their corresponding extensions of fields. Like in the field extension case, we have a chain of fields $K \subseteq E, L \subseteq F$ with $E \cap L = F$ (for simplicity). Then if we let R_K be a Dedikind ring whose quotient field is K, and let $R_W = \overline{int}(R_K, W)$ for each field, then it turns out that if either of E or L is Galois, then if you take the discriminant $\Delta(R_E/R_K)$ and multiply it by the ring R_F (the largest ring), that's a subset of the composite ring $R_E R_L$. This basically says that you only can divide by so much and get so far off from $R_E R_L$ before you can't be in the integral closure of R_K over F anymore. Moreover, if it just so happens that your two discriminants are relatively prime (i.e. the two ideals sum to the entire ring R_K) then it turns out you can show equality here!

I also learned about associated points and associated prime ideals, and in particular, that even though I don't know the definition yet, I know that they are the generic points of the irreducible components of the closed subsets that can be the support of some function (i.e. element) of the ring, and that there are only finitely many of these for Noetherian rings. Interesting.

(6/19/2017) Today I learned the proof that being Noetherian is an affine local property and wrote it up. It essentially comes from the fact that if you have a finite collection of $f_i \in A$ such that $(f_i) = A$ and your A isn't a Noetherian ring, then there exists a strictly increasing infinite chain of ideals. Now you can show that at each point in that chain, one of the "ideal of the numerators" in the A_{f_i} is also not equal. Therefore, an infinite amount of these must occur somewhere! I also learned about Chern classes and characteristic classes–characteristic classes are particular classes in cohomology which are natural with respect to pullback.

(6/20/2017) Today I learned what a Stiefel Whitney class is (although I can't necessarily spell it)–it's a theorem that says the colimit over all the orthogonal groups as vector bundles over a given manifold has cohomology some polynomial ring over the field with two elements with one coefficient in each degree. You can take the i^{th} Stiefel Whitney class to be the term x_i. You can also use axioms to compute it (like the Whitney sum or the fact that it's a characteristic class) and it turns out that a manifold is orientable (spin) if and only if its first (first and second) SW class vanishes!

(6/21/2017) Today I learned with a large chunk of work that if A is a ring and $\mathfrak{p} \in SpecA$ such that $A_{\mathfrak{p}}$ has nonzero nilpotent, then $\mathfrak{r} \in \overline{\mathfrak{p}}$ implies that $A_{\mathfrak{r}}$ also has a nonzero nilpotent element. To see this, let f be the nonzero element, and let $m \in \mathbb{N}$ and $g \notin \mathfrak{p}$ such that $gf^m = 0$ in A. Then you can show that in $A_{\mathfrak{r}}, gf \neq 0$ but clearly $(gf)^m = 0$. This proof seems trivial, but I took a damn good 3 hours to overwork it. But I understand how annihilators relate to these.

(6/22/2017) Today I learned that you can define associated points for modules over a Noetherian ring too. And those points are those prime ideals in the ring whose closure is an irreducible component of the support of some element. Moreover, for rings, localizing by a set S simply deletes the associated points that intersect with S–otherwise the associated points are identical! In particular, associated points do not change from a ring to its stalk, which in turn implies that you can define the associated point on any scheme by simply defining an associated point to be a point that is an associated point for some affine open set! And this equivalently, then, means for all open sets containing it.

(6/23/2017) Today I learned that the algebraic integers in any cyclotomic extension $\mathbb{Q}[\theta]/\mathbb{Q}$ are simply the elements of $\mathbb{Z}[\theta]$. You can first show this for cyclotomic extensions that are merely just powers of prime ideals (which is sort of the building blocks of a lot of things in number theory) by essentially computing things explicitly. You can show that $\Delta(1, \theta, \theta^2, ...)$ is a (possibly very large) power of $\pm p$ and then use that to show that you can't have any "strange" algebraic integers in prime power cyclotomic extensions. And then you can induct on the number of prime factors, using linear disjointness of fields of two cyclotomic extensions that don't divide one another.

(6/24/2017) Today I learned that for every irreducible closed subset K in some scheme, there exists a unique point in that subset such that the closure of that point is your entire subset. This follows because it is true for affine schemes, and then for an arbitrary point in your closed subset you can pick an affine closed subset containing that point. The resulting (necessarily) closed subset remaining after intersection is still irreducible, so in this affine scheme land you have a closed point! It turns out that if you could have gotten a different point this way (on ANY scheme), then they can't intersect at all, which you can use to show that your set is not really irreducible (since you can write it as the union of the closure of one point union the of the closure of each of the other points). Then you can show that suspicious possibly infinite union of closures of some points is still closed because none of the closures of two distinct points intersect so restricted to any of the affine schemes you picked earlier and K^C that suspicious set is still closed. Then I proved a small lemma which says it suffices to check closedness on an affine open cover.

(6/25/2017) Today I learned that if you have a composite of field extensions that are linearly disjoint and take the integral closures in all of them, it might not be that if you take the integral closure of the top field, then it's just the product of the two rings. There might be some more stuff in there, but what you do know is that if you multiply anything in that stuff with anything in the discriminant ideal of one of the intermediate rings, then you get in the product of the two rings. (*assuming one of the extensions is Galois) This helps if two ideals are coprime, for example, because then equality does hold. It essentially holds by a dual basis argument and the fact that

$Tr(y_ij_j)^{-1} = Tr(y_i'j_j')$, letting $Tr(y_iy_j') = \delta_{ij}$

(6/28/2017) Today I learned that you can obtain associated prime ideals of a ring/module through the composition series of that module. Essentially what's happening here is that in a composition series (which you can argue exists in any finitely generated module over any Noetherian ring) if you're an associated prime $\mathfrak{p}$, then there exists an m in the module where $\mathfrak{p}$ is precisely the annihilator of m. Then you ask, "where is the first time this m appears in the series?" Either that prime you mod out by the last one by to get is $\mathfrak{p}$, or you can show that $\mathfrak{p}$ is also the annihilator of some fm that appeared earlier in the series. Then you can use some kind of induction argument to show that you eventually make that prime appear!

(6/29/2017) Today I closed the book on associated points (well, at least mostly, I'm writing this a little early today). Essentially, here's the story of associated points. If M is a finitely generated module of a Noetherian ring A and $m \in M$, then $Supp(m)$ is the collection of prime ideals (that I like to think of as security guards or prisons) that hold back everything that can kill m, i.e. $Supp(m) = \{\mathfrak{q} : ann(m) \subseteq \mathfrak{q}\}$. This is because in $M_{\mathfrak{q}}$, you invert everything else, so if you inverted something that could have killed m you did. Associated primes, then, are the prime ideals that can do this with the least amount of work. That is, they are the prime ideals that are exactly the annihilators of some elements. It turns out that if you have any element annihilated, then you only need one good associated point to guard you–if the annihilator ideal of your element isn't prime, then there's a "concoction" that can kill you while the individual items remain harmless (like ammonia and bleach). All you need is to protect from one of them–i.e. you only need one of the associated primes. This is an explanation for why $Supp(m)$ is the union of the closures of the associated points $\mathfrak{p}$ where $m \neq 0$ in $M_{\mathfrak{p}}$.

I also finally sort of get handlebodies. Essentially they're the same thing as doing cells as in algebraic topology, but to make them manifolds, we have to add extra stuff as we attach to keep dimensions consistent.

(6/30/2017) Today I went through a lot of results involving modules over principal ideal domains. Essentially, all of these results boil down to the fact that given any submodule of a free module, it's free, and moreover, you can find a basis of the large module such that the first k terms, when multiplied by some ring elements a_i, give a basis for your submodule, such that $a_1|a_2|...|a_k$. Immediately this tells you that you can write your module isomorphic to a free module plus torsion $\oplus_i R/(a_i)$ and then you can argue via the Chinese Remainder Theorem that you can also write it as the direct sum of prime power modules. You can use the prime power modules to show that the invariant factor and the prime decompositions are unique, and then you can use that for the specific principal ideal domain $F[x]$ (where F is a field) to argue that there is a canonical form for any linear transformation, and, with more work, you can argue that a matrix satisfies its own characteristic polynomial.

May 2017

(5/1/2017) Today I learned a proof of Stokes Theorem! Which is really pretty cool. Essentially, to prove Stokes Theorem for a compactly supported form, you can argue by linearity that your form is supported on a parametrizable subset, and then if the subset doesn't intersect the boundary at all, since your form has compact support and since you can view your integral as a bunch of iterated integrals of partial derivatives, all of which eventually vanish, then by the fundamental theorem of calculus in one variable $\int_X d\omega = 0$, and since ω is not supported anywhere on the boundary, $\int_{\partial X} \omega = 0$. You can use a similar argument to show that if the support is on the boundary, you can reduce your integral down to all but one important integral (one where the last dx_n doesn't appear–on the boundary, this is zero) and then show that your forms are equal there. I'm actually being a little hand wavey here, but I don't have the book with me so I'm going to have to try and get this totally down tomorrow.

I also learned a bit about the homology of a Stein filling W of the unit cotangent bundle of a surface Σ_g, say, Y_g, where $g \geq 2$. It turns out that the fundamental group of this Stein filling is the same of that surface, which I haven't proved or seen the outline for yet. I also learned (but again, without proof) that $\pi_1(Y_g)$ is generated by all the generators of $\pi_1(\Sigma_g)$ and an extra element t which represents the fact it's a "circle bundle", with relations that this t commutes with all the other generators. I have seen the essential reason for why the inclusion map $i : Y_g \to W$, which turns out to be surjective, actually is surjective when restricted to the "$pi_1(\Sigma_g)$" part of $pi_1(Y_g)$. Essentially, we mod out by $pi_1(Y_g)/H] \to \pi_1(W)/i_*(H)$ and then use covering space theory to find a $k = [\pi_1(W) : i_*(H)]$ fold cover, which is a Stein domain since there's an analytic property you can classify Stein domains using compact sets and bounds, and then you can restrict your coverings to the boundary to argue that since the genus is less than $2, k = 1$.

Finally, I learned that you can glue sheafs together provided that you have a nice cocycle condition being met. I actually haven't fully written out the proof of gluability, but I imagine that the proof of gluability will follow since you can use the fact your isomorphisms are maps of sheafs to get a gluing trick to work for your large set. Identity follows as if two elements are the same up when they're hit with "restriction then isomorphism" then they're the same when hit with "restriction" by taking the inverse of your isomorphism.

(5/2/2017) Today I learned that you can take a collection of schemes which have subschemes that are isomorphic to each other and you can glue them together, essentially using a sheaf gluing construction, provided that the sheaf gluing construction "cocycle condition" holds. You can use this to construct some nonaffine schemes, such as a line with two origins. This is constructed by taking the open subset $U := D(t) \subseteq X := Spec(k[t])$ and $V := D(u) \subseteq Y = Spec(k[u])$. Now, I learned a lot more today, in particular I reviewed a good chunk about Dedikind rings, but I'm tired and going to bed.

(5/3/2017) Today I learned most of the proof of the Riemann Mapping Theorem, which says that any simply connected domain that is not the entire complex plane and has a point is conformally equivalent to the disc. Which is so freaking crazy! But here's why–if you avoid a point, you avoid an entire ray. If you avoid an entire ray, you can shift that ray to the origin, and then you can get an analytic logarithm defined there, and hence an analytic square root. Analytic square roots in particular avoid $-D$ if they hit D, so we have a function that avoids values in a disc, so we can traslate that disc over and then invert it. This shows that there's an analytic function mapping our domain into the disc. You can also rotate it to assume that the derivative at zero is a positive real number. Friday's class, which I'm not going to, will also show that we can get a surjective map by asking the derivative be maximized.

(5/4/2017) Today I learned the other two theorems that I have to give a talk on in a few weeks. They say that any exact filling (i.e. any symplectic manifold whose symplectic/non degenerate two form has differential zero) of a contact manifold that admits a *Calabi-Yau cap*, which is a strong concave filling which has torsion as its first Chern class. Also a strong concave filling means that there is a vector field pointing into your large manifold where the Lie derivative of the symplectic form along the vector field is a positive multiple of the symplectic field.

(5/7/2017) I reviewed a bunch of things about Dedikind rings and some alternative definitions that could be used for them. In particular, I learned something of note– if U is an ideal of a ring R (Dedikind or not), given a maximal ideal $\mathfrak{p}$, it's not necessarily the case that $U_\mathfrak{p} \cap R = U$. To see this, use the example of $R = \mathbb{Z}, U = 12\mathbb{Z}$, and $\mathfrak{p} = 2\mathbb{Z}$. In this case, 4 is in the left side but not the right. However, $4 = \frac{12}{3}$ in $R_\mathfrak{p}$ and $4R_\mathfrak{p} = \frac{12}{3}R_\mathfrak{p} = 12R_\mathfrak{p}$, which at least resolved the issue I had while working a specific example through the notes.

(5/8/2017) I learned what a spectral sequence was today, which is, essentially, given an $N \in \mathbb{Z}$, modules or abelian groups or whatever of the form $E^r_{p,q}$ with a differential map where $r \geq N$ where you can determine the $r+1$st "page" by looking at the homology of the r^{th} page. One application of this is computing homology–essentially, if you have an easy to compute quotient space and a class in your homology of that quotient, you can check to see if it lifts to the homology of your (possibly more difficult to work with) space. If it does, you can also check if it lives forever if your space is a filtered colimit, say $X = colim(... \to X_i \to X_{i+1} \to ...)$. If it lifts and then later dies at the $(n+t)^{th}$, it is taken care of by the t^{th} page. Similarly, this procedure can give us some fake stuff in homology class, but eventually that is quotiented out too.

(5/9/2017) Today I hammered down a lot of the definitions I needed to get down for the Calibi-Yau caps. In particular, a Calibi-Yau cap of a contact manifold is a strong concave filling whose first chern class is torsion. A strong filling means that if you take the differential of the contact form... well actually I learned I don't know the definition of a strong filling certainly well. But I also learned about the AH Spectral Sequence, and a related idea which talks about the convergence of a spectral sequence. This means that essentially, after taking enough differentials, your homology doesn't change.

(5/10/2017) Today I was destroyed by a Differential Topology exam, at least mostly. But I learned some things about it, including that you can argue that $\mathbb{RP}^2 \times \mathbb{RP}^3$ is not orientable, for if it were, the pullback of the inclusion map would induce some orientation on $\mathbb{RP}^2$. Also I learned about the Serre spectral sequence, and that you can use theorems about where things converge to not only to compute convergence, but you can work backwards–knowing that your spectral sequence converges to something tells you that elements not in that convergence have to be killed eventually by a differential from somewhere else. In quadrant one spectral sequences, you can argue that the "killing" has to happen reasonably soon, since if you're on the $(0,t)$ spot you only have $t-1$ chances to die before you're mapped into by zero.

(5/11/2017) I learned a technicality that I had looked over when I looked at the factoring of fractional ideals $\mathfrak{M}$ of a Dedikind domain R. In particular, choosing some nonzero $t \in \mathfrak{M}^{-1} \cap R$, then $t\mathfrak{M} \subseteq R$, so it can be factored into prime ideals, and so can Rt. But then what I hadn't noticed before was that $\mathfrak{M}t = \mathfrak{M}Rt = \mathfrak{M}Rt$, so "morally" $\mathfrak{M} = \frac{\mathfrak{M}t}{Rt}$, at least in the factoring sense.

(5/12/2017) Today I learned the idea of a right derived functor. Essentially, you can take an exact sequence in an abelian category $0 \to A \to B \to C \to 0$ and a left exact functor (so the sequence $0 \to FA \to FB \to FC$ is exact) and then there are "right derived functors" R^nF for all $n > 0$ such that the following sequence is exact $0 \to FA \to FB \to FC \to R^1FA \to R^1FB \to R^1FC \to R^2FA \to ....$ You then can define group cohomology for a given group G by noting the functor $(-)^G : G - Mod \to Ab$ is a left exact functor and then define the n^{th} group cohomology to be the n^{th} right derived functor with coefficients in a G module M to be $R^n(M)$.

I also learned about the Lyndon-Hochschild-Serre spectral sequence, which relates the cohomology of a group G to the cohomology of a normal subgroup $N < G$ and the quotient G/N. In particular, the LHS spectral sequence says for any fixed G module A there is a spectral sequence $H^p(G/N, H^q(N, A)) \implies H^{p+q}(G, A)$.

(5/13/2017) Today I learned stuff in the appendix of Algebraic Number Fields! In particular, I learned about the Normal Basis Theorem of Galois extensions (and learned how to prove the cyclic case), and Hilbert's Theorem 90. Hilbert's Theorem 90 says that if you have an element $\alpha \in K$ where K/F is Galois extension of degree n and the Galois group $G := Gal(K/F)$ is cyclic, generated by say, σ, and $N_{K/F}(\alpha) = 1$, then there exists an element $\psi \in K$ such that $\alpha = \psi/\sigma(\psi)$. You show this by considering elements of the form $\lambda_i = \alpha\sigma(\alpha)...\sigma^{i-1}(\alpha)$ for $i \in \{1, 2, ..., n\}$. Actually, writing this out, I discovered an inconsistency in my understanding about how everything connected, that I hope to rectify tomorrow. But alas, today, I'll just say that there's something called the Normal Basis Theorem which says in the above setup we can find a special element α such that $\alpha, \sigma^1(\alpha), ..., \sigma^{n-1}(\alpha)$ is a basis of K/F.

(5/14/2017) Today I learned about when two symmetric bilinear forms over $\mathbb{Z}$ are equivalent. The definition of equivalent means that you can find an isomorphism between the two vector spaces such that the pull back of one form is the other form. On the other hand, there's a theorem that says if you put the form into a matrix, then the matrices are isomorphic if and only if the rank, signature (i.e. the largest dimension you can make a subspace be positive definite - the largest dimension you can make a subspace be negative definite) and sign (meaning "even" if every diagonal entry is even, and odd otherwise) are equal.

(5/15/2017) Today I learned that if you have an oriented four manifold, then you have a fundamental class $[X] \in H^4(X;\mathbb{Z})$ (where hereafter we use integral homology) then you obtain a bilinear form, the *intersection form*, defined on $H^2(X) \times H^2(X)$ sending two elements to their cup product (and then to $\mathbb{Z}$ canonically). You can use this intersection form to show the thing I'm proving, which is that if you have any exact filling of the unit cotangent bundle of a surface of genus larger than one, its homology is that of the disc bundle. It's also related to an invariant called the signature of a manifold, and in fact it relates to the two matrices E_8 and H, which is the transposition matrix.

(5/16/2017) Today I learned a fact which I cannot prove yet. Let N be an exact filling of Y, the unit cotangent bundle of a surface of genus $g > 1$. Then you can show that the homology $H_2(N) = \langle S \rangle \cong \mathbb{Z}$ and that the map to $H(N, Y)$ is simply multiplication by $[S]^2 \cong k^2(2g-2)$. This in particular implies that all the torsion is killed off in the long exact sequence of a pair with $H_1(Y) = \mathbb{Z}^{2g} \oplus \mathbb{Z}/(2g-2)\mathbb{Z}$ so $H_1(N) \cong \mathbb{Z}^{2g}$, the same as the disc cotangent bundle. I also learned that I don't truly understand bilinear forms that don't have the notion of length attached to them. We'll hopefully work through that tomorrow.

(5/17/2017) I figured it out! It turns out that the fact that if you have any function $f : V \to F$ where V is an F vector space and $\langle , \rangle$, a nondegenerate bilinear form, then $f = (v, --)$ for some $v \in V$. This is simply because the vector space of all linear maps $V \to F$ is an $n := dimV$ dimensional vector space. Then the map sending $w \to \langle w, -- \rangle$ is an injection (which is where nondegeneracy comes in). I also learned the proof of the fact that if you take the integral closure of a Dedikind ring in a purely inseparable finite extension, the resulting ring is also Dedikind. This is essentially because in your finite extension, there is a power p^q where you can raise all of your elements in the large field to to get in the small field. You can then argue that you get a one to one correspondence between prime ideals of your integral closure and prime ideals of your old ring by intersecting the prime ideals with the old ring. This quickly gives you the fact that each element is contained in only finitely many prime ideals, and slowly gives you the fact that if you localize any maximal ideal you obtain a DVR.

(5/18/2017) Today I learned that if you are given Dedikind rings $R \subseteq R'$ then there is a natural way to define the rammification index of a nonzero prime ideal $\beta \in Spec(R')$, because $\beta \cap R$ is a nonzero prime ideal of R. The "prime" part of this proof is trivial, and the "nonzero part" is pretty hard, especially because it's not true in general. An obstruction is that $[L : K] < \infty$, where $K \subseteq L$ are the respective fraction fields (the rings $\mathbb{Z} \subseteq \mathbb{Z}[x]$ show this since $(x) \cap \mathbb{Z} = 0$. On the other hand, if $\alpha \in \beta$ is nonzero, then it is the root of some equation in $K[x]$, say $a_0 + a_1x + ... + x^n$. Clearing out with a common demoninator, we see that $r\alpha$ is the root of $r^n a_0 + r^{n-1}a_1(rx) + ... + (rx)^n$ so $r\alpha$ is a nonzero element of $\beta \cap R$, where $r\alpha \in R$ since R is integrally closed in its field of fractions. This isn't right, but I'm tired. I'll sort this out tomorrow. I also learned the basic idea that a map of affine schemes is determined by the map on $D(1)$. This is because a map is determined by how it operates on stalks, but on stalks there is only one prime ideal to map from and too.

(5/20/2017) Today I finally figured out my error above. Given an $\alpha \in \beta$, it's not necessarily true that a scalar multiple of α is in R, however, if you look at the polynomial expression above, $ra_0 = -a_1 r\alpha - ... - (r\alpha)^n \in R \cap \beta$. Also I just learned what the mapping torus is–given a map $\phi : X \to X$, you can take the cylinder $X \times I$ and glue $(0, x)$ to $(1, f(x))$. I also found out a picture of what handle sliding is–which is essentially when two rainbows are next to each other, and then the leftmost rainbow (say) decides that it wants its right side to move along the other rainbow and overtake it. I also learned the other move that doesn't change diffeomorphism, which is if you put a ball inside of a rainbow. Topology is weird.

(5/21/2017) Today I learned about projective space in algebraic geometry. Restricting discussion to the first projective space, I learned that projective space in one dimension is just taking $Spec[t]$ and $Spec[u]$ and gluing $D(t)$ to $D(u)$ via the isomorphism sending $t \to \frac{1}{u}$. This makes a picture that reminds me of the sphere where taking $\frac{1}{z}$ reflects the sphere about the center disk. I also learned what an almost complex structure was (an operator on the tangent space of a manifold which squares to negative 1–i.e. it looks like i.)

(5/22/2017) Today I learned about how homogeneous polynomials can determine a subscheme of projective space. Essentially the scaling can show that if you have a polynomial, like $x^2+y^2-z^2 = 0$, you can divide by z (say) and get the equation in one of the subschemes of projective space, say $(\frac{x}{z})^2 + (\frac{y}{z})^2 - 1 = 0$ and it turns out that the gluing maps make your choice not a real choice. Also, I learned facts about graded rings, including facts about how ideals of a graded ring are closed under addition, multiplication, intersection, radicalization, and if you're "prime" with homogeneous elements then you're a prime ideal. This last one comes from essentially the thing of the example if your rings are graded by $\mathbb{Z}_{\geq 0}$ and $\alpha = (\alpha_0, ..., \alpha_m, 0, ...)$ and $\beta = (\beta_0, \beta_1, ..., \beta_n, 0, ...)$ such that $\alpha\beta \in I$ then you let k, l be the minimal such that $\alpha_k, \beta_l \notin I$. Then everything smaller is in I so check the $k + l$ coordinate and split it into three parts to see that $\alpha_k \beta_l \in I$.

(5/23/2017) Today I learned the more general notion of the Projective space of a graded ring. You're supposed to think of the graded ring $k[x_1, ..., x_n]$ where grading is determined by the degree of each term in a polynomial expansion. Then when you want a "point," instead of just a prime ideal, what you want is a *homogeneous* prime ideal, where homogeneous means that the projection to any one particular grade is in the ideal if an element is in there (that's talked about above). The reason you want these is the same reason you want homogeneous equations–they are the equations that actually cut out solutions in projective space.

(5/24/2017) One thing I learned today was a very good interpretation of the fact that, in the spectrum of a ring, $V(I(L)) = \overline{L}$ where $L \subsetneq Spec(A)$ should be thought of as a list of prime ideals. Essentially what the heart of this statement is that if you have a point somewhere and it's not lying above any of the points on your list L (or on the list itself), then you can construct a function that makes you vanish on L but doesn't vanish at your point. At least if you have a finite list–although I'm pretty sure this adjusts for infinite lists. Like what if $L = \{(x - n) : n \in \mathbb{Z}\}$? I'm not sure. Let's see what this fact actually means in my notes.

(5/25/2017) Today I figured out what it truly means to be alone. Just kidding. I actually figured out why the topologies of regular Zariski topology on $Spec(S_f)_0$ (where S is a $\mathbb{Z} \geq 0$ graded ring and $f \in S$ is a homogeneous element) is just the restriction of the topology of $Spec(S)$ to $D(f)$. My hangup was that if we have some kind of ideal $V(I) \subsetneq Spec(S_f)_0$, we need to write the associated set of prime ideals in $Spec(S)$ as $V(J) \cap D(f)$, where J is a homogeneous ideal. Originally, I thought to set J as a graded ideal, via $\oplus_i \{\alpha \in S : \frac{\alpha^(degf)}{f^i} \in I\}$, but this isn't necessarily an ideal. However, you can take J to be the elements generated by these homogeneous elements and you get what you need.

(5/27/2017) Today I learned a lemma in number theory which says, among other things, that the relative degree is well defined for prime ideals a Dedikind ring $R' \supset R$ where R is also a Dedikind ring. This follows for two reasons–one is that if $\beta \subseteq R'$ is a nonzero prime ideal, then $\mathfrak{p} = R \cap \beta$ is also a nonzero prime ideal (as I've shown above with the "degree zero of a polynomial trick") and if $S := R - \mathfrak{p}$, then $R'/\beta \cong R'_S/\beta R_S$. This is because since we've already moduloed out by everything $\mathfrak{p} \subseteq \beta$, everything in $\mathfrak{p}$ already *has* an inverse. Therefore localizing by things that already have inverses doesn't change anything. Then you can work with a DVR to prove that $[R'/\beta : R/\mathfrak{p}] \leq [\mathfrak{F}(R') : \mathfrak{F}(R)]$, where $\mathfrak{F}(A)$ denotes the fraction field of A.

(5/28/2017) Today I learned some stuff relating relative degrees and rammification indices to dimension. In particular, if we are given a prime ideal $\mathfrak{p} \subseteq R \subseteq R'$ where R, R' are Dedikind rings where the quotient field of R' is a finite extension of the quotient field of R and $\mathfrak{p} = \beta_1^{e_1}...\beta_n^{e_n}$ then it turns out that $dim(R'/\mathfrak{p}R') = \sum_{i=1}^{n} e_i f(\beta_i/R)$, where f denotes the relative degree, i.e., $f(\beta_i/R) = [R'/\beta_i R' : R/\mathfrak{p}]$. You can take this a step further and argue that if you have that $\mathfrak{F}(R')/\mathfrak{F}(R)$ is separable finite extension of the two Dedikind rings then $\sum_{i=1}^{n} e_i f(\beta_i/R) = [\mathfrak{F}(R') : \mathfrak{F}(R)]$. The idea here is to use Chinese Remainder Theorem and then determine the dimension of each individual piece.

(5/29/2017) Today I learned an important theorem about rammification, which says that if R is a Dedikind domain with fraction field K and L/K is a finite separable extension with R' is the integral closure of R in L, then the primes that rammify (i.e. factor with some prime having exponent larger than one *or* have a quotient field that isn't separable over the small field) are precisely those contained in the *discriminant ideal*, that is, the ideal generated by the elements $det(Tr_{L/K}(x_i x_j))$, where $\{x_i\}$ ranges over the bases of L/K contained in R'.

(5/30/2017) Today I learned about reduced schemes, that is, schemes $(X, \mathfrak{O}_X)$ such that for all open $U \subseteq X, \mathfrak{O}_X(U)$ has no nonzero nilpotent elements. I also learned that this can be checked on the level of stalks, since if you're a nilpotent element on a stalk you can find an open set where this behavior occurs, and conversely obviously if you're nilpotent on an open set, you're nilpotent on the stalk, which is just the colimit of the restriction map diagram.

(5/31/2017) Today I that if I, J are two ideals in a Dedikind ring, there is a notion of a *greatest common divisor* of the two ideals, which can be viewed as the product of the prime ideals which divide both. If K is the common divisor, then $I + J = K$. This is because $\subseteq$ is obviously true, and then you can show $\frac{I}{K} + \frac{J}{K} = R$ by localizing by any prime ideal, since a prime ideal either isn't a factor of the I factor or either isn't a factor of the J ideal.

April 2017

(4/1/2017) Today I worked through the topological side of Spectrum. For example, I learned about a topological space being Notherian, which, like the ring definition, says that there's no infinitely descending chain of closed sets $V_1 \supset V_2 \supset ....$ You can use this to show that all open sets in a Noetherian topological space are compact, since this definition is equivalent to the definition that there's no infinitely *increasing* chain of open sets $U_1 \subseteq U_2 \subseteq ....$ Which is a pretty nice thing. I also learned the idea of Noetherian induction along those same lines, which is basically to construct an infinite chain and then use the Noetherian condition to argue that you've found an argument to break "maximality." You can use this to show that any closed set is the finite union of irreducible closed sets, and if no sets contain any other sets, this ordering is unique up to rearranging.

(4/2/2017) Today I proved one. hard. thing. At least it was hard for me. And that's that if you take $V(I(S)) = \overline{S}$, where V takes subsets of A to the prime ideals they vanish on, and I takes a set of prime ideals to the functions that vanish on each of them. The reason is because $\overline{S}$ is just the intersection of all closed sets containing S, but then since each closed set is a vanishing set of some set, we have $\overline{S} = \bigcap_{V(J_i)\supset S} V(J_i)$. But it turns out there's a "minimal set" among the sets $V(J_i)$–it's $V(I(S))$! This follows because if J_i is an ideal containing S and $\mathfrak{p} \in S$, then $\mathfrak{p} \in V(J_i)$, then $J_i \subseteq \mathfrak{p}$ which, intersecting over all $\mathfrak{p}$ and using the inclusion reversing nature of V establishes our claim. So the closure of any set is really just any prime ideal that is larger than the intersection of all the ideals in that set.

(4/3/2017) Today I learned a lot of cool stuff about the factorization of an ideal into a prime ideal. You can do this in a *Dedikind* domain–a notherian integral domain such that if you localize any nonzero prime ideal, you get a *Discrete Valuation Ring*–a PID with exactly one nonzero prime ideal. Then taking a Dedikind domain R and some nonzero ideal Q, you can consider the ring $R' = R/Q$. By Notherianness (which passes to quotients), every ideal has a product of prime ideals inside of it, so in particular zero does (which is Q in the original ring.) Then you can use the Chinese Remainder Theorem to argue since all of these ideals are maximal, you can write them as the direct sum of the quotient of each prime ideal to a power (with a technical Lemma that says the Chinese Remainder Theorem "coprime" condition still holds no matter what power you raise your maximal ideals to). Then you can basically argue that the prime ideals you got are ALL the ideals because of this direct sum notation. This gives you the factorization of an ideal $\mathfrak{U}$ you might want to factor–you can use the fact that there's a product of primes $Q := \mathfrak{p}_1^{a_1}...\mathfrak{p}_n^{a_n} \subseteq \mathfrak{U}$ and mod out by this, using the fact that any ideal is just an ideal of each slot in the quotient (CRT) ring–which are just powers of the prime ideal by our Dedikind domain assumption. You can also require this factorization be minimal on the power of all primes with a positive power–since for each prime ideal $\mathfrak{p}_i$ if we localize all of the other prime ideals, then we again get a DVR–$R_{\mathfrak{p}_i}\mathfrak{U}$ is an ideal of $R_{\mathfrak{p}_i}$ so by our DVR stuff it's just $R_{\mathfrak{p}_i}\mathfrak{p}_i^{a_i}$.

I also went through the proof of the Chinese Remainder Theorem for modules. And I saw something kinda cool. Let's work in $\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. Then in $\mathbb{Z}/6\mathbb{Z}, 2 * 5 = 10 = 4$. Now using the Chinese Remainder Theorem interpretation, $(0, 2) * (1, 2) = (0, 4)$, which is identified with four! I don't think I ever realized this before–it greatly simplifies a lot of the stuff I learned in my first undergraduate number theory class. Plus I learned some things about orientation theory (including that points are taken to be positive if they're the closed subset of the image manifold!)

(4/4/2017) Today I did two main things. The first thing was I learned some insight on what's really going into what Peter May calls the "Fundamental Theorem of Groupoids," which means that if $p : E \to B$ is a covering of two (connected, small-assume hereout today) groupoids, and $f : A \to B$ is a functor between two connected, small groupoids, then there's a lift functor $g : A \to E$ mapping a chosen basepoint $a \in A$ to a chosen point in $e \in p^{-1}(f(a))$ if and only if $f(\pi(A, a)) \subseteq p(\pi(E, e))$. One direction is straightforward and just compares images. But otherwise you can construct the functor g through this recipie. For any object $a' \in A$, where should we lift it to? Well, pick a path (really, a morphism) $\gamma : a \to a'$. Then $f(\gamma)$ is a path in B and we've specified the start point, so use the covering space to uniquely lift to a certain point. Then map a' to whatever endpoint the path got lifted to. The thing is, if we had chosen another path $\gamma' : a \to a'$, then $f(\gamma^{-1}\gamma')$, by assumption, is a loop in $\pi(E, e)$ after lifting. This is why the basepoint doesn't map. I also figured something that was obvious, but nonobvious why it was obvious. It's easier to show morphisms lifting are unique–that's essentially from the definition of covering–which tripped me up reading.

I also finally finished Chapter 3 of Vakil! I finished it off by noting that the vanishing set function $V()$ and the "Innihilated by everything in my subset of Spec" function $I()$ take prime ideals (for "vanishing sets") to irreducible closed sets, and moreover, a bijection between minimal prime ideals and irreducible compenents of the spectrum of a ring. The full proof is written on my printed copy–but for now I'm happy to say it's all sorted out! It's just a matter of remembering that a subset of $Spec(R)$ for a ring R is basically a list of prime ideals–another obvious thing you forget from time to time.

(4/5/2017) Today I did some differential topology, specifically Lefschetz intersection theory. In particular, if $f : X \to X$ is a map of a compact manifold to itself, then we can compute its Lefschetz number, defined as $I(\Delta, graph(f))$, where $\delta \subseteq X \times X$ is the diagonal. This in some slightly loose way measures the fixed points of a map (although, we are implicitly requiring that Δ is homotoped to be transverse to $graph(f)$–so this isn't so obvious for a map with infinite fixed points, like the identity map.) It turns out through some technical linear algebra details that the transversality condition at a fixed point $x = f(x)$ to be have the transversality condition satisfied is equivalent to 1 not being an eigenvalue for df_x. A way to interpret this is "fixed points are isolated" at the infinitesmal level. Then I worked through the case of the map being from $\mathbb{R}^2 \to \mathbb{R}^2$ (which extends to any two dimensional manifold since this is a local property), and got an idea of how to tell from the eigenvalues whether a Lefschetz point is a "sink" (in which liquid would go into), a "source" (in which liquid would always come from) or a "saddle," at least if the eigenvalues are positive. That interpretation also gives a semi-intuitive way to compute the Euler Characteristic of the torus of any genus, imaging the donus with many holes on its side and counting the one saddle, one sink (with Lefschetz number +1) and then the $2g$ (where g denotes the genus) saddles, which have Lefschetz number -1.

I also did a lot of algebraic topology. I translated the result I got yesterday (through the extensive help of Peter May's book, anyway) to show that the category of coverings of a connected, small groupoid is basically the same (read–equivalence of categories) as the orbit category of the fundamental group G at some basepoint. I also got that all G automorphisms of fibers are lifts of some loop in the fundamental group and vice versa, and the fact that a "covering of covers" exists if and only if a subconjugacy relation holds (not equality)–and this is essentially because your isomorphism need not match basepoints, so you need to conjugate to rectify that.

(4/6/2017) Today I learned the proof of the fact that the spectrum of a ring is a separated presheaf, as well glossed into the proof of the gluability axiom. Essentially, you can reduce the proof of the "identity" axiom to proving that if a function/ring element r restricts to zero on each $D(f_i)$ then it is zero in the whole ring. We then use a really nice fact that says at $D(f_i)$, if A is our ring, our corresponding ring at $Spec(D(f_i))$ is A_{f_i}, and then use compactness (or what Vakil calls quasicompactness) to reduce to i taking only a finite number of values. So then we get that $f_i^{n_i} r = 0$ after cross canceling some fractions. Then we use the fact that $Spec(A) = \bigcup_i D(f_i) = \bigcup_i D(f_i^{n_i})$ to note that $(f_i^{n_i}) = A$ so we can write $1 = a_1 f_1^{n_1} + ...$ and see that $s = 1s = 0$.

(4/11/2017) Today I learned some facts about knot theory and the cobordism of links. First, (which is actually something I learned a while ago technically), two knots are isotopic if and only if their knot diagrams are related by a sequence of the three Reidmeister moves–the first being "unkinking/adding" a loop, the second being crossing parallel strands or going in the opposite order, and the third being "moving a straight line that is completely over two other lines over the crossing (or under).

I also learned about the idea of "rammification," which essentially says that in an algebraic geometric sense, it's not helpful for an ideal to factor into prime ideals to a power. For example, factoring $Spec(\mathbb{Z})$ over $Spec(\mathbb{Z}[i])$, we see that the ideal $(2) = (1+i)^2$. This is in a different class than all of the other prime ideals in $\mathbb{Z}$–something I hope to learn more about later.

(4/12/2017) Today I learned a cool consequence of the Tubular neighborhood theorem. First, the Tubular Neighborhood Theorem says that if you are given manifold $Z \subseteq Y$, where Y is another manifold, then there's a diffeomorphism from a neighborhood of Z to a neighborhood of the subset of $Z \subseteq N(Z; Y)$. Now if Z also happens to be globally definable by independent functions, by a previous homework result we have that the normal bundle is actually trivial. Therefore using this tubular neighborhood theorem, if Z is a compact manifold globally definable by independent functions, we can slide Z off of itself in the neighborhood in the normal bundle, and hence in the manifold itself, which shows that $\chi(Z) = I(\Delta, \Delta) = 0$.

(4/13/2017) Today I learned a lot about Khovanov homology. In particular, if you have a knot diagram D and you designate one crossing, there are two ways you could have smoothed the crossing–call them D_0 and D_1 for, in the notation of Turner's Five Lectures on Khovanov Homology, the resulting diagram obtained by 0-smoothing and 1-smoothing our diagram respectively. But "basically," we have that the complex $C^{*,*}(D)$ we get is a direct sum $C^{*,*}(D_0) \oplus C^{*,*}(D_1)$. Now, the "basically" part comes in because we haven't taken into account grading. But by following the receipe of Khovanov homology, there's an easy way to put grading back into the picture, depending on whether your crossing is positively or negatively oriented. Then you obtain a short exact sequence from your fake direct sum, which then immediately gives you a long exact sequence you can use to compute Khovanov homology of! One application of this is that, knowing that the Khovanov homology of the unknot is zero except in the $(0, \pm 1)$ bigrading, we can use the sequence to compute the Khovanov homology of the Hopf link more easily than computing it explicitly.

(4/14/2017) Today I learned a lot about Dedikind rings. The original definition of a Dedikind ring is a Noetherian integral domain R such that if $\mathfrak{p}$ is a prime ideal, then the ring $R_{\mathfrak{p}}$ is a Discrete Valuation Ring (DVR), meaning that it is a PID exactly one maximal ideal. This satisfies the idea of "good" in some sense because, as I learned in the course of the two equivalent definitions of Dedikind ring, it means that we could work locally and locally, according to the definition, we have an easy ring to work with. One of the equivalent definitions of a Dedikind ring is a ring R such that $\mathfrak{p}$ is a *maximal* ideal, then the ring $R_{\mathfrak{p}}$ is a Discrete Valuation Ring (DVR), with the additional requirement that every ideal contains only a finite number of prime ideals. Without totally going through the proof of why these two make sense, one way to sort of think about it is to note that another "looser" definition of a Dedikind ring is a place where ideals can be factored into products of prime ideals.

Then I learned the proof of the other definition of a Dedikind ring, which is "a Noetherian, integrally closed domain with every nonzero prime ideal maximal." This sort of makes sense too if you take the "factor into primes" being natural definition, considering if you have primes of height two, say, what would you factor them as in your natural factoring? In this proof, I also learned some Lemmas that really hammered home the point of "work locally and see how to apply globally." For example, one of the things I learned is that $R = \bigcap_{\mathfrak{p}} R_{\mathfrak{p}}$, which is used to show that R is integrally closed (since $R_{\mathfrak{p}}$ is, in particular, a PID, and the intersection of integrally closed rings are integrally closed).

(4/17/2017) Today I learned a few things. To be brief on at least one of them, I learned about infiinte products. In particular, you can define whether a product converges if and only if it stops being zero eventually, and then the product $\Pi \omega_k$ converges absolutely if and only if the sum $\sigma |\omega_k - 1|$ does. That's pretty neat, and loosely put, you can identify an entire function $\mathbb{C}$ with its genus, which roughly measures what happens after the zeroes of the function occur (more to come on that on Friday).

I also learned this amazing fact about why one forms are the way they are. In the case of a two dimensional vector space, if we want some notion of area form we want the area function $\alpha : V \times V \to \mathbb{R}$ to be linear in each variable, and we want it to send v, v to 0. This *forces* $\alpha(v_1, v_2) = -\alpha(v_2, v_1)$. Which makes so much sense and is much more motivated than just asserting that the determinant is anticommutative. Since we don't have normalization on an "abstract vector space," we have a one dimensional subspace of choices of what to make our area function. This leads to the notion of a two form on a vector space, and the wedge product and all the other fun stuff.

I also finally went through the definition of a goddamn scheme! It's been like 6 months of me reading algebraic geometry, and I finally get to that point. Yay. And I have my first example of a scheme that is not an affine scheme, which is the infinite disjoint union of affine schemes. This isn't an affine scheme because in an affine scheme, the whole space is compact, whereas if we have an infinite number of schemes whose topologies are disjoint, we can have an open cover with no subcover.

And finally I touched up on some Algebraic Number Fields stuff. In particular, I learned more about fractional ideals, which included taking a detour into knowing what a Notherian module meant. And here's what it means–there's a similar "ascending chain condition," but this time on submodules. Which makes a lot of sense because of the insight I had today that *ideals are just submodules of the ring itself.* It makes so much sense in hindsight. This is why the inverse of a fractional ideal is finitely generated by the way, since it's contained in a finitely generated module (since if I is my fractional ideal and m is in the I, $I^{-1}m \subseteq R$, so $I^{-1} \subseteq Rm^{-1}$. The equivalence of the ACC of submodules and all submodules being finitely generated show that our inverse submodule is also a fractoinal ideal, since it's finitely generated and trivially an R submodule. Speaking of, I also learned some technical deails which are kind of important–fractional ideals must be finitely generated as modules and nonzero.

(4/18/2017) Today I learned that when determining the long exact sequence traditionally used in Khovanov homology, we are typically just exploiting a particular i (homology) grading, and the fact that if you take into account grading correctly, you get a direct sum, and hence a short exact sequence, and hence a long exact sequence induced from that. I've also been working hard on a problem that says if $\mathfrak{M}, \mathfrak{N}$ are fractional ideals, then $(\mathfrak{M}\mathfrak{N})^{-1} = \mathfrak{M}^{-1}\mathfrak{N}^{-1}$. I hope to type up a complete solution tomorrow.

(4/19/2017) Today I learned/went through problems regarding the flows on a vector space, and sorted out the difference between a flow and a vector field. So a *flow* is just a family of diffeomorphisms on a manifold, say f_t such that $f_{t+s} = f_t f_s$ ("If you drop a leaf in a river and wait $t + s$ seconds, it'll end up in the same spot as if you wait s seconds and then drop the leaf into the $f_s(leaf)$ spot and wait t seconds both leaves will end up at the same spot. Now, given a flow, one can partition a manifold via flow lines, which are simply fixing a point x on the manifold and considering the curve $t \to f_t(x)$. So putting the tangent vector of the flow line at each point onto the manifold, we get a vector field on our manifold. And going backwards, at least in the compact case, if we're given a vector field we can get a flow using the theory of ODE. We can also talk about the *index* of an isolated zero x of a vector field, which is cooked up via "take a small ball not containing any zeroes of the vector field other than at x and then on the boundary define a sphere to sphere map taking the point and mapping it to the direction of the sphere. I drew some pretty pictures today.

I haven't totally gotten the number theory problem yet, but I've solidified some shit. For example, I think I subconsciously believed that any irreducible element p has the property that (p) is prime. But this isn't true for all rings (although it is true for PIDs). For example, (2) isn't a prime ideal in $\mathbb{Z}[\sqrt{5}]$, for $2|6 = (1+\sqrt{5})(1-\sqrt{5})$ but doesn't divide either term in the product (which can be seen by modding our ring out by 2.) Maybe tomorrow. Maybe.

Or maybe today. First I'm going to prove a lemma that says if U and B are fractional ideals with $U \subseteq B$ and for all maximal $\mathfrak{q}$ $U_\mathfrak{q} = B_\mathfrak{q}, U = B$. For if $b \in B$, we let $I = \{y \in R : yb \subseteq U\}$ Then I is an ideal of R since U is a fractional ideal, and since $b \in U_\mathfrak{q}, b = a/s$ for some $a \in U, s \in R \setminus \mathfrak{q}$. Therefore $s \in I$ and therefore I isn't contained in any maximal ideal, and therefore $I = R$. Therefore $1b \in U$.

Great. Now that we have that Lemma, note by exercise 1 we have $[(MN)^{-1}]_\mathfrak{q} = [(MN)_\mathfrak{q}]^{-1} = [M_\mathfrak{q}N_\mathfrak{q}]^{-1} == [M_\mathfrak{q}]^{-1}[N_\mathfrak{q}]^{-1} = (M_\mathfrak{q}^{-1})(N^{-1})_\mathfrak{q} = (M^{-1}N^{-1})_\mathfrak{q}$, where the double equals sign comes from the fact that the statement is true for a PID.

Now all I have to do is show that this is true for a PID. But given a fractional ideal M with generators $\{\frac{a_j}{s_j}\}$, I bet you $M = R\frac{r}{s}$ where r is a principal generator of $M \cap R$ and $s = gcd\{s_j\}$. Oh and what do you know–that's exercise true. Every fractional ideal is principal.

(4/20/2017) Today I did a LOT of number theory. In fact, I did exclusively number theory today–specifically I focused on the linear algebra you can do on a field extension L/F. For example, for any element $x \in L$, you can view "multiplication by x" as an $F-$linear map from L to L. Picking an F basis of L, you can write a matrix expression for the linear map. Usually though, we don't like to pick bases arbitrarily, and so we compute things which aren't dependent on bases. The two we care about are the trace and norm/determinant of the map.

The first thing I learned about this today is that since you can factor any F term out of the trace, there is a symmetric F bilinear form on L defined via $(x, y) = T_{L/F}(xy)$. Now one thing you could ask about this form is if it is *nondegenerate* or not–meaning whether or not $(x, L) = 0 \implies x = 0$. Today I learned the proof that this bilinear form is nondegenerate if and only if the field extension L/F is separable. To see this in one direction, note that if L/F isn't separable, then $L^p \neq L$ but L^p contains any separable element. Therefore you can find an element $x \in L$ that isn't a p^{th} power, where p denotes the characteristic of our nonseparable field, since you can argue that x is in the purely inseparable part of the field, meaning that $t^{p^k} - x^{p^k}$ is the minimal polynomial over F for some k. So if $y \in L$ and $xy \notin L^p$, then $Tr_{L/F}(xy) = 0$ since the minimal polynomial of xy is of the above form–which has second term zero. But if $xy \in L^p$, we use the fact that we can just factor it out/view it as a scalar over L^p and use the transitivity of the trace–i.e. $Tr_{L/F}(xy) = Tr_{L_p/F}(Tr_{L/L^p}(xy)) = Tr_{L_p/F}(xyTr_{L/L_p}(1)) = Tr_{L^p/F}(xy0) = 0$ since $[L : L^p]$ must be a power of p.

But even more cool, we have some theorems relating the trace and the norm to Galois Theory! The specific version says that if L/F is Galois, $G = Gal(L/F)$ and $x \in L$, then $Tr_{L/F}(x) = \sum_{\sigma \in G} \sigma(x)$ and $N_{L/F} = \Pi_{\sigma \in G}\sigma(x)$. There's a really cool proof of it–essentially you break it down into a chain of field extensions $F \subseteq F(x) \subseteq L$ and you exploit the fact that over $F(x)$, the characteristic polynomial of x is just $q(x)^{[L:F(x)]}$, which relates the trace/determinant to the characteristic polynomial, which can be related to roots of the characteristic polynomial (which is now approaching Galois Theory) and then if you break down G into rising chain of subgroups corresponding to the above field extension, you can compute the trace/determinant and notice those also involve the roots of the characteristic polynomial. Woah!

(4/21/2017) Today I learned some things in preparation for Chandrashekar Khare's job talk (which I hope he gets). One thing I learned was that given any field F, you can take the *absolute Galois group* G_F, which is defined to be the automorphisms of F_{sep}/F, where F_{sep} is any element whose minimal polynomial over F repeats no roots. Khare's talk went into the representation theory of it, which connected it to $SL_2(\mathbb{Z})$. I also learned from Shalin's talk that for almost every number between zero and one, if you take the geometric mean of the first n numbers in the continued fraction expansion of a real number, you approach $K_0 := \Pi_{r=1}^{\infty}(1 + \frac{1}{r(r+2)})^{log_2(r)} = 2.685452..$ However, there is no specific number that we work with that we know this holds for–i.e. the only numbers we have that actually do satisfy this property were written specifically to satisfy this. I learned more things today, but went to the bar and forgot to write shit up.

(4/22/2017) Today I learned more about forms! In particular, I hammered down the definition of the alternating product of two given elements in the tensor algebra and the wedge product–in particular $S \wedge T := Alt(S \otimes T)$, at least up to a factorial sign. I also learned that given a positively oriented basis chosen, you can construct a cannonical volume form, which is an element in the top exterior power which evaluates to one on the ON basis. I also learned (or discovered) today that the Galois Theory/linear transformation norms that can be applied to any field extension are the exact same norms that I learned for quadratic field extensions–which is pretty interesting.

(4/23/2017) Even more about forms today. This time, we applied forms to manifolds! In particular, you can define the notion of a p form ω, which, given some point x on your manifold, $\omega(x)$ returns a p form on the vector space T_xX (where X, of course, is your manifold.) Then you can define the pullback of a given form via a smooth function $f : X \to Y$, which takes in forms on Y and returns a form on X as follows. Given a form ω on Y, after it's fed a point $y \in Y$, it returns a functional on T_yY. How can we move our form backwards to a point $x \in X$? Well, we need $(f^*\omega)(x)$ to be a functional on T_xX. How do we get such a functional? Well, we can push the tangent space forward through the derivative map, and then use our old form to send that into your ground field. Also, in Euclidean space, I learned that the forms dx_i, where x_i are the coordinate functions, form a basis for any form, and we can define *smoothness* of a form $\sum_I f_I x_I$ by requiring that all f_I are smooth (where I is a strictly increasing index set), and furthermore the smoothness of a form on a manifold by requiring that the pull back of a local parametrization yields a smooth form on Euclidean space.

I also created my first realistic exercise and something original, which I haven't done much of. In particular, I proved that for all nontrivial field extensions K/F, there exists a nonzero $\alpha \in K$ such that $Tr_{K/F}(\alpha) = 0$. I then showed that furthermore if $[K : F] > 2$ (at least, not sure about the two case), then there exists a $\theta \in K$ such that $(\theta, \theta) := Tr_{K/F}(\theta) = 0$. This is different and weirder than other inner products, say over $\mathbb{R}$. However, in $\mathbb{Q}[i]$, $(a + bi)^2 = a^2 - b^2 + 2abi$, which in particular says that $(a + bi, a + bi) = Tr_{\mathbb{Q}(i)/\mathbb{Q}}((a + bi)^2) = a^2 - b^2 - 2ab \neq 0$, since if equality held we would have $2a^2 = b^2 + 2ab + a^2 = (a + b)^2$. This says that either $a = 0$ (so $b = 0$ also) or 2 is rational.

(4/24/2017) Today, I learned (finally) what an analytic continuation is, and what a maximal analytic continuation is. Essentially, the idea is a dumb thing to define would be a "two analytic continuation" on a function $f_1 : \Omega_1 \to \mathbb{C}$ and another $f_2 : \Omega_2 \to \mathbb{C}$ as long as the functions agree on the intersection of the two domains Ω_i. But then you basically have a function split into two pieces. However, say with the logarithm example, you might have an $f_3 : \Omega_3 \to \mathbb{C}$ which makes a "two analytic continuation" (I think the actual term is "direct") which agrees with f_2 on the common domain of intersection, but not f_1. This leads to the notion of multiple valued function, and to require you have the largest possible sets of $(f_\alpha, \Omega_\alpha)$ such that any two f have a chain of analytic continuations connecting them define a maximal analytic continuation.

I also learned how to integrate today–finally. You can integrate a top form on an open set in Euclidean space U, that must have the form $\omega f dx_1 \wedge ... \wedge dx_k$ by merely defining $\int_U \omega = \int_U f dx_1...dx_n$. It turns out that forms respect the pullback rule (at least for orientation preserving diffeomorphisms), and that fact is built into how forms were constructed. Therefore we can define an integral of a form on a compactly supported function as just pulling back a parametrization to Euclidean space, and then (independent of choice of partition of unity) define the integral on the total space as summing over a partition of unity. I'm actually not 100 percent on these details yet. Maybe tomorrow!

I also learned how to prove that, given a Dedikind ring R and its quotient field K, then if E/K is a separable field extension then R', defined to be the integral closure of R over E, is also a Dedikind ring. Now, you can show it's integrally closed since the integral closure of an integral closure is just the integral closure. Notherianness follows from the fact that you can cleverly choose a basis of E/K to ensure that all of your basis elements are actually in R' (which, by the way, it turns out that E is the field of fractions of R') and then use a dual basis argument to essentially argue that $R' \subseteq \sum Rb_i$ where b_i is the dual basis of your freshly chosen basis in R'. Then you get Notherianness because it's the subset of a finitely generated R module (which, by the way, I reviewed today). Also you get the fact that every nonzero ideal is maximal from a lemma which essentially says that if you have an integrally closed ring A and B is integral over A then every nonzero ideal $\mathfrak{p} \subseteq B$ has an associated nonzero prime ideal $\mathfrak{p} \cap A \subseteq A$. This in particular says that if A above is also a field, then so must B be–which is essentially how you show modding out by any nonzero prime ideal of R' gives you a field.

(4/25/2017) Today I learned the idea of what a moduli space. It seems to be sort of like the idea of a fiber bundle, and in fact, I can't really tell the difference at this point. But I know that a moduli space is essentially the idea of a space where each point has a "insert your favorite mathematical thing" living above it. The classical example is $\mathbb{C} - 0 \to \mathbb{RP}^1$. I also learned the generalization of this, the *Grassmannian*, which is simply the k dimensional subspace of a vector space V, when k and V are given. I also solidified what is going on with respect to the isomorphism as schemes $Spec(A) \coprod Spec(B) \cong Spec(A \times B)$. In the course of this isomorphism, I essentially showed that each closed set in $Spec(A \times B)$ is mapped to the union of two closed sets in $Spec(A) \coprod Spec(B)$, which sort of explains why the infinite disjoint union of affine schemes isn't an affine scheme.

(4/26/2017) Today I learned (well, technically solidified, but I basically learned) what a contact form is on a manifold. A contact form on a three manifold is a form α on the manifold such that $(\alpha \wedge d\alpha)(x) \neq 0$ for all x. You can generalize this definition to any $2n+1$ manifold by requiring that $\alpha \wedge d\alpha \wedge ... \wedge d\alpha \neq 0$, where $d\alpha$ appears n times. This is one way to define three and four manifolds–simply take a surface of genus g, say Σ_g and then the unit cotangent or disc cotangent bundle are three and four manifolds respectively. I learned what the paper I'm presenting at Kylerec is going to be talking about–essentially the question is, if we define a Stein filling to be a "nice" filling (okay, I'm not 100 percent sure on the actual definitions, but essentially a Stein manifold admits a proper biholomorphic embedding and say that a Stein filling of a manifold is essentially a Stein manifold having that as a boundary). The statement my authors of my paper plan to explore and prove is that if $g > 2$, then any Stein filling of the unit cotangent bundle is s-cobordant (a cobordism whose inclusions are simple homotopy equivalences) to the "trivial" disc filling.

I also learned an easy fact that if you take the talk of $Spec(A)$ at a point $\mathfrak{p}$, you obtain the ring $A_{\mathfrak{p}}$. This is simply because if $f \notin \mathfrak{p}$, by definition $\mathfrak{p} \in D(f)$, which in particular tells us that we have an open set containing $\mathfrak{p}$ where f is invertible. Thus for any open set U contained in $D(f)$, f will be invertible on U as well, which in particular implies f is invertible in the stalk.

(4/27/2017) I learned there's a canonical structure on the *cotangent bundle* of a manifold M^k, denoted $T * M$, which is the manifold defined as a set as $\{(x, \tau) : x \in M, w \in T *_x M\}$, and topologized via the product topology, with atlas defined via maps of atlases on X crossed with $\mathbb{R}^k *$. In particular, there's a canonical one form on the cotangent bundle! How, you might ask? Well, the cotangent bundle comes with a canonical projection $\pi : T * M \to M$, and let's say we want our form to be ω. Then ω should eat a point (x, τ) as above, and spit out a map from the tangent space of the cotangent space at (x, τ), which is simply the tangent space at X times $\mathbb{R}^k$. But we already have a projection map, which in particular tells us that we can project down to the tangent space. So why not use $d\pi$ to project onto the first coordinate, and then take τ, which is already a linear map into the ground field? Which is exactly what it's defined to be.

I also learned why it's a little weird/deeper than you might think that there's a bijection between ring isomorphisms $A \to A'$ and scheme isomorphisms $Spec(A') \to Spec(A)$. Basically, the problem and the "soft deep truth" is that your whole scheme isomorphism is basically determined by how it operates on $D(1)$, since that's the whole ring. I suspect this has to do with localization and the fact that $A = \cap_{\mathfrak{p}} A_{\mathfrak{p}}$ or the stalk version of that, but we'll see. In algebraic geometry I also just did an exercise which essentially says that functions vanish on a closed set in locally ringed spaces. This is because in ringed spaces, if you vanish on the stalk, you can find a small enough open set to find an inverse, and then you get an inverse in that whole open set. Then since in a *locally* ringed space the stalks are fields, so you're either zero or invertible. Then you can use this to argue that in a locally ringed space, if you don't vanish anywhere then you're invertible. This is essentially because for every point you can locally cook up an inverse, so you use gluability and then identity to show that it actually is the inverse.

(4/28/2017) Today I learned (formally) what the exterior derivative is, at least in Euclidean space. Essentially we want a linear operator that satisfies some sort of product rule on the form $\omega = \sum f_I \wedge dx_i$, but also, since dx_i is basically a constant, all those terms should evaluate to zero. So we can define $d\omega = \sum df_I \wedge dx_i$. From this, you can unravel some stuff to prove that it satisfies an "almost" product rule (which essentially means you have to account for the fact that a minus sign might appear), and more importantly, that $d^2 = 0$, which actually later means you can define a cohomology theory with it! Wooh! Another nice thing about the exterior derivative operator is that if you have another operator that distributes over sums, has the "fake" product rule, and squares to zero, it's determined by how it operates on functions, i.e. zero forms. So in particular there's only one way to define a D operator on forms that agrees with the three properties above and agrees on zero forms! You can use this to show that you can do exterior derivatives on manifolds. Wooh!

I also learned an example of a nonaffine scheme today, which I would have thought before should be an affine scheme. Let $X = \mathbb{A}^2$ and $U = \mathbb{A}^2 - \{(0,0)\}$, which really is $\mathbb{A}^2 - \{(x,y)\}$. I learned how to compute the scheme at this open set U, which isn't an affine open set, since if $f \in (x,y)$, it's in another prime ideal–pick an irreducible element g dividing f and then $f \in (g)$ as well. But you can show that the functions actually don't change if you restrict to U, which actually follows more formally for some result about complex geometry if the codimension is larger than one, but in this particular case, follows as $U = D(x) \cup D(y)$ and for a function to be a function on U it must be an element of $k[x,y,1/x]$ which when projected agrees with the prechosen element of $k[x,y,1/y]$ in $k[x,y,1/(xy)]$. This is just $k[x,y]$ again, so we don't gain any functions. This actually shows that the restriction to U isn't an affine scheme–for if it were, what its ring would have to be $k[x,y]$ by above, and you would ask, what point is associated to $(x,y) \subseteq k[x,y]$?

(4/29/2017) Today I learned (well, unfortunately, had to rebuild the foundations of), the function $I()$, which takes a set (or a way I felt works better for me for some reason, "list") of prime ideals, and returns all of the functions (i.e. ring elements) that vanish at that point. In particular, I showed today in a better manner that $V(I(S)) = \overline{S}$. This is because you can do some definition chasing to show that $\subseteq$ holds, and to show that $\supset$ holds, you can argue that it holds for S and then take the closure of both sides, which helps since the right hand side is closed. I also came up with a slightly better way to work with the closure of a set in $SpecA$ for some ring A–you can think of $\overline{S}$ as $\bigcap_{g_\alpha : V(g_\alpha) \supset S} V(g_\alpha)$, since the intersection of all the closed sets are vanishing sets of ideals, and then $V(I) = \cap_{g \in I} V(g)$ since you vanish on the ideal if and only if you vanish on every point of the ideal.

(4/30/2017) Today I reviewed the foundations of the Algebraic Number Fields book I'm reading. One of the first things I realized is that although this what I learned today is great, I should really not try to bullshit it. Some days you just need to go back through things and make sure you have a solid foundation on them–and that's what I'm doing with algebraic number fields right now. One of the things I learned was a sort of "determinant trick." Essentially, if you have a set of generators $\{m_1, ..., m_n\}$ for an R module M, and you need a relation in terms of those generators (in particular, in the book, this is used if $\mathfrak{p}M = M$ and R is a local ring, or in the case where you have an element b in an $R[b]$ module finitely generated as an R module which isn't annihilated by any nonzero element and you want to determine the integral relation that your element solves), you can use this method. In the case of finding an integral polynomial some b solves, simply write each $bm_i = \sum_j a_{ij} m_j$ and then consider the matrix $Q = bI - [a_{ij}]$. The summation above implies that $Q(m_1, ..., m_n)^T = 0$. Note–the word "basis" doesn't come up here because this is a module.

So I don't think this means Q is the zero transformation. Letting B be the adjoint matrix of Q. Then if $d = det(Q)$, $BQ = dI$. Thus $d(m_1, ..., m_n) = 0$ so $d = 0$ so b is a root of the polynomial $det(xI - [a_{ij}])$. Oh you know what though? I don't think this transformation needs to be $R[b]$ linear, which explains why it need not be the zero transformation. Okay. Cool.

March 2017

(3/20/17) Today I learned that every if R is a ring and if $S \subseteq R$ is a multiplicative subset of R not containing 0, then the prime ideals of R whose intersection with S is empty are in one to one correspondence with the prime ideals of R_S, the ring localizing S. The way I look at this is to think, well, if I want a prime ideal in R_S from an ideal in R, say, I, if $I \cap S = \emptyset$, then you really didn't change the ideal. I'm not 100 percent sure how right that is-we'll see.

(3/21/17) Today on a homework problem I learned about the *Tubular Neighborhood Theorem*, which says that if one is given manifolds $Z \subseteq Y$, then there is a neighborhood in the normal bundle of Z in Y,written $N(Z;Y)$, containing Z, that is diffeomorphic to a neighborhood of Z. The picture I drew of this was a central circle for Z and a sphere for Y, which makes it seem super obvious, but I'm sure it isn't in general.

(3/22/17) So fun fact. The fact about prime ideals and localization in (3/20/17) was in the book I was reading for fun at night called Algebraic Number Fields. But today, I learned a very similar fact about prime ideals, but this time relating to quotienting, in Vakil's notes. In particular, there's a natural bijective correspondence between prime ideals in a ring R containing an ideal I and prime ideals in the quotient ring R/I. This in particular gives some good visualization of Spec(R/I) as a subset of $Spec(R)$. This sort of explains how we can relate ideals–for example, we can view any element of $Spec(\mathbb{C}[x,y]/(x^2 - y^2))$ as the set of ideals $I \subseteq \mathbb{C}[x,y]$ containing $(x^2 - y^2)$. So for example, the point $(x-1, y-2)$ isn't considered. This will probably have a similar story with the above localization and prime ideals.

Also today I learned a fact in DiffTop which will probably serve well in the future, well, maybe. A submanifold $Z \subseteq Y$ is *globally* cut out by the zeroes of independent coordinate functions if and only if its normal bundle in $YN(Z;Y)$ is trivial. I feel like now that we're getting into intersection theory stuff that might come in handy? I should think about why this applies to the example I learned in class about projecting a donut down to its height–why one circle by its lonesome can't be cut out in this case.

(3/23/17) The first thing I can think of that I learned today is the Van Kampen theorem for groupoids. In particular, if X is a topological space, then $Pi(X)$ is a groupoid whose objects are the points of x and whose morphisms between objects are homotopy classes of paths between the two points. In particular, if $S = \{U\}$ is an open cover of X that is closed under finite intersection such that all of those open sets are path connected*, we obtain an S shaped diagram if we view S as a category whose morphisms are "inclusion". Then the Van Kampen theorem for groupoids says that $\Pi(X) \cong colim_S(\Pi(U))$. *Although to be honest, I'm not sure if this condition was necessary. I combed through the proof a few times to see, and it didn't seem to. Then I saw this Stack-exchange post. http://math.stackexchange.com/questions/198348/does-mays-version-of-groupoid-seifert-van-kampen-need-path-connectivity-as-a-hy Also, the "closed under finite intersection" thing might be a similar thing to being a "filtered category."

(3/24/17) I don't know if there's a theorem I can say I learned new today, although I did learn some new insight (I imagine that will happen more if I'm doing research). There was a DiffTop homework problem about the four dimensional manifold $S^2 \times S^2$. Now, I had no idea how to visualize this manifold, but I realized that I could abstract the idea of the torus being $S^1 \times S^1$ and work with that. It turns out that, if $a \in S^2$ is chosen, then $\{a\} \times S^2$ is not homotopic or cobordant to $S^2 \times \{a\}$. Basically, we showed on an earlier homework that if two closed manifolds $X, Z \subseteq Y$ are cobordant, then for any compact submanifold $C, I_2(X,C) = I_2(Z,C)$. However, those two above manifolds aren't cobordant because for some $b \in S^2$ that isn't a, $\{b\} \times S^2$ intersects one manifold one time and one manifold zero times. I also learned the formal definition of cobordism, at least in the smooth category, and learned how a manifold that is the boundary of another inherits the orientation from the large manifold.

Also also I learned something with my DRP student in graph theory. The statement is "If G is a graph with n vertices and chromatic number k and G^c has chromatic number l then $n \leq kl$." The proof is slick as fuck. Basically, color G via the colors $a_1, ..., a_k$ and color its compliment with $b_1, ..., b_l$. Then $\{(a_i, b_j)\}$ colors the complete graph.

(3/25/17) Today I learned that if you have a region with n holes, then you can pick a homology basis and compute any complex integral you want by just summing over the winding number of the homology basis times the integral over the specified point in the basis. That's super cool–it basically says any weird integral can just be reduced to whatever its homology is. I also learned that you can give a prime ideal a height in a ring, and saw my first PhD defense ever! Plus, I learned more Morse theory. In particular, a Morse function is a function f from a manifold M to $\mathbb{R}$ such that any critical point has invertible matrix of second derivatives. Then there are no critical points in $[a,b]$ if and only if $f^{-1}((-\infty,a))$ is homeomorphic to $f^{-1}((\infty,b))$.

(3/26/17) I learned two main things today–one of them is a technical lemma involving the inherited dimension of a map $f : X \to Y \supset Z$ where $f \pitchfork Z$ and $\partial f \pitchfork Z$, we get a submanifold $S = f^{-1}(Z)$ where $\partial S = \partial X \cap S$. First, I learned how to put an orientation on S. Basically, we use the direct sum orientation, noting that if $x \in S$, then $df_x(N(S;X)) \oplus T_{f(x)}Z = T_{f(x)}Y$ so we can use the induced orientation plus the orientation property of df_x to give orientation to the normal bundle, and thus the tangent bundle. Now, to get the orientation on ∂S, we could either do this procedure first to get orientation of S and then give ∂S the boundary orientation, or we can immediately note that $\partial f : \partial X \to Y$ satisfies the above procedure so we could orient ∂S that way. It turns out, these differ by a factor of $(-1)^{codim(Z)}$, basically because we need to permute the outward pointing normal vector with a basis of $N(Z;Y)$ to put them in the same order in direct sum notation.

I also learned that, at least with my stupid hour of trying, it's nonobvious why if p, q are primes and the zetas are primitive roots of unity, $\mathbb{Q}(\zeta_p) \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$, although for some reason it seems obvious that it's true. I'll find out tomorrow.

(3/27/2017) I learned so much today! Although I did not learn that thing above that I said I was going to learn tomorrow. However, I did learn the definition of an elliptic function today finally–it's just a function $f : \mathbb{C} \to \mathbb{C}$ that has two linearly independent periods. Then it relates to lattices and the like.

I also learned a lot about orientation today. In particular, you can't orient $\mathbb{R}P^k$ when k is an even number, since its pullback under the map $S^k \to \mathbb{R}P^k$ yields the two different orientations on the two different poles. On the other hand, he did mention that you can always orient $\mathbb{C}P^k$, and I don't remember the explanation. Also, since for odd k the map $S^k \to \mathbb{R}P^k$ pulls back to the orientation, then you can orient $\mathbb{R}P^k$ for odd k. Also topologically related, I learned the idea of a satellite knot, which has a formal definition about incompressible discs, but as explained by Duncan, it's better thought of as a knot put in a torus, and then the torus put in a knot itself. Weirddddddd. I also finally learned what an isotopy is for real this time–it's just a homotopy where at each time t the map is an embedding!

And I did a good chunk of algebraic geometry today too. In particular, I learned that geometric intuition of localizing things is the best way to see localizing zerodivisors, and more importantly, given a ring map $\phi : B \to A$, then we obtain an induced map $\phi^{-1} : Spec(A) \to Spec(B)$. Also, you can use this method to generalize a given map of points from a space to another and determine how to extend it to the spectrum. Although I haven't totally done the full exercise to show why the algorithm works fully. Oh well. Also, I learned that a function $f \in R$ is not necessarily determined by its value at all points, but on the other hand, f is zero at every point if and only if it is nilpotent. Also, I learned that this thing might actually work for morale boosting. I've learend two pages of things. I'm really excited about how much I'm learning!

(3/28/17) I also learned a lot today. I mean, I worked pretty hard all day. The first thing I did was some Algebraic Topology. It was nice, because I learned about how to prove the most general case of the Van Kampen's theorem–where any open cover can be infinite. And essentially this works because colimits commute with colimits. This is basically because if you have two diagrams you're varying over, you can show they're both isomorphic to what I imagine is called the "bicategory." May only actually did it for two specific categories, both of which were in set whose maps were inclusion, but it seems like an easy generalization. I also learned (through an awful failure of computing the fundamental group of the Klein bottle) that the coproduct in the category of Groups is the free group. Can't believe I didn't connect it to what I learned in Hatcher. And for more category things, I learned that for any map $f : A \to B, p : E \to B$ where p is a covering space map, then the pullback map $f^*p : A \times_f E \to A$ is also a covering space! Basically given any a, you can choose the neighborhood around $f(a)$ to be homeomorphic, and then pull back those corresponding open sets to A. I'm probably not 100 percent sure on the details of this though, although I worked through it for a little while. I'll probably work more on Thursday.

I also learned a good chunk of Algebraic Geometry today. I'm really liking it. The main thing I learned was the idea of a vanishing set, and the corresponding Zariski topology you can put on a given Spectrum. Basically, the idea is that if you have a set S you can ask, "where does S vanish?" Well if you want to say S vanishes at a point $\mathfrak{p} \in Spec[A]$, thinking of a point $f \in S$ as a function, we want $f \equiv 0 mod \mathfrak{p}$, or equivalently, $S \subseteq \mathfrak{p}$. So the vanishing set of a set S in a ring A in Algebraic Geometry is "the set where S is zero" or by above, $\{\mathfrak{q} \in Spec(A) : S \subseteq \mathfrak{q}\}$. I learned some nifty lemmas, including a way to think about these sorts of vanishing sets that makes it clear that the complement of these form open sets. Basically, if you want $V(S) \cup V(T)$, you want the places where $S = 0$ or $T = 0$. Well then that's where $ST = 0$! So $V(S) \cup V(T) = (ST)$. Similarly, $\bigcap_i V(S_i) = (S_i) = \sum_i (S_i)$ since we want it to be zero everywhere.

(3/29/17) One of the things I learned was a nice proposition an an application. If R denotes an integral domain and K denotes its quotient field, an element $a \in R$ is integral over R, then the coefficients of the minimal polynomial of a over K are also integral over R! In particular, in an integrally closed domain R, the minimal polynomial of a is in $R[x]$ if and only if a is integral over R. This can prove a pretty powerful thing–that if d is a squarefree integer, then the integral closure of $\mathbb{Z}$ in $\mathbb{Q}[\sqrt{d}]$ is $\mathbb{Z} + \mathbb{Z}\sqrt{d}$ if $4 \not| d - 1$ and it is $\mathbb{Z} + \mathbb{Z}\frac{1+\sqrt{d}}{2}$ otherwise. This above proposition makes it easy to show, since you can show all of that stuff is in the corresponding integral closure, and then you can show that if $q_1 + q_2\sqrt{d} \in \mathbb{Q}[\sqrt{d}]$ solves an integer polynomial, then $q_1 \in \mathbb{Z}$ since you simply take the "integer part" of expanding the polynomial it solves plugging it in. Then you get that via subtracting an integer that $q_2\sqrt{d}$ solves some integer polynomial. I also learned something today on the level of practical advice–I should ask somebody to do a reading course in the fall before the summer starts.

(3/30/17) Today I got a major insight from a number theory talk that happened at UT given by someone named Samit Dasgupta. Well, I'm not sure if it's major, but I did learn the idea that a lot of things in number theory are proven by going from local to global principles. For example, one of the first applications of this is the Chinese Remainder Theorem. An extension of this is that

$\mathbb{Z} = \bigoplus_{pprime} \mathbb{Z}/p\mathbb{Z}$. From this talk, I also learned you can extend the zeta function to a function on any number field simply due to the fact that if you mod out by a maximal ideal in your number field, you get a finite number of elements, so you can "multiply" over all maximal ideals $_\mathfrak{}$ where $\frac{1}{1-N(\mathfrak{p})}$ where $N(\mathfrak{p})$ is the number of elements in $\mathfrak{p}$.

(3/31/17) Along with a few small things and an april fools talk, there were two main things I learned today. One was a few technical details in differential topology. For example, I disambiguated a phrase in one of the homework questions about if $i : X \to Y \supset Z$ is an inclusion map on manifolds suitable for intersection theory i "perscribes points," which I spent a pretty decent time trying to figure out what it meant. It turns out that it simply means the preimage orientation via $i^{-1}(Z)$. Also, I learned a definition and a good picture to have in my head to prove the relation between $I(X, Z)$ and $I(Z, X)$–two maps $f : X \to Y, g : Z \to Y$ are transverse at $y \in Y$ if for every $(x, z) \in f^{-1}(y) \times g^{-1}(y)$ that $df_x(T_xX) + dg_z(T_zZ) = T_yY$. The informal picture was f being a north-south circle going through itself on a torus, and g also winding around the circle twice hitting that same intersection point two times, but east-west (I should learn the technical terms for these), mostly. Then you have four pairs of points to check transversality.

I also learned a pretty good categorification of covering spaces, I have to admit. Essentially, you can say that a connected groupoid E covers another B if there's a functor $P : E \to B$ surjective on objects and such that if $St(x)$ denotes the morphisms starting at x, the function $f : St(x) \to St(P(x))$ is bijective. For groupoids, this means that every path with a specified start point lifts to a unique path, and formally implies statements about conjugacy relations in $St(x, x) = Aut(x, x)$ for a point x.