# Recovering $\mathfrak{g}$ Modules from The Weyl Group Action

Tom Gannon

University of Texas at Austin

November 24, 2020

# Representations of Semisimple Lie Algebras

- To study finite dimensional, complex representations of $G$ $(= \mathrm{SL}_n)$, it suffices to study representations of its *Lie algebra* $\mathfrak{g} := T_1(G)$.
- Conjugation of $G$ on itself $\mapsto$ *Lie bracket* on $\mathfrak{g}$.
- *Example*: If $G = SL_n$, $\mathfrak{sl}_n = \{\text{traceless } n \times n \text{ matrices}\}$, with Lie bracket of two matrices given by their *commutator*, i.e. $[X, Y] = XY - YX$.

**Definition**

A (finite dimensional) *representation of* $\mathfrak{g}$ is a vector space map $\mathfrak{g} \to \mathfrak{gl}_n$ sending the Lie bracket on $\mathfrak{g}$ to the Lie bracket on $\mathfrak{gl}_n$, i.e. the commutator.

# Example: $\mathfrak{sl}_2$

A useful basis:

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

With Lie bracket determined by:

$$[h, e] = 2e,\ [h, f] = -2f,\ [e, f] = h$$

*Note*: $[X, X] = 0$ for all $X \in \mathfrak{g}$.

*Remark*: $\{$Representations of $\mathfrak{sl}_2\} = \{$Modules for $\mathrm{U}(\mathfrak{sl}_2)\}$, where

$$\mathrm{U}(\mathfrak{sl}_2) := \mathbb{C}\langle h, e, f\rangle/(he - ef = 2e, hf - fh = -2f, ef - fe = h).$$

# Example: Representations of $\mathfrak{sl}_2$

## Proposition

The finite dimensional representations of $\mathfrak{sl}_n$ are *semisimple*, meaning any finite dimensional subrepresentation is a direct summand.

$\implies$ to classify finite dimensional $\mathfrak{sl}_2$ representations, it suffices to classify the irreducible ones.

## Theorem - Classification of Irreducible $\mathfrak{sl}_2$ Representations

For any irreducible $\mathfrak{sl}_2$ representation $V$, there exists a unique nonnegative integer $n$ and nonzero $v \in V$ such that:

- $hv = nv$
- $ev = 0$

and this $n$ classifies the irreducible $\mathfrak{sl}_2$ representations!

In other words, $\{\text{Finite Dimensional Irreducible Reps of } \mathfrak{sl}_2\} \longleftrightarrow \mathbb{Z}/\langle w \rangle$, where $w$ is reflection about 0.

# Representations of $\mathfrak{sl}_3$

- *Idea*: Use *all* diagonal matrices $\mathfrak{t} := \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} : a + b + c = 0 \right\}$.
- Eigenvalues for $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are replaced by eigenvalues for $\mathfrak{t}$, i.e.

## Definition

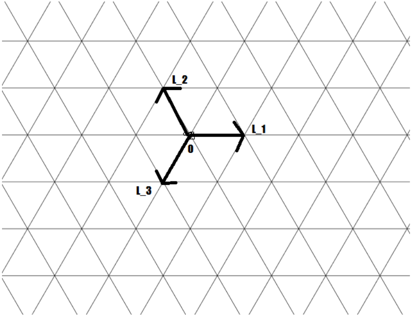
We say a nonzero vector in a $\mathfrak{sl}_3$ representation $v \in V$ is an *eigenvector* if there exists a linear map $\lambda : \mathfrak{t} \to \mathbb{C}$ such that for all $t \in \mathfrak{t}$,

$$tv = \lambda(t)v.$$

# Picturing the Eigenvalues $\mathfrak{t}^\vee$

Set $L_1$ such that $L_1(\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}) = a$.

Set $L_2, L_3$ similarly. $\implies \mathfrak{t}^\vee = \{aL_1 + bL_2 + cL_3 : a + b + c = 0\}$

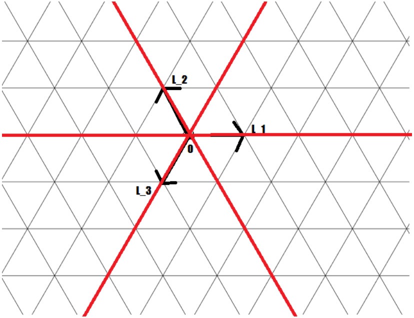



*Pictured*: Real points of $\mathfrak{t}^\vee$

*Source*: Matthew Fayers Coxeter Graph Paper

# The Weyl Group and the Integers

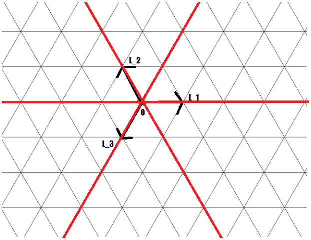
Note also that $S_3$ acts $\mathfrak{t}^\vee$:




Role of integers in $\mathfrak{sl}_2$ is replaced by

$$\Lambda := \mathbb{Z}L_1 + \mathbb{Z}L_2 + \mathbb{Z}L_3$$

# Irreducible Representations of $\mathfrak{sl}_3$




**Theorem - Classification of Irreducible $\mathfrak{sl}_3$ Representations**

We have a bijective correspondence

$$\{\text{Finite Dimensional Irreducible Representations of } \mathfrak{sl}_3\} \longleftrightarrow \Lambda/S_3.$$

*Remark*: $S_3 \cong \mathrm{N}_{\mathrm{SL}_3}(T)/T$.

# A Theme of Representation Theory

**Theme**

Data about $G$ and $\mathfrak{g}$ representations can be recovered from:

- A maximal torus $T$, or its associated Lie algebra $\mathfrak{t}$
- The 'integers' $\Lambda \subset \mathfrak{t}^\vee$
- The action of the *Weyl group* $W := \mathrm{N}_G(T)/T$ on $T$ or $\mathfrak{t}$.

*Example*: The Weyl group of $\mathrm{SL}_n$ is the symmetric group on $n$ letters.

# The Category $\mathcal{O}_0$

We want a category that captures all highest weights.

## Definition

Let $\mathcal{O}_0$ denote the full (abelian) subcategory of ($\mathfrak{g}$-Mod) of modules $M$ which have a highest weight vector and have *central character zero*, meaning elements like $h^2 + 2h + 4ef + 44 \in \mathrm{Z}(U(\mathfrak{sl}_2))$ scale by 44.

*Remark*: Often times we also ask objects of $\mathcal{O}_0$ to also be finitely generated.

*Example*: The *Verma module* $\Delta_0 := \mathrm{U}(\mathfrak{sl}_2) \otimes_{\mathrm{U}(\mathfrak{b})} \mathbb{C} \in \mathcal{O}_0$, since

$$0 \to L_{-2} \to \Delta_0 \to L_0 \to 0$$

$\implies$ $\mathcal{O}_0$ is not semisimple!

*Question*: Can we recover $\mathcal{O}_0$ from $\Lambda$, $T$, and $W$?

# Soergel's Endomorphismensatz

Soergel used combinatorics to reduce the study of $\mathcal{O}_0$ to a certain quotient category, $\mathcal{O}_{0,nondeg}$.

**Theorem - Soergel 1990**

We have an equivalence of categories

$$\mathcal{O}_{0,nondeg} \simeq (\mathrm{Sym}(\mathfrak{t}) \otimes_{\mathrm{Sym}(\mathfrak{t})^W} \mathbb{C})\text{-Mod}.$$

Geometrically, if $\chi : \mathfrak{t} \to \mathfrak{t}//W$ is the (GIT) quotient map, $\mathrm{Sym}(\mathfrak{t}) \otimes_{\mathrm{Sym}(\mathfrak{t})^W} \mathbb{C}$ is functions on $\chi^{-1}(0)$.

*Example:* If $\mathfrak{g} = \mathfrak{sl}_2$, $\chi^{-1}(0) = \mathit{Spec}(\mathbb{C}[\epsilon]/\epsilon^2)$.

*Good News*: There is an analogous version of this theorem for every central character, which 'seem to fit together'.

*Bad News*: The methods use to prove this result are very finite length.

# Replacing $\mathcal{O}_0$ with Its Derived Category

*Notation:* $(\mathfrak{g}\text{-Mod})_0^N := \mathcal{D}(\mathcal{O}_0)$.

**Theorem - Soergel 1990, Derived Version**

We have an equivalence

$$(\mathfrak{g}\text{-Mod})^N_{0,\text{nondeg}} \simeq \mathrm{QCoh}^{\text{ren}}(\chi^{-1}(0))$$

where $\chi : \mathfrak{t} \to \mathfrak{t}//W$ is the quotient map.

*Remark*: Renormalization occurs because some objects in $\mathcal{O}_{0,nondeg}$ don't have finite length projective resolutions.

*Question*: Can we recover the full category $(\mathfrak{g}\text{-Mod})^N_{nondeg}$?

# Removing Central Character Restriction

*Answer*: Yes, but we need to use $\Lambda$:

**Definition**

The *affine Weyl group* is the group $W^{\text{aff}} := \Lambda \rtimes W$.

**Theorem - G. 2020**

We have an equivalence

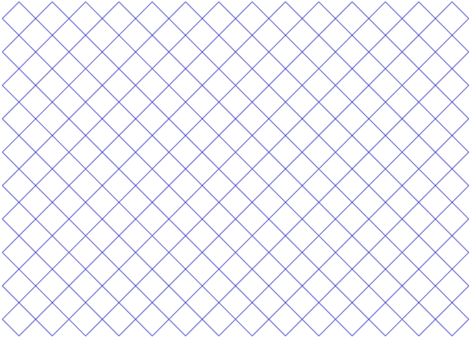
$$(\mathfrak{g}\text{-Mod})^N_{nondeg} \simeq \text{QCoh}^{\text{ren}}(\mathfrak{t}//W \times_{\mathfrak{t}//W^{\text{aff}}} \mathfrak{t}/\Lambda).$$

Equivalently, we have

$$(\mathfrak{g}\text{-Mod})^N_{nondeg} \simeq \text{QCoh}^{\text{ren}}(\mathfrak{t}//W \times_{\mathfrak{t}//W^{\text{aff}}} \mathfrak{t})^{\Lambda}.$$

# Example: $\mathfrak{sl}_2$

*Example:* $(\mathfrak{sl}_2\text{-Mod})^N_{nondeg}$ can be identified with $\mathbb{Z}$-equivariant sheaves on



$//\langle w \rangle$

where $w$ denotes reflection across $y = 0$.

# Concluding Remarks

**Theorem - G. 2020**

We have an equivalence

$$(\mathfrak{g}\text{-Mod})^N_{nondeg} \simeq \mathrm{QCoh}^{\mathrm{ren}}(\mathfrak{t}//W \times_{\mathfrak{t}//W^{\mathrm{aff}}} \mathfrak{t}/\Lambda).$$

Equivalently, we have

$$(\mathfrak{g}\text{-Mod})^N_{nondeg} \simeq \mathrm{QCoh}^{\mathrm{ren}}(\mathfrak{t}//W \times_{\mathfrak{t}//W^{\mathrm{aff}}} \mathfrak{t})^{\Lambda}.$$

*Remark:* We can also further identify the $W$ action on the right category.

*Remark:* We also have versions for $\mathcal{D}(N\backslash G/N)_{nondeg}$, for the weakly $T$ equivariant, and the weakly $T \times T$ equivariant category.

*Remark:* Recovering $(\mathfrak{g}\text{-Mod})^N$ from the quotient category is a work in progress with Gurbir Dhillon and Sam Raskin.

*Thank you!*