Categorical Representation Theory and the Coarse Quotient

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Setup

Today, let

- G denote a reductive algebraic group over $\mathbb{C}$
- B denote some choice of Borel subgroup.

Example

We can take $G := \mathrm{SL}_n$ with choice of $B := \{\text{upper triangular matrices}\}$

and let

- $\mathfrak{g}$ denote the *Lie algebra* $T_1(G)$
- $U(\mathfrak{g})$ denote the *universal enveloping algebra.*

Example

If $G = \mathrm{SL}_2$, $\mathfrak{sl}_2 \cong \{\text{traceless } 2 \times 2 \text{ matrices}\}$ and

$$U(\mathfrak{sl}_2) \cong \mathbb{C}\langle e, f, h\rangle/(ef - fe = h, he - eh = 2e, hf - fe = -2f).$$

Representation Theory Via Sheaves and Symmetries

Theorem (Beilinson-Bernstein '81)

The global sections functor $\Gamma : \mathcal{D}(G/B) \to \mathfrak{g}\text{-mod}_0$ is an exact equivalence of categories.

- $\mathcal{D}(G/B)$ is the category of *$\mathcal{D}$-modules* on G/B
- $\mathfrak{g}\text{-mod}_0$ denotes the $\mathfrak{g}$-representations with *central character zero*

Idea: Use the symmetries of $\mathcal{D}(G/B)$.

- Left G-action on $G/B \rightsquigarrow$ essential image of Γ.
- Right $B\backslash G$-action on G/B? Not defined.
- *Residual symmetries:* If $\mathcal{F} \in \mathcal{D}(G/B) \simeq \mathcal{D}(G)^B$ and $\mathcal{G} \in \mathcal{D}(G)^{B\times B} \simeq \mathcal{D}(B\backslash G/B)$, can define $\mathcal{F} \star^B \mathcal{G} \in \mathcal{D}(G/B)$.
- Action of $\mathcal{D}(B\backslash G/B) \rightsquigarrow$ Hecke algebra symmetries $\rightsquigarrow$ multiplicities of simples in Vermas

Categorical Symmetries

Theme: We can obtain concrete, representation theoretic information via sheaves and categorical symmetries.

Definition

A group G *acting on a category* $\mathcal{C}$ is the data exhibiting $\mathcal{C}$ as a module category for $(\mathcal{D}(G), \star)$.

Remark: Our categories are *DG categories* or $\mathbb{C}$-linear *stable* ∞*-categories*.

Examples:

- If G acts on a variety X, then G acts on $\mathcal{D}(X)$.
- G acts on $\mathfrak{g}$-mod.

Invariants

Definition: If $G \curvearrowright \mathcal{C}$, the *invariants* are $\mathcal{C}^G := \underline{\mathrm{Hom}}_G(\mathrm{triv}, \mathcal{C})$.

Examples:

- If $G \curvearrowright X$, $\mathcal{D}(X)^G \simeq \mathcal{D}(X/G)$.
- $(\mathfrak{g}\text{-mod})^G \simeq \mathrm{Rep}(G)$.
- $\mathcal{D}(G/B)^N \simeq \mathcal{O}_0$.
- $\mathcal{D}(G)^{G_{ad}} \ni$ character sheaves

Proposition: If G acts on $\mathcal{C}$ and $H \leq G$, then $\mathcal{D}(H \backslash G/H)$ acts on $\mathcal{C}^H$.

Remark: $\mathcal{D}(H \backslash G/H) \simeq \underline{\mathrm{End}}_G(\mathcal{D}(G/H))$.

Nondegenerate PGL_2-Categories and Classification

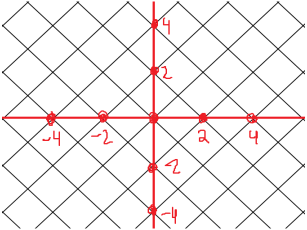
Definition: Let $G := \mathrm{PGL}_2$. A *nondegenerate G-category* is a G-category $\mathcal{C}$ such that $\mathcal{C}^{\mathrm{SL}_2} \simeq 0$.

Theorem - G.

We have an equivalence of categories

$$G\text{-cat}_{\text{nondeg}} \simeq \mathrm{IndCoh}(\Gamma_{W^{\mathrm{aff}}})\text{-modcat}$$

$\implies$ nondegenerate G-categories sheafify over $(\mathbb{A}^1/\mathbb{Z})//W$.



$\Gamma_{W^{\mathrm{aff}}}$ - The union of graphs of the affine Weyl group.

Whittaker Invariants

Definition: Given $G \curvearrowright V$, its *Whittaker invariants* are $V^{N,\psi} := \{v \in V : mv = \psi(m)v\}$ for a generic character $\psi : N \to \mathbb{C}$.

Definition: Given $G \curvearrowright \mathcal{C}$, its *Whittaker invariants* are $\mathcal{C}^{N,\psi} := \underline{Hom}_N(\mathrm{Vect}_\psi, \mathcal{C})$.

Remark: $\mathcal{H}_\psi := \mathcal{D}(G)^{N \times N, \psi \times \psi} \curvearrowright \mathcal{C}^{N,\psi}$.

Theorem - (Ginzburg, Lonergan, G.)

There is an equivalence of categories

$$\mathcal{H}_\psi \simeq \mathrm{IndCoh}(\mathfrak{t}^*//\tilde{W}^{\mathrm{aff}})$$

which is monoidal and which is t-exact up to cohomological shift.

$\implies \mathcal{H}_\psi$ is symmetric monoidal!

The Coarse Quotient

Assumption: G is adjoint.

Definition: The *stack quotient* $\mathfrak{t}^*/W^{\mathrm{aff}}$ is

$$\operatorname{colim}\left(\cdots \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} W^{\mathrm{aff}} \times W^{\mathrm{aff}} \times \mathfrak{t}^* \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} W^{\mathrm{aff}} \times \mathfrak{t}^* \underset{\mathrm{proj}}{\overset{\mathrm{act}}{\rightrightarrows}} \mathfrak{t}^* \right)$$

Definition: The *coarse quotient* $\mathfrak{t}^*//W^{\mathrm{aff}}$ is:

$$\operatorname{colim}\left(\cdots \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} \Gamma_{W^{\mathrm{aff}}} \times_{\mathfrak{t}^*} \Gamma_{W^{\mathrm{aff}}} \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \Gamma_{W^{\mathrm{aff}}} \underset{\mathrm{s}}{\overset{\mathrm{t}}{\rightrightarrows}} \mathfrak{t}^* \right)$$

Proposition - G.

The projection map $\pi : \mathfrak{t}^*/W^{\mathrm{aff}} \to \mathfrak{t}^*//W^{\mathrm{aff}}$ induces

$$\mathrm{IndCoh}(\mathfrak{t}^*//W^{\mathrm{aff}}) \xrightarrow{\pi^!} \mathrm{IndCoh}(\mathfrak{t}^*/W^{\mathrm{aff}}) \simeq \mathrm{IndCoh}(\mathfrak{t}^*)^{W^{\mathrm{aff}}}$$

which is fully faithful, t-exact, and whose essential image is sheaves *satisfying Coxeter descent*.

Nondegenerate G-Categories from Invariants

Theorem (Ben-Zvi–Gunningham–Orem)

The functor $\mathcal{C} \mapsto \mathcal{C}^N$ is conservative, and moreover induces

$$G\text{-cat} \xrightarrow{\sim} \mathcal{D}(N\backslash G/N)\text{-modcat}.$$

Definition: A G-category is *nondegenerate* if for every rank one parabolic P, $\mathcal{C}^{[P,P]} \simeq 0$.

Proposition: Every G-category $\mathcal{C}$ has a G-subcategory $\mathcal{C}_{\text{nondeg}} \xhookrightarrow{J_*} \mathcal{C}$ and has a functor $\mathcal{C} \xrightarrow{J^!} \mathcal{C}_{\text{nondeg}}$ which behaves like a quotient functor.

Theorem - G.

We have an equivalence of categories

$$G\text{-cat}_{\text{nondeg}} \simeq \text{IndCoh}(\Gamma_{W^{\text{aff}}})\text{-cat}$$

$\Longrightarrow$ nondegenerate G-categories sheafify over $\mathfrak{t}^*//W^{\text{aff}} \simeq (\mathfrak{t}^*/X^\bullet)//W$.

Explicit Monoidal Equivalence

If G acts on $\mathcal{C}$, we can define the *weak invariants* $\mathcal{C}^{G,w}$.

Corollary: (Ben-Zvi–Gunningham) G-cat $\xrightarrow{\sim} \mathcal{D}(N\backslash G/N)^{T\times T,w}$-modcat.

Theorem - G.

There are t-exact, monoidal, equivalences of categories

$$\mathcal{D}(N\backslash G/N)^{(T\times T,w)}_{\text{nondeg}} \simeq \text{IndCoh}(\Gamma_{W^{\text{aff}}})$$

$$\mathcal{D}(N\backslash G/N)_{\text{nondeg}} \simeq \text{IndCoh}(\Gamma_{W^{\text{aff}}})^{X^\bullet \times X^\bullet}$$

where $X^\bullet$ is the character lattice of T.

- Right hand side only depends on the root datum.
- Koszul duality for $\mathcal{D}(N\backslash G/N)$?

Parabolic Restriction

Definition: *Parabolic restriction* is $Res : \mathcal{D}(G)^G \to \mathcal{D}(T)^T$ given by pull-push along $G/G \leftarrow B/B \to T/T$.

Definition: The *horocycle functor*, for any $G \times G$ category $\mathcal{C}$, is the composite: $\mathcal{C}^{G_\Delta} \xrightarrow{\text{oblv}} \mathcal{C}^{B_\Delta} \xrightarrow{\text{Av}_*^N} \mathcal{C}^{(N \times N)T_\Delta}$.

Proposition: The *parabolic restriction* functor is the composite $\mathcal{D}(G)^G \xrightarrow{hc_{\mathcal{D}(G)}} \mathcal{D}(N\backslash G/N)^{T_\Delta} \xrightarrow{i^!} \mathcal{D}(T)^T$.

Conjecture - Ben-Zvi–Gunningham

Parabolic restriction lifts to a functor $\text{WRes} : \mathcal{D}(G)^{G,\heartsuit} \to \mathcal{D}(T)^{W,\heartsuit}$ such that if $\mathcal{F} \in \mathcal{D}(G)^{G,\heartsuit}$ has $hc(\mathcal{F})$ supported on T, then $W\text{Res}(\mathcal{F})$ *satisfies Coxeter descent.*

Theorem - G.

There is a lift $\tilde{hc}_{\mathcal{D}(G)_{\text{nondeg}}} : \mathcal{D}(G)^G_{\text{nondeg}} \to \mathcal{D}(N\backslash G/N)^{T \rtimes W}_{\text{nondeg}}$ such that if $\mathcal{F} \in \mathcal{D}(G)^{G,\heartsuit}_{\text{nondeg}}$ has $hc(\mathcal{F})$ supported on the torus, $\tilde{hc}(\mathcal{F})$ satisfies Coxeter descent.