Introduction to Representation Theory of Lie Algebras

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These are the notes for the summer 2020 mini course on the representation theory of Lie algebras. We'll first define Lie groups, and then discuss why the study of representations of simply connected Lie groups reduces to studying representations of their *Lie algebras* (obtained as the tangent spaces of the groups at the identity). We'll then discuss a very important class of Lie algebras, called *semisimple* Lie algebras, and we'll examine the representation theory of two of the most basic Lie algebras: $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$. Using these examples, we will develop the vocabulary needed to classify representations of all semisimple Lie algebras!

Please email me any typos you find in these notes! Thanks to Arun Debray, Joakim Færgeman, Aaron Mazel-Gee, and Pablo Sanchez Ocal for doing this. Also–thanks to Saad Slaoui and Max Riestenberg for agreeing to be teaching assistants for this course and for many, many helpful edits.

Contents

1 From Lie Groups to Lie Algebras

1.1 Lie Groups and Their Representations

Definition 1.1.1. A *complex (real) Lie group* is a group G equipped with a complex (real) manifold structure, such that the multiplication map $G \times G \xrightarrow{m} G$ and the inversion map $G \xrightarrow{i} G^{-1}$ (which sends $g \mapsto g^{-1}$) are both maps of complex (real) manifolds.

Example 1.1.1. The groups $GL_n(\mathbb{C})$ and $GL_n(\mathbb{R})$ are complex and real Lie groups, respectively. This is because both of these groups are $\det^{-1}(U)$ for some open $U \subset \mathbb{R}$, so they are open subsets of affine space $\mathbb{A}^{n^2}$. Similarly, the groups $SL_n(\mathbb{C})$ and $SL_n(\mathbb{R})$ are complex and real Lie groups, respectively, because $SL_n = \det^{-1}(\{1\})$ and is a regular level set of the determinant function with respect to the regular value $1 \in \mathbb{R}$. Notice that in both cases, matrix multiplication and matrix inversion are manifold maps (the first is a polynomial in each entry and the second is a non-vanishing rational function in each entry by Cramer's rule).

Example 1.1.2. The groups $(\mathbb{R}, +)$, $(\mathbb{R}_{>0}, \times)$, and S^1, are all one dimensional real Lie groups (where we define the group structure on $S^1 = \{e^{i\theta} \in \mathbb{C} : 0 \leq \theta < 2\pi\}$ via complex multiplication).

Remark 1.1.1. The definition of a Lie group is a bit stronger than 'a group which is also a manifold', because we are requiring the maps that are part of the group data to be maps of manifolds (i.e. smooth or holomorphic maps). This reflects a general theme throughout algebra and topology, which says that if we have certain structure (for example, a group structure, ring structure, manifold structure, etc), then the maps to consider between any two objects should be those that preserve that structure (for example, group homomorphisms, ring homomorphisms, smooth maps, etc). This leads us to defining:

Definition 1.1.2. A *map of Lie groups* $\phi : G \to H$ is a map of real/complex manifolds which is also a group homomorphism. A *representation of a Lie group* G is a map of Lie groups $G \to GL(V)$ for some finite dimensional[1] vector space V.

Our goal in this course will be to study representations of Lie groups. The first step in doing this is to note that our requirement that the group multiplication map is manifold map buys us a lot of mileage. For instance, any two representations of a connected Lie group which agree in some neighborhood of the identity are actually the same representation, as we will see exercise 1.1 below.

We can go even further than that. One important feature of manifolds is that many properties of maps between them are determined locally by their associated maps (derivatives) at the level of tangent spaces. Combined with the fact that group multiplication on G can be used to carry information from one tangent space to another, it turns out that much can be learned about G by simply looking at its tangent space at the identity T_eG. This leads us to make the following definition:

Definition 1.1.3. Given a Lie group G, define the *Lie algebra* of G to be the tangent space of G at the identity, and denote it by $\mathfrak{g} := T_e(G)$. (Similarly for a Lie group H, write $\mathfrak{h} := T_e(H)$, and so on.)

Remark 1.1.2. So far, we have only defined the Lie algebra $\mathfrak{g}$ as a vector space. The choice of terminology will be justified when we define a multiplicative structure on $\mathfrak{g}$ called the *Lie bracket* in the next section.

Now, for any map of Lie groups $\phi : G \to H$, we get an induced map $d\phi_e : \mathfrak{g} \to \mathfrak{h}$ (since ϕ is a group homomorphism!). One of the first major theorems in the subject, which we won't prove in this course, is that ϕ is actually determined by its derivative at the identity $d\phi_e$, provided that G is connected:

Theorem 1.1.1. If G is a connected Lie group, H a Lie group, then any two maps of Lie groups $\phi_1, \phi_2 : G \to H$ inducing equal maps $d(\phi_1)_e = d(\phi_2)_e : \mathfrak{g} \to \mathfrak{h}$ at the level of Lie algebras must be equal: $\phi_1 = \phi_2$.

Example 1.1.3. The multiplicative group $G = (\mathbb{R}_{>0}, \times)$ has Lie algebra given by $\mathfrak{g} \cong (\mathbb{R}, +)$. Viewed as a connected Lie group, $\mathfrak{g}$ is its own Lie algebra, and the exponential map $\exp : \mathfrak{g} \to G$ identifies these two groups, so the theorem holds in this case.

Broadly speaking, the proof of theorem 1.1.1 involves working with the *exponential map* $\exp : \mathfrak{g} \to G$, defined for an arbitrary Lie group by following integral flows of vector fields (see exercise 1.5). One can argue that this exponential map identifies a small neighborhood of $0 \in \mathfrak{g}$ with a small neighborhood of $e \in G$, and then we can appeal to exercise 1.1 to obtain theorem 1.1.1.

[1] This assumption is made so that $GL(V)$ is a manifold as defined in example 1.1.1 above – some care is needed to define infinite dimensional representations of Lie groups.

1.2 Exercises

Exercise 1.1. Show that if two maps of Lie groups $\phi_1, \phi_2 : G \to H$ agree on any nonempty neighborhood of $e \in G$, then $\phi_1 = \phi_2$ provided G is connected.

Exercise 1.2. Show that $\mathfrak{sl}_n$ can be identified with the subspace of $\mathfrak{gl}_n$ consisting of traceless matrices.

Exercise 1.3. Show that $\mathfrak{so}_n$ can be identified as a vector space with the set of matrices A with $A^T = -A$ (i.e. skew-symmetric matrices). As SO_n is a subgroup of GL_n, it follows from a general fact that the Lie bracket on $\mathfrak{so}_n$ is given by the commutator of the matrices.

Exercise 1.4. Show that $\mathfrak{pgl}_n$ can be identified with the vector space of traceless matrices. (Hint: Consider the map $SL_n \to PGL_n$.) We will see that $\mathfrak{sl}_n \cong \mathfrak{pgl}_n$ as *Lie algebras* tomorrow, hence this exercise shows that two distinct Lie groups can have the same Lie algebra!

Exercise 1.5. Fix a tangent vector $X \in \mathfrak{g}$. Show that there exists a (smooth or algebraic– whichever context you want to work in) vector field on G which returns the vector X at the identity. Show that the vector field you constructed is either invariant under the right action of G on itself, or invariant under the left action of G on itself, depending on how you defined it. (This exercise is not as critical as the other four).

1.3 Bonus Exercises

These exercises are not required for the course and are for those interested in going deeper into the ideas of representation theory touched on in this course or in the very related representation theory of algebraic groups. As such, in the bonus exercises, we freely use the language of algebraic geometry (and you are welcome to ask us for clarification!)

Exercise 1.6. In the spirit of today's lecture, define an 'algebraic group' as the algebraic variety analogue of a Lie group[2]. Show that GL_n is an algebraic group and define the notion of a representation of an algebraic group G.

Exercise 1.7. Show that one dimensional representations of $\mathbb{G}_m := GL_1$ are classified by the integers.

For the following exercises, it helps to know why affine algebraic groups are classified by *Hopf algebras*.

Exercise 1.8. Show that a representation of $\mathbb{G}_m$ is the same thing as a graded vector space.

Exercise 1.9. Show that all irreducible representations of affine algebraic groups are finite dimensional (!). (Hint: Show that any finite dimensional subspace of this representation is contained in a finite dimensional G subspace.)

[2]Please ask us if this is unclear!

Exercise 1.10. Let $\mathbb{G}_a$ denote the *additive group*, defined by $Spec(\mathbb{C}[x])$ with the addition law $\mathbb{G}_a \times \mathbb{G}_a \to \mathbb{G}_a$ or, equivalently, as the forgetful functor *Rings* $\to$ *Groups* given by $A \mapsto (A, +)$. Show that the category of representations of $\mathbb{G}_a$ is equivalent to the category of vector spaces V equipped with a nilpotent endomorphism $T : V \to V$, i.e. an endomorphism such that $T^n = 0$ for $n \gg 0$. (Hint: Show that over any characteristic a representation of $\mathbb{G}_a$ is equivalent to a module V over the *divided power algebra* $k[M_1, M_2, ...]/(\binom{i+j}{i} M_{i+j} = M_i M_j)$. How does the classification of $\mathbb{G}_a$ reps change in characteristic p?)

2 Examples and Semisimple Lie Algebras

2.1 The Bracket Structure on Lie Algebras

Since vector space maps are generally easier to work with than maps of groups, a natural question to ask next is, 'When does a map of tangent spaces $\mathfrak{g} \to \mathfrak{h}$ actually induce a map of the associated Lie groups $G \to H$?' [3]

There are a few ways to go about this, and here is one. Given a group G, we automatically obtain three different canonical actions of G on itself–G acts on itself by left multiplication, right multiplication, and conjugation. By exercise 1.1, we have, in essence, 'used up' the first two actions by requiring that Lie group maps preserve the identity. However, we haven't touched on the conjugation action yet (because conjugation preserves the identity). This is pretty important, because any map of groups should in particular preserve conjugation.

Concretely, fix an element $g \in G$. Then 'conjugation by g' is a map $G \to G$ which fixes the identity, so we can take its derivative at the identity to get a map $\mathfrak{g} := T_e(G) \xrightarrow{d(g - g^{-1})_e} T_e(G) = \mathfrak{g}$. This gives a (set theoretic) assignment $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$, which is a map of real/complex manifolds because it was defined exclusively in terms of d, multiplication, and inversion, which are all manifold maps because G is a Lie group. Since we want our condition to be entirely in terms of $\mathfrak{g}$ only, we can take the derivative of Ad at the identity to obtain the *adjoint action* $\mathrm{ad} := d(\mathrm{Ad})_e : \mathfrak{g} \to T_{\mathrm{id}}(\mathrm{Aut}(\mathfrak{g})) = \mathrm{End}_{Vect}(\mathfrak{g})$.

Remark 2.1.1. In this summer mini course we will almost exclusively work with matrix Lie groups $G \subset GL_n$, in which case $\mathfrak{g}$ is naturally a subspace of $\mathfrak{gl}_n \cong \{n \times n \text{ matrices}\}$, and for $X \in \mathfrak{gl}_n$, $\mathrm{ad}(X) : \mathfrak{gl}_n \to \mathrm{End}_{Vect}(\mathfrak{gl}_n)$ is given by $\mathrm{ad}(X)(Y) = XY - YX$, the usual commutator of matrices (see [2]). Note that for a general Lie algebra, XY may not even be defined!

Because of remark 2.1.1, for $X \in \mathfrak{g}$ associated to a general Lie group G, we often denote $\mathrm{ad}(X)$ by $[X, -] : \mathfrak{g} \to \mathfrak{g}$ and refer to it as the *Lie bracket*. This is further justified by the following properties, which can be readily verified for the commutator of matrices, and can be proven in general by carefully combing through the construction above:

Proposition 2.1.1. The Lie bracket is linear in each variable, it is *antisymmetric* (meaning that for any $X, Y \in \mathfrak{g}$, we have $[X, Y] = -[Y, X]$), and it satisfies the *Jacobi identity* (meaning that for any $X, Y, Z \in \mathfrak{g}$, we have $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$).

[3] A small technical point that will come up later–we've still started this whole discussion by fixing Lie groups G and H, because different Lie groups can have the same tangent space, see exercise 1.4.

As discussed earlier, given a map of Lie groups $\varphi : G \to H$, we obtain an induced map of Lie algebras $\mathfrak{g} \to \mathfrak{h}$, and it turns out that this map must take the Lie bracket of $\mathfrak{g}$ to the Lie bracket of $\mathfrak{h}$, i.e.

$$d\varphi_e([X,Y]_{\mathfrak{g}}) = [d\varphi_e(X), d\varphi_e(Y)]_{\mathfrak{h}} \text{ for all } X, Y \in \mathfrak{g}.$$

This also the most information that we can extract about a map of Lie groups from its Lie algebra:

Theorem 2.1.1. Given two Lie groups G, H such that G is simply connected, any map $\mathfrak{g} \to \mathfrak{h}$ which preserves the Lie bracket induces a map $G \to H$.

Remark 2.1.2. The hypothesis that G be simply connected is necessary. Coming back to exercise 1.4, recall that there is an inclusion map $\mathfrak{pgl}_2 = \mathfrak{sl}_2 \hookrightarrow \mathfrak{gl}_2$ that lifts to an inclusion $SL_2 \hookrightarrow GL_2$, yet does not lift to a map $PGL_2 \to GL_2$. A general pattern in the representation theory of Lie groups is to first work with the universal cover $\pi : \tilde{G} \to G$, obtain a representation $\rho : \tilde{G} \to GL_n$, and then show that $\rho(ker(\pi)) = 1$ to obtain an induced representation of G.

This leads to the following definition:

Definition 2.1.1. A *Lie algebra* is a vector space $\mathfrak{g}$ equipped with a bilinear map

$$[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

which is antisymmetric and satisfies the Jacobi identity. A *map of Lie algebras* $\phi : \mathfrak{g} \to \mathfrak{h}$ is a linear map which preserves the Lie bracket, i.e. for all $X, Y \in \mathfrak{g}$, $\phi([X,Y]_{\mathfrak{g}}) = [\phi(X), \phi(Y)]_{\mathfrak{h}}$. A (finite dimensional) *representation of a Lie algebra* $\mathfrak{g}$ is a map of Lie algebras $\mathfrak{g} \to \mathfrak{gl}(V) = \text{End}_{Vect}(V)$ for some finite dimensional vector space V, where the Lie bracket structure on $\text{End}_{Vect}(V)$ is given by the commutator of linear maps (i.e. $[A,B] := AB - BA$).

Remark 2.1.3. In the language of category theory, the fact that maps of Lie groups induce maps of the corresponding Lie algebras together with the chain rule tell us that the assignment $G \mapsto \mathfrak{g}$ extends to a *functor* Lie : LieGrp $\to$ LieAlg from the category of (smooth, resp. complex) Lie groups and maps of Lie groups to the category of Lie algebras (over $\mathbb{R}$, resp. $\mathbb{C}$) and maps of Lie algebras.

2.2 Ideals and Simplicity of Lie Algebras

Fix a Lie algebra $\mathfrak{g}$ associated to a Lie group G. Some of the properties of G readily translate over to the Lie algebra $\mathfrak{g}$. One example is that if G is an abelian Lie group, then its Lie bracket always returns 0. For another example, one can verify that the notion of $N \trianglelefteq G$ being a (closed) normal Lie subgroup of translates to the notion of an *ideal* $\mathfrak{n} \subseteq \mathfrak{g}$ of the Lie algebra:

Definition 2.2.1. A *ideal* of $\mathfrak{g}$ is a linear subspace $I \subset \mathfrak{g}$ such that $[X,Y] \in I$ for all $X \in \mathfrak{g}, Y \in I$ (i.e. $[\mathfrak{g}, I] \subseteq I$).

One useful fact from the representation theory of finite groups is the phenomenon of *complete reducibility*–that is, if V is a subrepresentation of some representation W of a finite group H (over a field of characteristic zero), then one can obtain a decomposition $V \cong W \oplus V/W$ as H-representations. It follows by induction on dimension that every H-representation decomposes (essentially uniquely) into a direct sum of irreducible ones.

Example 2.2.1. Let $\mathfrak{b} \subset \mathfrak{sl}_2$ denote the Lie algebra of traceless upper triangular matrices, with coefficients taken in a field k. We have a two dimensional representation of $\mathfrak{b}$ by its standard action on k^2, which admits a $\mathfrak{b}$-subrepresentation given by the subspace spanned by the first canonical basis vector, but it turns out that there is no other one dimensional subrepresentation, so complete reducibility of $\mathfrak{b}$ reps fails.

It turns out that the existence of abelian ideals is essentially what determines this failure.

Theorem 2.2.1. If a Lie algebra $\mathfrak{g}$ has no nonzero abelian ideals, then all finite dimensional(!) representations of $\mathfrak{g}$ are completely reducible.

This leads to the following definition:

Definition 2.2.2. A *semisimple Lie algebra* is a Lie algebra with no nonzero abelian ideals.

2.3 Exercises

Exercise 2.1. Fix a Lie algebra $(\mathfrak{g}, [-,-]_{\mathfrak{g}})$. Show that for any object $X \in \mathfrak{g}$, $[X, X]_{\mathfrak{g}} = 0$.

Exercise 2.2. Fix a Lie algebra $(\mathfrak{g}, [-,-]_{\mathfrak{g}})$. Show that the map $\mathfrak{g} \to \mathrm{End}_{Vect}(\mathfrak{g})$ given by $X \mapsto [X, -]_{\mathfrak{g}}$ is a representation of Lie algebras (as introduced in definition 2.1.1).

Exercise 2.3. Show that $\mathfrak{b} \subset \mathfrak{sl}_2$ (as in example 2.2.1) admits a nonzero abelian ideal, and thus is not semisimple.

Exercise 2.4. Show that the category of abelian groups has an object which is *not* completely reducible (Hint: Look at $2\mathbb{Z} \subset \mathbb{Z}$). This exercise is not as important as the other two, and is given mostly to make you feel happy about the fact that the category of finite dimensional $\mathfrak{g}$ modules is completely reducible for a semisimple $\mathfrak{g}$.

2.4 Bonus Exercises

Exercise 2.5. Show that the condition of a Lie algebra $\mathfrak{g}$ being semisimple is equivalent to the weaker condition that $\mathfrak{g}$ contains no nonzero *solvable* ideals, i.e. no ideal I for which the series determined by $I_0 := I, I_n := [I_{n-1}, I_{n-1}]$ eventually vanishes.

Exercise 2.6. Show that the category $Rep(SL_2)$ fails to be semisimple over a field k of characteristic p by showing that the representation $\nabla_p := k[x, y]_p$ (the homogeneous polynomials of degree p) defined in exercise 3.9 is a nontrivial extension of $L_p := k[x^p, y^p]_p$ by another subrep ∇_m. What is m? (All you'll need from exercise 3.9 is the formula, but depending on your experience you may want to wait until tomorrow to solve this exercise.)

Exercise 2.7. Show that any algebraic group in characteristic zero is smooth. (Hint: Characteristic zero buys generic smoothness.) Show this is not true in characteristic p, possibly by using the kernel of the Frobenius morphism $\mathbb{G}_a \xrightarrow{Frob} \mathbb{G}_a$.

3 Representation Theory of $\mathfrak{sl}_2$

This will largely follow Lecture 11.1 from [2], but will be written mostly in exercise form. Before we begin, we'll need to cite one theorem (that we won't prove in this course), and discuss how we can use it in classifying the representations of semisimple Lie algebras.

3.1 Diagonalizability

Today, we'll study the smallest semisimple Lie algebra $\mathfrak{sl}_2$ and its representations. Recall that $\mathfrak{sl}_2$ consists of 2×2 traceless matrices. It is a three dimensional Lie algebra which admits the following basis:

$$f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

with Lie bracket determined[4] by $[h, e] = 2e, [h, f] = -2f$, and $[e, f] = h$. To obtain a representation of $\mathfrak{sl}_2$, it therefore suffices to give the data of a vector space V equipped with three endomorphisms $e, f, g : V \to V$ which are compatible with the relations stated above (i.e. $h \circ e - e \circ h = 2e$, and so on).

To start, we'll cite one of the most useful theorems which stems from the 'general theory' of representations of a semisimple Lie algebra. We'll discuss its generalization below, but for now we'll just give the statement:

Theorem 3.1.1. (Diagonalizability) Let V be a finite dimensional $\mathfrak{sl}_2$-representation over $\mathbb{C}$. Then we can write V as a direct sum of h-eigenspaces, i.e.

$$V \cong \oplus_{\lambda \in \mathbb{C}} V_\lambda,$$

where $V_\lambda := \{v \in V : hv = \lambda v\}$.

Remark 3.1.1. The existence of eigenvalues is so important that, for the rest of the course, we will assume that our ground field is $\mathbb{C}$, or at least an algebraically closed field of characteristic zero. There are alternative techniques for working with real representations, and these will hopefully be discussed in Max's mini course!

3.2 Classification of the Irreducible Representations of $\mathfrak{sl}_2$

We'll now classify the reps of $\mathfrak{sl}_2$. To do this, we'll employ a convenient language shift from representations to modules over a certain non-commutative algebra known as the *universal enveloping algebra* of $\mathfrak{sl}_2$:

Exercise 3.1. Convince yourself until you are satisfied that representations of $\mathfrak{sl}_2$ are the same thing as finite dimensional[5] vector spaces which are modules over the *universal enveloping algebra* of $\mathfrak{sl}_2$, defined as follows:

$$U(\mathfrak{sl}_2) := \mathbb{C}\langle e, f, h\rangle/(ef - fe = h, he - eh = 2e, hf - fh = -2f).$$

[4] along with exercise 2.1

[5] Recall from definition 1.1.2 that we defined our representations to be finite dimensional.

(This exercise is optional, but may help you translate from the possibly less familiar concept of Lie algebras and their representations to a possibly more familiar one, (non-commutative) rings and modules over them.)

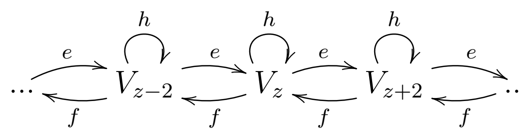
Now fix a finite dimensional irreducible representation V of $\mathfrak{sl}_2$ for the rest of this section. By theorem 3.1.1, we can write V as a direct sum of h-eigenspaces, $V \cong \oplus_{\lambda \in \mathbb{C}} V_\lambda$.

Exercise 3.2. Show that there are only finitely many $\lambda \in \mathbb{C}$ for which $V_\lambda \neq 0$. Choose some λ for which $Re(\lambda)$ is maximal, breaking ties if necessary.[6] Prove that $ev = 0$ (Hint: Try to determine which $h-$eigenspace ev lives in).

For the rest of the section, we fix a λ as in the previous exercise and a nonzero eigenvector $v \in V_\lambda$.

Exercise 3.3. In which eigenspace does the vector fv live? In which eigenspace does the vector efv live?

Exercise 3.4. More generally, show that if $w \in V_z$ for some $z \in \mathbb{C}$, then $fw \in V_{z-2}$ and $ew \in V_{z+2}$. The following diagram (modeled after the one on page 148 of [2]) is a helpful visualization:



The previous two exercises show that the set $S := \{v, fv, f^2v, f^3v, ..., f^{m-1}v\}$, where m is the smallest nonnegative integer such that $f^m v = 0$, is linearly independent.

Exercise 3.5. Why must such an m exist? Show by induction on n that

$$ef^n v = n(\lambda - n + 1) f^{n-1} v,$$

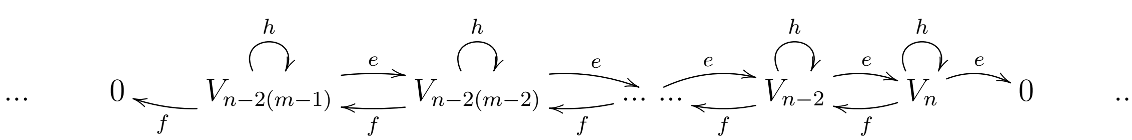
and conclude that S spans V, hence is a basis for V.

Exercise 3.6. Show that $\lambda \in \mathbb{Z}^{\geq 0}$ (Hint: Substitute $n = m$ in your formula for exercise 3.5.) Note that by exercise 3.4, it follows that all eigenvalues of h are integers.

There we have it!

Theorem 3.2.1. All irreducible finite dimensional representations of $\mathfrak{sl}_2$ are classified by their *highest weight*, i.e. the largest positive integer n such that $V_n \neq 0$.

This allows us to complete the above picture illustrating the structure of V as an $\mathfrak{sl}_2$-representation:



[6] At the risk of spoiling the surprise, we will see that λ was in fact a positive integer the whole time, so we never needed to break this tie!

Existence is not always easy to verify directly for general semisimple Lie algebras, but it can be easily done in the case of $\mathfrak{sl}_2$: the vector space $\nabla_n := \mathbb{C}[x,y]_n$ of homogeneous polynomials of degree n viewed as an $(n+1)$-dimensional $\mathfrak{sl}_2$representation under the action induced by matrix multiplication on $\begin{pmatrix} x \\ y \end{pmatrix}$ gives the highest weight $\mathfrak{sl}_2$-representation of highest weight n.

For general semisimple Lie algebras, existence of these highest weight representations can be established pretty easily by studying line bundles on the *flag variety* associated to the Lie group G. This is examined for SL_2 in exercise 3.8. One can also show existence of these irreps as quotients of *Verma modules*, which are introduced in exercise 3.10 and further discussed in exercise 3.11.

Note that by theorem 2.2.1, we have actually just classified *all* finite dimensional representations of $\mathfrak{sl}_2$! We state a corollary of our results below which uses the complete reducibility of finite dimensional representations of $\mathfrak{sl}_2$:

Exercise 3.7. Let U be a finite dimensional $\mathfrak{sl}_2$-representation, such that $U_n \neq \emptyset$ for some nonnegative integer n. Show that $U_{n-2} \neq \emptyset$, $U_{n-4} \neq \emptyset, ..., U_{-n} \neq \emptyset$. In particular, this says that the eigenvalues of any finite dimensional $\mathfrak{sl}_2$ representation are symmetric about zero (Hint: look at the subrepresentation of U generated by the images of some nonzero $v \in U_n$ under the $\mathfrak{sl}_2$ action and use exercise 3.5 and exercise 3.4).

3.3 Bonus Exercises

The next few exercises will give a geometric interpretation of the above irreps as line bundles on $\mathbb{P}^1$. This will be phrased in the language of algebraic geometry, as that is the language with which the author is most familiar. The real/complex analytic viewpoint on this story can be found in 11.3 of [2].

Exercise 3.8. Let B be the set of upper triangular matrices in the group SL_2. Show that we can identify $SL_2(\mathbb{C})/B(\mathbb{C}) \cong \mathbb{P}^1(\mathbb{C})$. We call SL_2/B the *flag variety* of SL_2. We will explore these ideas in more detail in exercise 4.9.

Exercise 3.9. Show (or remember) that the global sections of the line bundle $\mathcal{O}(n)$ on $\mathbb{P}^1$ are given by $\nabla_n = k[x,y]_n$ (the homogeneous polynomials of degree $n \in \mathbb{Z}$). Define a *right* SL_2 action by exercise 3.8 above, via translation. Specifically, given $f\begin{pmatrix} x \\ y \end{pmatrix} \in \nabla_n$, we define the action via the formula

$$(fg)\begin{pmatrix} x \\ y \end{pmatrix} := f(g\begin{pmatrix} x \\ y \end{pmatrix}).$$

Show that this is a right (!) action and that there is an object $p\begin{pmatrix} x \\ y \end{pmatrix} \in \nabla_n$ on which any element of $N := \{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\}$ acts by the identity and any element of $B/N \cong \{\begin{pmatrix} \alpha & * \\ 0 & \alpha^{-1} \end{pmatrix}\}$ acts on p by scaling by[7] α^{-n}. We'll show how to interpret this construction in a way that

[7] This pesky inverse can be taken care of with a dual, which we won't define here.

generalizes more readily in exercise 3.12 (which doesn't require any exercises other than this one).

Exercise 3.10. Fix some $\lambda \in \mathbb{C}$. Define the *Verma module* associated to λ to be the tensor product representation:

$$M_\lambda := U(\mathfrak{sl}_2) \otimes_{U(\mathfrak{b})} \mathbb{C},$$

where $U(\mathfrak{b})$ is the subring of $U(\mathfrak{sl}_2)$ generated by h and e, and $\mathbb{C}$ is given a $U(\mathfrak{b})$ module structure by letting h act by scalar multiplication by λ and e act by 0. Show that M_λ is a simple $U(\mathfrak{sl}_2)$-module if and only if λ is *not* a nonnegative integer.

Exercise 3.11. In the notation above, show that for $n \in \mathbb{Z}_{\geq 0}$, M_n surjects onto the finite dimensional irreducible representation ∇_n of highest weight n. In the language used in this lecture, what is the kernel of this map? (Once you show that this kernel actually is a $\mathfrak{sl}_2$-submodule, this also proves existence of these finite dimensional representations, where the analysis above shows that the quotient actually is finite dimensional!)

Exercise 3.12. Keeping the notation of exercise 3.8, fix an integer n, and show that the above construction has a more methodical interpretation as follows:

Define a map $\lambda : B \to \mathbb{G}_m$ via[8] $\begin{pmatrix} \alpha & * \\ 0 & \alpha^{-1} \end{pmatrix} \mapsto \alpha^n$, where we set $T := B/[B,B] \cong \{\begin{pmatrix} \alpha & * \\ 0 & \alpha^{-1} \end{pmatrix} : \alpha \in \mathbb{C}^\times\}$. Show that because SL_2 acts on $SL_2/B \cong \mathbb{P}^1$, SL_2 acts on the global sections of the line bundle $SL_2 \times_B \mathbb{A}^1$ and that these global sections are precisely the representation of highest weight $-n$ (note the minus sign, which can again be taken care of by dualizing).

4 Representation Theory of $\mathfrak{sl}_3$

Today, we'll chat about the representation theory of $\mathfrak{sl}_3$. The really cool part about this is that the representation theory of $\mathfrak{sl}_3$ accounts for a large percentage of the complexity that goes into the study of representations of a general semisimple Lie algebra $\mathfrak{g}$ (and in fact suggests how to proceed with the general theory.) Recall that $\mathfrak{sl}_3$ can be identified with traceless 3×3 matrices.

4.1 The Generalization of Eigenvalues

The main jump from studying the representations of $\mathfrak{sl}_2$ to studying representations of a general semisimple Lie algebra $\mathfrak{g}$ is generalizing the role of h. Because of the fact that $\mathfrak{sl}_2$ sits naturally inside the upper left 2×2 block of $\mathfrak{sl}_3$, one might hope that $h_{1,2} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ plays the same role as h did in the previous section. It turns out that it does–one can show that any finite dimensional $\mathfrak{sl}_3$ rep splits as a direct sum of its $h_{1,2}$-eigenspaces.

[8]Here, $\mathbb{G}_m := Spec(\mathbb{C}[x, x^{-1}])$ can be replaced with the complex analytic version $\mathbb{C}^\times$.

But why stop there? We can also embed $\mathfrak{sl}_2$ inside $\mathfrak{sl}_3$ in the bottom right 2×2 block, or we can identify it as $\{\begin{pmatrix} a & 0 & c \\ 0 & 0 & 0 \\ b & 0 & d \end{pmatrix} : a + d = 0\}$. These embeddings lead to corresponding elements $h_{2,3}, h_{1,3} \in \mathfrak{sl}_3$ respectively. But even further still–why choose $h_{1,3}$ over $h_{3,1}$?

Note also that any two diagonal matrices h and h' commute, so that $[h, h'] = 0$. The solution to our problem of choosing the analogue of h will come from the linear algebra fact that pairwise commuting diagonalizable matrices can be simultaneously diagonalized (see exercise 4.7). Specifically, let $\mathfrak{h} \subset \mathfrak{sl}_3$ be the subspace of diagonal matrices in $\mathfrak{sl}_3$. The notion of eigenvalues will be replaced in our settings by the generalized notion of *weights*.

Definition 4.1.1. For a given Lie algebra $\mathfrak{g}$ with a choice of maximal abelian Lie subalgebra[9] $\mathfrak{h}$, the *weight space* of $\mathfrak{g}$ is defined to be $\mathfrak{h}^* := \mathrm{Maps}_{Vect}(\mathfrak{h}, \mathbb{C})$.

In this particular case, $\mathfrak{h}^*$ is spanned by the functionals L_i, $i = 1, 2, 3$, where $L_i(M) := M_{ii}$ picks out the ii^{th} entry, subject to the relation $L_1 + L_2 + L_3 = 0$:

$$\mathfrak{h}^* \cong \mathbb{C}\langle L_1, L_2, L_3\rangle/(L_1 + L_2 + L_3 = 0).$$

The notion of weights leads to a description of the Lie algebra structure on $\mathfrak{sl}_3$ that generalizes well to an arbitrary semisimple Lie algebra $\mathfrak{g}$.

Exercise 4.1. Convince yourself that $\mathfrak{sl}_3$ is an eight dimensional vector space spanned by three embedded $\mathfrak{sl}_2$'s as described above. Specifically, $\mathfrak{sl}_3$ admits a basis given by $e_{1,2}, e_{1,3}$, $e_{2,3}, f_{2,1}, f_{3,1}, f_{3,2}$, together with a choice of two linearly independent elements in $\mathfrak{h}$, and the corresponding relations are given by $[h, e_{i,j}] = (L_i - L_j)(h)e_{i,j}$ and $[h, f_{j,i}] = (L_j - L_i)(h)f_{j,i}$ for all $h \in \mathfrak{h}$ and $1 \leq i < j \leq 3$.

We now formulate the analogue of theorem 3.1.1 in the case of $\mathfrak{sl}_3$–in fact, we state it for a general semisimple Lie algebra:

Theorem 4.1.1. (Diagonalizability) Let V be a finite dimensional representation of a semisimple Lie algebra $\mathfrak{g}$ with a choice of a maximal abelian subalgebra $\mathfrak{h}$. Then we can write

$$V \cong \oplus_{\lambda \in \mathfrak{h}^*} V_\lambda,$$

where the sum is taken over finitely many $\lambda \in \mathfrak{h}^*$ and $V_\lambda := \{v \in V : hv = \lambda(h)v \ \forall h \in \mathfrak{h}\}$.

4.2 Highest Weights for $\mathfrak{sl}_3$

Remark 4.2.1. For the following section, the reader may wish to have a few copies of (finely graded) A_2 graph paper at hand, available at Matthew Fayers's web page **here**.

We will classify representations of $\mathfrak{sl}_3$ through exercises, similarly to our classification of the irreps of $\mathfrak{sl}_2$. We'll start with the canonical nontrivial example of a representation, the adjoint representation.

[9] It turns out that diagonal matrices give such a choice for $\mathfrak{sl}_n$.

Exercise 4.2. Show that the adjoint representation of $\mathfrak{sl}_3$ has nonzero eigenvalues[10] given by $L_i - L_j$ for distinct $i, j \in \{1, 2, 3\}$. We call these six eigenvalues the *roots* of $\mathfrak{sl}_3$, and sometimes denote the set of roots by R. On your A_2 graph paper, draw $\mathfrak{h}^*$ as follows: choose an origin, declare that the rightward arrow is L_1, the up and left line L_2, and the down and left line L_3, then plot the six eigenspaces.

Exercise 4.3. Let V be an $\mathfrak{sl}_3$-representation and let $v \in V_\lambda$ for some $\lambda \in \mathfrak{h}^*$. Recall the construction of $e_{i,j}$ and $f_{i,j}$ from exercise 4.1. In what eigenspace does $e_{1,2}v$ live? Generalize this to the other $e_{i,j}$'s and to the $f_{j,i}$'s.

Back in $\mathfrak{sl}_2$, our choice of $h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ dictated that e had a positive h-eigenvalue, and our notion of highest weight vector was precisely that of a nonzero vector $v \in V_\lambda$ for some $\lambda \in \mathbb{C}$ such that $ev = 0$. With that in mind, let V be an $\mathfrak{sl}_3$ rep. We will declare[11] that a *highest weight vector* for $\mathfrak{sl}_3$ is a nonzero vector $v \in V_\lambda$ for some $\lambda \in \mathfrak{h}^*$ such that $e_{i,j}v = 0$ for all $i, j \in \{1, 2, 3\}$ with $i < j$. Accordingly, we introduce a partition on the set of roots $R = R^+ \cup R^-$ of $\mathfrak{sl}_3$ into *positive roots* $R^+ := \{L_i - L_j : i < j\}$ and *negative roots* $R^- := \{L_i - L_j : i > j\}$. In this terminology, a highest weight vector is an eigenvector (again, labeled by $\mathfrak{h}^*$) that is killed by the action of $e_{1,2}, e_{2,3}$, and $e_{1,3}$ (or, equivalently, it turns out, all the eigenvectors associated to the positive roots).

Exercise 4.4. Show that such a highest weight vector exists. (Hint: Fix a nonzero $v \in V_\lambda$ for some $\lambda \in \mathfrak{h}^*$. Let $t \in \mathbb{Z}^{\geq 0}$ be the maximal element for which $(e_{1,3})^t v \neq 0$. Next, let $s \in \mathbb{Z}^{\geq 0}$ be the maximal element for which $w := (e_{2,3})^s (e_{1,3})^t v \neq 0$. Show that $e_{1,3}w = 0$ and $e_{2,3}w = 0$. Finally, let $r \in \mathbb{Z}^{\geq 0}$ be maximal so that $u := (e_{1,2})^r w \neq 0$, and check that u is a highest weight vector for V.)

Remark 4.2.2. As an alternative to exercise 4.4, one can draw a line L with irrational slope in $\mathfrak{h}^*$ which separates the 'positive' weights $L_i - L_j$ for $i < j$ from the 'negative' weights $L_j - L_i$, thereby inducing a well-defined ordering on the lattice spanned by these six elements. The ordering is given by looking at the real part of the value of the weights [12] of our representation under application of the functional l defining L, and one can show that multiplication by the $e_{i,j}$'s increases weight (see [2] for more details on this). In this approach, positive (resp. negative) roots are declared to be those $\alpha \in R$ such that $\mathrm{Re}(l(\alpha)) > 0$ (resp. $\mathrm{Re}(l(\alpha)) < 0$).

By exercise 4.4, we can fix a highest weight vector $v \in V_\lambda$ for some $\lambda \in \mathfrak{h}^*$. When plotting this on your graph paper, it may help to assume $\lambda = 2L_1 - L_3$.

Now, we'll use one of the most important techniques in the representation theory of semisimple Lie algebras – namely, reduction of the analysis to the various copies of $\mathfrak{sl}_2$

[10] Although these might more descriptively be called the $\mathfrak{sl}_3$ eigenvalues with respect to $\mathfrak{h}$, the choice of maximal abelian subalgebra $\mathfrak{h}$ is often left implicit.

[11] Note that here we make a **choice**! Just as we could have given the whole $\mathfrak{sl}_2$ story with h instead being $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and e replaced by $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ in the highest weight discussion, there are different choices for the notion of 'highest weight' in general. The amount of freedom we have in making these choices is captured by the *Weyl group*, which we will discuss later.

[12] And, similarly to $\mathfrak{sl}_2$, we will see that these weights are always integral linear combinations of the L_i!

embedded in $\mathfrak{sl}_3$. Specifically, let $\alpha := L_1 - L_2$ (one of the positive roots of $\mathfrak{sl}_3$), and consider the subalgebra $\mathfrak{sl}_{2,\alpha} \subset \mathfrak{sl}_3$ obtained by embedding $\mathfrak{sl}_2$ into the upper left 2×2 block. Then, we observe that *the $\mathfrak{sl}_{2,\alpha}$-span of v is an $\mathfrak{sl}_2$-representation with highest weight v.* By using exercise 3.7, we thus obtain that $\lambda - 2\alpha, \lambda - 4\alpha, ..., w_\alpha(\lambda)$ are all weights of V, where w_α denotes the operation of reflection across the hyperplane cut out by the condition $h_{1,2} = 0$ (i.e. the hyperplane defined by the equation $h_{1,2}X = 0$ for $X \in \mathfrak{h}^*$, geometrically perpendicular to $\alpha = L_1 - L_2$) – draw this!

Exercise 4.5. Repeat the above $\mathfrak{sl}_{2,\alpha}$ discussion for $\alpha = L_2 - L_3$ and $\alpha = L_1 - L_3$, the two other positive roots of $\mathfrak{sl}_3$. Show that if $\mu \in \mathfrak{h}^*$ is any eigenvalue of the representation V, then for any positive root $\alpha \in R^+$, $w_\alpha(\mu)$ is also an eigenvalue, where w_α is reflection across the hyperplane defined by $h_\alpha = 0$ in $\mathfrak{h}^*$ for the corresponding $h_\alpha \in \mathfrak{sl}_{2,\alpha} \cap \mathfrak{h}$.

Using the above exercise with the choice of highest weight $\lambda = 2L_1 - L_3$, you should be able to draw a series of eigenvalues which trace out a sort of 'hexagon[13]' in $\mathfrak{h}^*$.

We thus see that studying the embedded copies of $\mathfrak{sl}_2$ can tell us where the eigenvalues of our representation live in $\mathfrak{h}^*$. It can also give us restrictions on what the highest weight λ can be!

Exercise 4.6. Show that if $\lambda = aL_1 + bL_2 + cL_3$ for some $a, b, c \in \mathbb{C}$ is a highest weight for the representation V, then $a, b, c \in \mathbb{Z}_{\geq 0}$, and furthermore $a \geq b \geq c$. (Hint: Show that Λ_W, the set of all elements of $\mathfrak{h}^*$ which take integer values on the $h_{i,j}$, is a lattice and is generated as a lattice by the L_i's.)

This is enough to state the classification theorem, where we write in parenthesis terms which will be generalized to the case of an arbitrary semisimple Lie algebra later:

Theorem 4.2.1. Every finite dimensional irreducible representation V of $\mathfrak{sl}_3$ has a highest weight $\lambda \in \mathfrak{h}^*$ which is in the '$\frac{1}{6}^{th}$' plane cut out by the conditions $h_{1,2} \geq 0$ and $h_{2,3} \geq 0$ (the *dominant Weyl chamber* associated to the *simple weights* $L_1 - L_2$ and $L_2 - L_3$). This λ determines the representation. Furthermore, for all λ satisfying the above conditions, there is an irreducible representation of highest weight λ. If $v \in V$ is a highest weight vector, then any vector in V can be obtained as the $U(\mathfrak{n}^-)$ span of V, where $\mathfrak{n}^-$ is the Lie algebra spanned by $f_{2,1}, f_{3,2},, f_{3,1}$ (the *negative root vectors*), and $U(\mathfrak{n}^-)$ is the associated universal enveloping algebra.

The last statement is exercise 4.8 below. Existence, as usual, is not as fun to do by hand, but it can be done by realizing this representation as a line bundle on the space G/B (as was the case for $\mathfrak{sl}_2$) – see exercise 4.10 for more.

4.3 Bonus Exercises

Exercise 4.7. Show that 'commuting diagonalizable matrices can be simultaneously diagonalized', i.e., if A and B are two commuting linear maps of a finite dimensional vector space V to itself which are diagonalizable, then there is a basis of V consisting of simultaneous eigenvectors for both A and B (Hint: Show that A preserves the B-eigenspaces.)

[13]Although if you had picked a different highest weight λ, you may have obtained a triangle!

Exercise 4.8. Prove that the multiplication map $U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}) \to U(\mathfrak{sl}_3)$ is a surjection of vector spaces, and show this implies the final claim of theorem 4.2.1, where $\mathfrak{n}$ is the Lie algebra generated by the positive root vectors $e_{i,j}$ for $i > j$. (The *Poincaré Birkhoff Witt (PBW) theorem* for $\mathfrak{sl}_3$ says that this map is in fact an isomorphism of vector spaces).

Exercise 4.9. For a Lie (resp. algebraic) group G, the *flag manifold (variety)* is the quotient manifold (variety) G/B, where B is a Borel subgroup of G. Show that G/B can be identified with the manifold (variety) $\{(u,v) \in \mathbb{P}^2_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}} : u \perp v\}$ (Hint: SL_3 acts on the set of *complete flags* of $\mathbb{C}^3$, i.e. sequences of i-dimensional subspaces $V_i \subset \mathbb{C}^3$, $i = 1,2,3$, such that $0 \subset V_1 \subset V_2 \subset V_3 = \mathbb{C}^3$. What is the stabilizer of the flag $0 \subset \mathbb{C} \subset \mathbb{C}^2 \subset \mathbb{C}^3$ under the action of SL_3?)

Exercise 4.10. Imitate the constructions in exercise 3.12 to give an algorithm to construct line bundles on G/B giving rise to the irreducible representations of $\mathfrak{sl}_3$ stated in theorem 4.2.1.[14] The *Borel-Weil theorem* states that all finite dimensional representations of a semisimple Lie algebra $\mathfrak{g}$ can be realized as line bundles on the flag variety of an associated semisimple simply connected Lie group G.

5 The Geometry of Semisimple Lie Algebras

Fix an arbitrary semisimple Lie algebra $\mathfrak{g}$ and a maximal diagonalizable subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Last time, we saw that when $\mathfrak{g} = \mathfrak{sl}_3$ and $\mathfrak{h}$ is the subspace of traceless diagonal matrices, we can determine all finite dimensional representations of $\mathfrak{g}$ by studying embedded $\mathfrak{sl}_{2,\alpha}$'s for each positive root $\alpha \in \{L_1 - L_2, L_1 - L_3, L_2 - L_3\}$ and generalizing the notion of highest weight. We also saw that the other choices of highest weight could be obtained from a given one by reflecting across the hyperplanes in $\mathfrak{h}^*$ cut out by the condition that evaluation at some $H_\alpha \in \mathfrak{h}_\alpha \subset \mathfrak{sl}_{2,\alpha}$ must be zero.

Today, we'll talk about how these facts generalize to a general $\mathfrak{g}$. First, we will need to generalize the embeddings of $\mathfrak{sl}_2 \subset \mathfrak{g}$. The following theorem is one of the main theorems of the basic theory of semisimple Lie algebras, which we will take as a black box for the remainder of the course (recall that a *root* of $\mathfrak{g}$ is a nonzero (generalized) eigenvalue of $\mathfrak{h}$ under the adjoint representation of $\mathfrak{g}$ on itself):

Theorem 5.0.1. Let α be a root of $\mathfrak{g}$. Then

- $-\alpha$ is also a root of $\mathfrak{g}$,
- The root space $\mathfrak{g}_\alpha := \{x \in \mathfrak{g} : [h,x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$ is one dimensional, and
- There exists an embedding of Lie algebras $\mathfrak{sl}_{2,\alpha} \hookrightarrow \mathfrak{g}$, where the $\mathfrak{sl}_2$ element $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ maps to an element of $\mathfrak{h}$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ to an element of $\mathfrak{g}_\alpha$, and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ to an element of $\mathfrak{g}_{-\alpha}$.

Furthermore, the roots of $\mathfrak{g}$ span $\mathfrak{h}^*$.

[14]No need to prove these actually do have the global sections–A fun proof that these line bundles actually have these global sections is given in [4]!

5.1 The Killing Form

We saw that much of the representation theory of $\mathfrak{sl}_3$ came from the geometry of the maximal diagonalizable subalgebra $\mathfrak{h}$. We'll soon see that this geometry came implicitly from the Lie algebra structure on $\mathfrak{sl}_3$.

Definition 5.1.1. Fix any (not necessarily semisimple!) finite dimensional Lie algebra $\mathfrak{f}$. The *Killing form* on $\mathfrak{f}$ is defined to be the pairing $\kappa : \mathfrak{f} \times \mathfrak{f} \to \mathbb{C}$ given by

$$\kappa(x, y) := \mathrm{Tr}(\mathrm{ad}(x) \circ \mathrm{ad}(y)),$$

where we view $\mathrm{ad}(x) \circ ad(y) : \mathfrak{f} \to \mathfrak{f}$ as a linear endomorphism.

The cyclicity of trace ($\mathrm{Tr}(XY) = \mathrm{Tr}(YX)$) implies that κ is symmetric. A few more linear algebra tricks (see [3]) involving properties of the trace show that the Killing form is *ad-invariant*, i.e.:

Proposition 5.1.1. Fix a finite dimensional Lie algebra $\mathfrak{f}$. Then for all $x, y, z \in \mathfrak{f}$,

$$\kappa([x, y], z) = \kappa(x, [y, z]).$$

Example 5.1.1. If $\mathfrak{g} = \mathfrak{sl}_2$, then $\kappa(h, e) = \kappa(h, f) = 0$, since $\mathrm{ad}(h) \circ \mathrm{ad}(e)$ and $\mathrm{ad}(h) \circ \mathrm{ad}(f)$ permute the three eigenspaces $\mathfrak{sl}_2 = \mathbb{C}f \oplus \mathbb{C}h \oplus \mathbb{C}e = (\mathfrak{sl}_2)_{-2} \oplus (\mathfrak{sl}_2)_0 \oplus (\mathfrak{sl}_2)_2$. Similarly, $\kappa(e, e) = \kappa(f, f) = 0$. Direct computation shows that $\kappa(e, f) = 4$ and $\kappa(h, h) = (-2)^2 + 0^2 + 2^2 = 8$.

The above method for computing $\kappa(h, h)$ can be generalized to any semisimple Lie algebra. Specifically, write $\mathfrak{g}$ as the sum of its root spaces, i.e. $\mathfrak{g} = \oplus_{\alpha \in R} \mathfrak{g}_\alpha$ where R denotes the set of roots of $\mathfrak{g}$. Then if $h \in \mathfrak{h} \subset \mathfrak{g}$, then $\kappa(h, h) = \sum_{\beta \in R} (\beta(h))^2$. In particular, since $\alpha(h_\alpha) \neq 0$, we see that each $\kappa(h_\alpha, h_\alpha) \neq 0$:

Proposition 5.1.2. The Killing form restricted to the real span of the roots inside $\mathfrak{h} \times \mathfrak{h}$ is positive definite.

When $\mathfrak{g} = \mathfrak{sl}_2$, the example also shows that the Killing form on $\mathfrak{g}$ is *non-degenerate* (i.e. if $\kappa(x, y) = 0$ for all $y \in \mathfrak{g}$ then $x = 0$), hence that it induces an isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$. This is, in fact, is an equivalent characterization of semisimple Lie algebras:

Proposition 5.1.3. A Lie algebra $\mathfrak{f}$ is semisimple if and only if the Killing form on $\mathfrak{f}$ is non-degenerate.

5.2 The Weyl Group

Consider again the adjoint representation of $\mathfrak{sl}_2$. When working with the representation theory of $\mathfrak{sl}_3$, we saw that it was better to make our eigenvalues functionals on *all* diagonal matrices, instead of just singling out one (or a few) diagonal matrices. In particular, this implies that for the representations of $\mathfrak{sl}_2$, our entire discussion should work with $h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ replaced by $-h$. In carrying out that discussion, we would find that $f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, the vector we originally took to be in the 'negative' direction, would now be pointing in the positive direction.

In the $\mathfrak{sl}_3$ case, this discussion amounted to the fact that reflections across the hyperplanes cut out by the various h_α's for all three positive roots of the adjoint representation (i.e. $\alpha = L_i - L_j$ for $i < j$) preserved the set of eigenvalues of a representation. The fact that the roots always span $\mathfrak{h}^*$ (see theorem 5.0.1) leads us to a general definition:

Definition 5.2.1. The *Weyl group* of $\mathfrak{g}$ is defined to be the group generated by the reflections in the various hyperplanes $\{\beta : \beta(h_\alpha) = 0\}$, where β varies over the real span of the roots inside $\mathfrak{h}^*$ and α varies over the roots of $\mathfrak{g}$.

Remark 5.2.1. We could also take this definition for general $\beta \in \mathfrak{h}^*$, but the emphasis here is on the *real* geometry inside of $\mathfrak{h}^*$. We'll discuss this more when we discuss the notion of *root systems* in section 6.2.

Remark 5.2.2. Geometrically, the reflection corresponding to a given $\alpha \in R$ is taken across the hyperplane perpendicular to α in $\mathfrak{h}$, hence in particular it sends α to $-\alpha$.

Example 5.2.1. The Weyl group of $\mathfrak{sl}_3$ is the symmetric group on three letters S_3, where $\sigma \in S_3$ acts via $\sigma(L_i) := L_{\sigma(i)}$.

Example 5.2.2. The Weyl group of $\mathfrak{sl}_2$ is $S_2 \cong \mathbb{Z}/2\mathbb{Z}$.

By definition, the Weyl group is a *Coxeter group*[15]. One of the reasons for which the Weyl group is important is the analogue of the fact that the weights of any $\mathfrak{sl}_3$ representation are preserved under the action of the Weyl group (i.e. the Weyl group induces *symmetries* on the set of weights). This generalizes to any $\mathfrak{g}$:

Theorem 5.2.1. If V is any finite dimensional representation of $\mathfrak{g}$, then the weights of V are preserved under the action of the Weyl group W.

5.3 Exercises

These exercises are meant to give a taste of how the general theory of semisimple Lie algebras is built. Readers who have not finished the exercises for $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$ may wish to work on those!

Exercise 5.1. Fix a semisimple Lie algebra $\mathfrak{g}$. Show that if $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_\beta$ are such that $\kappa(x, y) \neq 0$, then $\alpha = -\beta$.

Exercise 5.2. Prove the 'if' direction of proposition 5.1.3. (Hint: Assume the Killing form is nondegenerate on $\mathfrak{g}$, and let $\mathfrak{a}$ be some abelian ideal of $\mathfrak{g}$. Then for any $a \in \mathfrak{a}, x \in \mathfrak{g}$, prove that $ad(x) \circ ad(a)$ maps entirely into $\mathfrak{a}$ and kills $\mathfrak{a}$, so this map has trace zero.)

[15] In fact, W is a finite Coxeter group, but not all finite Coxeter groups arise in this way, see section 6.2.

5.4 Bonus Exercises

These next two exercises introduce the *Casimir operator* of a semisimple Lie algebra $\mathfrak{g}$, a nontrivial element in the center of $U\mathfrak{g}$ which, among other things, is used to prove that all finite dimensional representations of $\mathfrak{g}$ are completely reducible.

Exercise 5.3. Let $\mathfrak{g}$ be a semisimple Lie algebra. Prove that we have identifications of $\mathfrak{g}$-modules $\mathfrak{g} \otimes \mathfrak{g} \cong \mathfrak{g}^* \otimes \mathfrak{g} \cong \underline{\mathrm{Hom}}_{Vect}(\mathfrak{g}, \mathfrak{g})$, where the tensor product of two Lie algebras is given a $\mathfrak{g}$-module structure by the 'product rule' and the vector space of maps between two Lie algebras is given a $\mathfrak{g}$-module structure via $(xf)(v) = xf(v) - f(xv)$.

Exercise 5.4. Show that there is an element $C \notin \mathbb{C} \subset U\mathfrak{g}$ with $C \in Z(\mathfrak{g}) := Z(U\mathfrak{g})$. (Hint: For a $\mathfrak{g}$ representation V, define the *invariants* $V^{\mathfrak{g}} := \{v \in V : xv = 0 \text{ for all } x \in \mathfrak{g}\}$. Use the chain of isomorphisms from exercise 5.3, and the map of $\mathfrak{g}$-modules $\mathfrak{g} \otimes \mathfrak{g} \to U\mathfrak{g}$ to show that $C \in (U\mathfrak{g})^{\mathfrak{g}}$.) (For this exercise, it may help to use the 'adjoint' definition of the universal enveloping algebra, namely that $U : Lie \to \mathbb{C}$-Alg is the left adjoint to the forgetful functor $\mathbb{C}$-Alg $\to Lie$.)

Exercise 5.5. Note that the Weyl group action $W = S_3$ preserves the root lattice Λ_R of $\mathfrak{sl}_3$, namely the lattice spanned by the roots (and this phenomenon holds for all semisimple Lie algebras). In particular, we can define the *affine Weyl group* $W^{\mathrm{aff}} := W \rtimes \Lambda_R$.

It is a theorem that for an algebraically closed field of characteristic p, the category $\mathrm{Rep}(G)$ is not semisimple (i.e. there are nontrivial extensions of simple objects by other simple objects), but that certain "blocks" of the category which do not have no nontrivial maps or extensions between them. Specifically, define the *p-dilated $\cdot$ action* ("p-dilated dot action") on Λ via the map $(nr, w) \cdot_p \lambda := w(\lambda + pr + \rho) - \rho$, where $n \in \mathbb{Z}, r$ a root, $w \in W$, and ρ denotes half the sum of the positive roots. There is a theorem that for a semisimple algebraic group G, the category of representations breaks up as a direct sum of categories labeled by the W^{aff} orbits on Λ. In symbols,

$$Rep(G) = \bigoplus_{\overline{\lambda} \in \Lambda / W^{aff}} Rep_{\lambda}(G).$$

Compute the W^{aff} orbits of this action for SL_2 and SL_3. Compare your answer to that of exercise 2.6.

Exercise 5.6. Fix a semisimple Lie algebra $\mathfrak{g}$ with a choice of $\mathfrak{b}$ and $\mathfrak{t}$. Define the *BGG Category $\mathcal{O}$* to be the full subcategory of finitely generated $U\mathfrak{g} - \mathrm{Mod}$ on which $\mathfrak{t}$ acts semisimply (i.e. "diagonalizably") and $\mathfrak{n}$ acts locally nilpotently (i.e. the $U\mathfrak{n}$ span of any element in any module of $\mathcal{O}$ is finite dimensional).

Are all objects of $\mathcal{O}$ generated by highest weight vectors? That is, given $M \in \mathcal{O}$, can we find a direct sum of Verma modules which surjects onto M? (This exercise will force you to contemplate an important but annoying point about category $\mathcal{O}$ and may not be the best exercise if this is the first time you are seeing this notion.)

6 Toward the General Theory

6.1 The Representation Theory of Semisimple Lie Algebras

We have almost all of the terminology we will need to classify all representations of an arbitrary semisimple Lie algebra. We won't prove many of the statements made here, but it may help to verify each statement in the special case of $\mathfrak{sl}_3$, for which we've already developed much of the theory (for a translation guide, see appendix A). We'll need to make a few more definitions in the general case, starting with the generalization of the notions of weights of finite dimensional representations of $\mathfrak{sl}_3$ (recall that the set R of roots of $\mathfrak{g}$ denotes the set of generalized $\mathfrak{h}$-eigenvalues of the adjoint representation):

Definition 6.1.1. The *weight lattice* Λ of a semisimple Lie algebra $\mathfrak{g}$ is defined to be the lattice of *integral weights*:

$$\Lambda := \{\lambda \in \mathfrak{h}^* : \lambda(h_\alpha) \in \mathbb{Z} \text{ for all } \alpha \in R\}.$$

Next, we generalize the notion of positive roots. Recall what we did for $\mathfrak{sl}_3$: we made the choice of taking the *upper triangular* (traceless) matrices as giving us the positive directions for each embedded $\mathfrak{sl}_2$. The correct analogue in the general case of an arbitrary semisimple Lie algebra $\mathfrak{g}$ is the notion of a maximal *solvable* Lie algebra.

Definition 6.1.2. The *derived central series* of a Lie algebra $\mathfrak{b}$ is the sequence defined recursively via $D_0 = \mathfrak{b}$, $D_n := [D_{n-1}, D_{n-1}]$. If the derived central series for $\mathfrak{b}$ terminates at zero, $\mathfrak{b}$ is said to be *solvable.* A maximal solvable subalgebra of $\mathfrak{g}$ is called a *Borel subalgebra* of $\mathfrak{g}$, and usually denoted by $\mathfrak{b}$.

One can check (see exercise 6.2) that if $\mathfrak{g}_\alpha \subset \mathfrak{b}$ is in a Borel subalgebra of $\mathfrak{g}$, then any nonzero vector in $\mathfrak{g}_{-\alpha}$ is not in the Borel subalgebra. It is therefore plausible that a choice of Borel subalgebra induces a notion of positivitiy on the roots. From here on out, we choose a Borel subalgebra $\mathfrak{b}$ which contains $\mathfrak{h}$ (this can always be done), and we make the following definition:

Definition 6.1.3. A root α is said to be *positive* if $\mathfrak{g}_\alpha \subset \mathfrak{b}$. The set of positive roots for $\mathfrak{g}$ is denoted by R^+. A positive root is said to be *simple* if it cannot be written as the sum of two positive roots.

We can then define the notion of a highest weight for an arbitrary finite-dimensional representation of $\mathfrak{g}$ as follows (recall from theorem 5.0.1 that we can associate to each root $\alpha \in R$ an embedding $\mathfrak{sl}_2 \hookrightarrow \mathfrak{g}$ with images $e_\alpha \in \mathfrak{g}_\alpha, f_\alpha \in \mathfrak{g}_{-\alpha}$, and $h_\alpha \in \mathfrak{h}$):

Definition 6.1.4. Let V be a finite-dimensional representation of $\mathfrak{g}$. Say that $\lambda \in \mathfrak{h}^*$ is a *highest weight* for V with *highest weight vector* $v \in V_\lambda \setminus 0$ if $e_\alpha v = 0$ for each positive root $\alpha \in R^+$.

Finally, we'll need a notion that generalizes the part of the plane for $\mathfrak{sl}_3$ in which we found our highest weights:

Definition 6.1.5. Choose embeddings of $\mathfrak{sl}_2$ into $\mathfrak{g}$ for each positive root $\alpha \in R^+$, mapping $e \in \mathfrak{sl}_2$ into $\mathfrak{g}_\alpha$ and h to $h_\alpha \in \mathfrak{h}$. The *dominant Weyl chamber* is defined to be the set $\{\lambda \in \mathfrak{h}^* : \lambda(h_\alpha) \geq 0 \text{ for all } \alpha \in R^+\}$.

Remark 6.1.1. Geometrically, the dominant Weyl chamber corresponds to one of the connected components of the complement of the union of the hyperplanes involved in the definition of the Weyl group of $\mathfrak{g}$.

We are finally in a position to state the classification theorem of irreducible representations of a general finite dimensional semisimple Lie algebra:

Theorem 6.1.1. Fix a finite dimensional semisimple Lie algebra $\mathfrak{g}$ with a choice of maximal diagonalizable subalgebra $\mathfrak{h}$ and maximal solvable subalgebra $\mathfrak{b}$ containing $\mathfrak{h}$. Then all finite dimensional irreducible representations of $\mathfrak{g}$ correspond to and are uniquely determined by a highest weight $\lambda \in \mathfrak{h}^*$ located in the dominant Weyl chamber. Furthermore, this weight is integral.

If you are in the mood for more definitions, you can define the *opposite Borel* $\mathfrak{b}^-$ to be the unique Borel subalgebra for which $\mathfrak{b} \cap \mathfrak{b}^- = \mathfrak{h}$, and construct the opposite unipotent radical $\mathfrak{n}^- := [\mathfrak{b}^-, \mathfrak{b}^-]$. We then have the following:

Theorem 6.1.2. Fix an irreducible representation V of a semisimple Lie algebra $\mathfrak{g}$. All vectors in V can be obtained as the $U(\mathfrak{n}^-)$-span of the highest weight vector.

6.2 Root Systems and Dynkin Diagrams

We've seen that much about a semisimple Lie algebra $\mathfrak{g}$ can be learned by studying the geometry of a (mild) choice of maximal diagonalizable subalgebra $\mathfrak{h}$. It turns out that the geometry of this diagonalizable subalgebra determines the Lie algebra itself! We won't have time to prove this, but it's good to verify in the case of $\mathfrak{sl}_3$ that $\mathfrak{h}^*$ has the structure of a *root system* (with κ taken to be the killing form and $\Delta = R$ taken to be the roots of $\mathfrak{sl}_3$):

Definition 6.2.1. A *root system* is a finite dimensional real vector space E equipped with an inner product (i.e. a positive definite symmetric bilinear form) $\kappa : E \times E \to \mathbb{R}$ and a finite set of nonzero vectors $\Delta \subset E$, called *roots*, satisfying the following properties:

1. If $\alpha \in \Delta$, $n \in \mathbb{Z}$, then $n\alpha \in \Delta$ if and only if $n = \pm 1$.
2. For any $\alpha \in \Delta$, reflection across the hyperplane cut out by α preserves the roots.
3. For any $\alpha, \beta \in \Delta$, we have that $2\frac{\kappa(\alpha,\beta)}{\kappa(\alpha,\alpha)} \in \mathbb{Z}$.
4. The roots span E.

Remark 6.2.1. Some sources define a root system as a triple (E, κ, Δ) satisfying only the first two of the above axioms, referring to those that satisfy the third as *crystallographic* and those which satisfy the fourth as *essential*.

Theorem 6.2.1. A choice of Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ for a semisimple Lie algebra $\mathfrak{g}$ defines a root system, and each root system determines a Lie algebra.

The idea behind the proof of this theorem is that the properties of a root system impose tight geometric conditions on the angles that roots can have. For example, if α, β are two roots, then axiom 3 of a root system (applied twice) tells us that if θ is the angle between two roots α and β, then $4\cos^2(\theta) \in \mathbb{Z}$! This implies that $\theta \in \{0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}\} + \frac{\pi}{2}\mathbb{Z}$.

The geometry of a root system is further restricted when we choose a simple system, analogous to choosing positive roots.

Definition 6.2.2. Let (E, κ, Δ) be a root system. A choice of *positive roots* corresponds to a choice of subset $\Delta^+ \subset \Delta$, closed under addition, such that for each $\alpha \in \Delta$, precisely one of $\{\alpha, -\alpha\}$ is an element of Δ^+. Given a choice of positive roots, the associated *simple roots* are those positive roots which cannot be written as a sum of two other positive roots.

Once we choose a simple system, we obtain an even stronger restriction on the angles that can be formed between two simple roots , and from this, one can construct the *Dynkin diagram* of a root system.

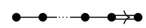
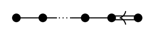
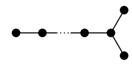
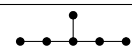
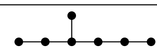
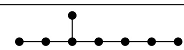
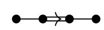
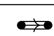
Definition 6.2.3. Fix a root system Δ with a choice of simple roots. The *Dynkin diagram* associated to Δ is the (oriented, multi)graph with vertices labeled by the set of simple roots and edges given by the following rules:

- No edge if the angle is $\frac{\pi}{2}$, i.e. • •,
- One edge if the angle is $\frac{2\pi}{3}$, i.e. •—•,
- Two edges with the longer root pointing to the shorter root if the angle is $\frac{3\pi}{4}$, i.e. •⇒•, with the convention that the arrow points to the shorter root,
- Three edges if the angle between the two simple roots is $\frac{5\pi}{6}$, i.e. •⇛•, with the convention that the arrow points to the shorter root.

It turns out that these are all the angles that can appear between two simple roots, as we will see.

Remark 6.2.2. One reason that we don't label any vertices between the edges is that the only time the Killing form returns $\frac{\pi}{2}$ is when the associated semisimple Lie algebra was a product of two smaller semisimple Lie algebras. Since semisimple Lie algebras are classified by simple ones, we'll restrict to the classification of **simple** Lie algebras from here on out.

Theorem 6.2.2. Root systems are classified by their associated Dynkin diagram, and all connected Dynkin diagrams either fall into one of four families or correspond to one of five *exceptional root systems*, all labeled below. Therefore, simple Lie algebras are classified by their associated Dynkin diagrams, which fall into one of the four families or one of the five *exceptional Lie algebras* listed below:

Label	**Lie Algebra**	**Diagram**	Restriction on n
A_n	$\mathfrak{sl}_{(n+1)}$		$n \geq 1$
B_n	$\mathfrak{so}_{2n+1}$		$n \geq 2$
C_n	$\mathfrak{sp}_{2n}$		$n \geq 3$
D_n	$\mathfrak{so}_{2n}$		$n \geq 4$
E_6	$\mathfrak{e}_6$		
E_7	$\mathfrak{e}_7$		
E_8	$\mathfrak{e}_8$		
F_4	$\mathfrak{f}_4$		
G_2	$\mathfrak{g}_2$		

In the above list, $\mathfrak{sp}_{2n}$ is the Lie algebra of the symplectic group $Sp(2n)$, given by

$$\mathfrak{sp}_{2n} = \{A \in \mathfrak{gl}_{2n} : JA + A^t J = 0\}, \text{ where } J := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

6.3 Exercises

Exercise 6.1. Verify that the Lie algebra of upper triangular $n \times n$ matrices is solvable.

Exercise 6.2. Fix a semisimple Lie algebra $\mathfrak{g}$ and choose a Borel subalgebra (i.e. a maximal solvable subalgebra) $\mathfrak{b}$. Show that if $\mathfrak{g}_\alpha \subset \mathfrak{b}$ for some root α, then $\mathfrak{b} \cap \mathfrak{g}_{-\alpha} = \{0\}$.

Exercise 6.3. (Exceptional Isomorphisms) Identify the Dynkin diagrams for B_n, C_n and D_n for n smaller than their constraint. (eg, B_1, C_1, C_2, D_2, D_3–note that $\mathfrak{so}_2$ is an abelian Lie algebra). What isomorphisms of semisimple Lie algebras do these identifications suggest? If you want, you can also prove these exceptional isomorphisms directly. (We also have the isomorphisms $E_3 \cong A_1 \times A_2, E_4 \cong A_4$ and $E_5 \cong D_5$, which you can also interpret via the Dynkin diagrams.)

Exercise 6.4. Using the definition of root systems, show the only Dynkin diagrams induced by rank two root systems are given by $A_1 \times A_1, A_2, B_2, G_2$ (The *rank* of a root system (E, κ, Δ) is defined to be the dimension of E).

6.4 Bonus Exercises

The next few exercises will motivate and define the *Langlands dual* of a semisimple (in fact, any *reductive*) algebraic group. For more details, see [1]. It turns out that reductive algebraic groups are classified by their *root datum*, a mild generalization of the idea of a root system which requires the additional notion of *coroots*.

Exercise 6.5. Verify that for a semisimple Lie algebra $\mathfrak{g}$ with maximal diagonalizable subalgebra $\mathfrak{h}$ and root $\alpha \in R$, if we define the *coroot* $\alpha^\vee \in \mathfrak{h}^*$ via $\alpha^\vee := 2\frac{\alpha}{\kappa(\alpha,\alpha)}$, then $\kappa(\alpha, \alpha^\vee) = 2$.

Definition 6.4.1. A *root datum* is defined to be a quadruple $(X, R, X^\vee, R^\vee)$ where $X, X^\vee$ are free finite rank $\mathbb{Z}$-modules equipped with a perfect duality pairing $\langle -, - \rangle : X \times X^\vee \to \mathbb{Z}$ (i.e. the map $\varphi : X \to \mathrm{Hom}_{\mathbb{Z}}(X^\vee, \mathbb{Z})$, $\varphi(x)(y) := \langle x, y \rangle$ is an isomorphism), and where $R \subset X$, $R^\vee \subset X^\vee$ are both finite subsets such that there exists[16] a bijection $R \to R^\vee$, written $\alpha \mapsto \alpha^\vee$, satisfying:

1. For all $\alpha \in R$, $\langle a, a^\vee \rangle = 2$.
2. For all $\alpha \in R$, the endomorphism $s_{\alpha,\alpha^\vee} : X \to X$ given by $s_{\alpha,\alpha^\vee}(x) := x - \langle x, \alpha^\vee \rangle \alpha$ has the property that $s_{\alpha,\alpha^\vee}(R) = R$.
3. Symmetrically, for all $a \in R$, the endomorphism $s_{a^\vee,\alpha} : X^\vee \to X^\vee$ given by $s_{\alpha^\vee,\alpha}(y) = y - \langle y, \alpha \rangle \alpha^\vee$ has the property that $s_{\alpha^\vee,\alpha}(R^\vee) = R^\vee$.

A root datum is defined only up to isomorphism of this data (i.e. a map of $\mathbb{Z}$-modules which maps the roots to the roots and the coroots to the coroots). For any reductive algebraic group G with choice of maximal torus T, we define the associated root datum to be $(\Lambda := \mathrm{Hom}(T, \mathbb{G}_m), R, \Lambda^\vee = \mathrm{Hom}(\mathbb{G}_m, T), R^\vee)$ with the pairing given by exercise 1.7, $R \subset \Lambda$ the set of roots of the adjoint representation T on $\mathfrak{g}$, and the coroots $R^\vee \subset \Lambda^\vee$ given by the map $\mathbb{G}_m \cong T_{SL_2} \to T$ in the following theorem:

Theorem 6.4.1. Let G be a reductive algebraic groups with a choice of maximal torus T, and let $\alpha \in R$ be a root of G. Then there exists a map of algebraic groups $SL_2 \to G$ whose induced map on Lie algebras sends $e \in \mathfrak{sl}_2$ to some eigenvector of α.

It is a theorem that the above construction gives a bijection

$$\{\text{Reductive Algebraic Groups}\} \xrightarrow{\sim} \{\text{Root Datum}\}.$$

Exercise 6.6. The root datum of SL_2 is isomorphic to $(\mathbb{Z}, \{\pm m\}, \mathbb{Z}, \{\pm n\})$ for some $m, n \in \mathbb{Z}$. What are m and n?

Note that the definition of root datum is symmetric–i.e. if $(X, R, X^\vee, R^\vee)$ is a root datum, then so is $(X^\vee, R^\vee, X, R)$. In particular, there is an endomorphism of {reductive algebraic groups} which squares to the identity, usually denoted $G \mapsto G^\vee$ or $G \mapsto {}^L G$.

Definition 6.4.2. For a reductive algebraic group G, we call the group $G^\vee$ obtained by reversing the root datum the *Langlands dual* of G.

Exercise 6.7. Show that the Langlands dual of SL_2 is PGL_2. More generally, show that the Langlands dual of SL_n is PGL_n. This illustrates a general phenomenon–'simply connected algebraic groups go to adjoint type algebraic groups under Langlands duality.'.

Exercise 6.8. Show that the Langlands dual of GL_n (which is a reductive algebraic group) is GL_n.

In particular, $\mathbb{G}_m = GL_1$ is its own Langlands dual. Many phenomena which go under the name 'class field theory' generalize (often with great difficulty!) to a more general theory relating G and $G^\vee$.

[16] It turns out that if such a bijection exists, then it is uniquely determined.

A Cheat Sheet for $\mathfrak{sl}_3$

The following list includes terms which we haven't discussed in this course, but are useful to know:

Term	(Usual) Value for $\mathfrak{sl}_3$	Determined By
$\mathfrak{sl}_3$	{traceless 3×3 matrices}	No Choices
Borel Subalgebra $\mathfrak{b}$	{upper triangular traceless matrices}	Choice
Unipotent Radical $\mathfrak{n} := [\mathfrak{b}, \mathfrak{b}]$	{strictly upper triangular matrices}	$\mathfrak{b}$
$\mathfrak{h}$ (or $\mathfrak{t}$)	{diagonal traceless matrices}	*
Opposite Borel $\mathfrak{b}^-$	{lower triangular traceless matrices}	$\mathfrak{b}$
Opposite Unipotent Radical $\mathfrak{n}^-$	{strictly lower triangular matrices}	$\mathfrak{b}$
$\mathfrak{h}^* = Hom_{Vect}(\mathfrak{h}, \mathbb{C})$	$\{(L_1, L_2, L_3) : L_1 + L_2 + L_3 = 0\}$	$\mathfrak{h}$
Weight Lattice Λ	$\mathbb{Z}$-$Span\{L_1, L_2, L_3\}$	$\mathfrak{h}$
Roots R	$\{L_i - L_j\}$ $i, j \in \{1, 2, 3\}, i \neq j$	$\mathfrak{h}$
$\mathfrak{sl}_{2,\alpha}$	$\mathfrak{sl}_2$ Lie-subalgebra spanned by $f_\alpha, h_\alpha, e_\alpha$	Root α
Root Lattice Λ_R	$\mathbb{Z}$-span of roots	$\mathfrak{h}$
Positive Roots R^+	$\{L_i - L_j\}$ $i, j \in \{1, 2, 3\}, i < j$	$\mathfrak{b}, \mathfrak{h}$
Simple Roots	$\{L_1 - L_2, L_2 - L_3\}$	$\mathfrak{b}, \mathfrak{h}$
Weyl Group	S_3 via $\sigma(L_i) = L_{\sigma(i)}$	$\mathfrak{b}, \mathfrak{h}$
Simple Reflections	$\{(1, 2), (2, 3)\}$	$\mathfrak{b}, \mathfrak{h}$
Dominant Weyl Chamber	$\{aL_1 + bL_2 + cL_3 : a \geq b \geq c\}$	$\mathfrak{b}, \mathfrak{h}$
Parabolic Subalgebra** $\mathfrak{p}_s$	$\mathbb{C}$-Span$(\mathfrak{b}, f_s)$	$\mathfrak{b}$, simple root s
Killing Form κ	$\kappa(M, N) = 6\text{Tr}(MN)$	No Choices
Coroot $\alpha^\vee := 2\alpha/\kappa(\alpha, \alpha)$	$\alpha^\vee = h_\alpha \in \mathfrak{h}$	Root α
$\rho := \sum_{r \in R^+} r/2$	$L_1 - L_3$	$\mathfrak{b}, \mathfrak{h}$

*Canonically $\mathfrak{b}/\mathfrak{n}$ for any choice of Borel $\mathfrak{b}$ but a choice makes it a subset of $\mathfrak{g}$

**Caution: These are the nontrivial parabolic subalgebras ($\mathfrak{b}$ and $\mathfrak{g}$ are parabolic subalgebras) and in general parabolic subalgebras correspond in an order preserving bijection with subsets of the simple roots.

References

[1] Brian Conrad. Root datum for split reductive groups. `http://http://virtualmath1.stanford.edu/~conrad/249BW16Page/handouts/rootdatum.pdf`.

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[3] James E. Humphreys. *Introduction to Lie algebras and representation theory*, volume 9 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1978. Second printing, revised.

[4] Jacob Lurie. A proof of the borel-weil-bott theorem. `http://people.math.harvard.edu/~lurie/papers/bwb.pdf`.