Lecture 1.

Overview

Let G be a semisimple algebraic group, and let $\mathfrak{g}$ be its Lie algebra. If you're not familiar with these terms, you can take $G = SL_n$, the group of matrices of determinant 1, and then its Lie algebra is $\mathfrak{sl}_n$, the vector space of traceless matrices. Choose a Borel subalgebra $\mathfrak{b}$ and let $\mathfrak{t}$ and $\mathfrak{n}$ be a Cartan subalgebra and the nilpotent radical of $\mathfrak{b}$, respectively. We're going to talk about extensions of the notion of *highest weight vectors*, a crucial notion in the theory of finite dimensional representations of Lie algebras:

Theorem 1.1. *Any irreducible, finite dimensional representation of $\mathfrak{g}$ is given by a highest weight vector, i.e. for any irreducible representation V of $\mathfrak{g}$, there exists some $\lambda : \mathfrak{t} \to \mathbb{C}$ and a vector $v \in V$ such that $\mathfrak{n}v = 0$ and for any $\xi \in \mathfrak{t}$, $\xi v = \lambda(\xi)v$.*

You might hope that the theorem applies as is to infinite dimensional representations of $\mathfrak{g}$. However, an inspection of the universal enveloping algebra (as a $\mathfrak{g}$ representation) quickly shows that that hope fails. Furthermore, you have a more fundamental problem here–you don't have just one notion of highest weight module:

Example 1.2. If $U(\mathfrak{sl}_2) := k\langle e, f, h\rangle/(ef - fe = h, he - eh = 2e, hf - fh = -2f)$, then the representation $U(\mathfrak{sl}_2) \otimes_{U(\mathfrak{b})} k$ (where $U(\mathfrak{b}) \to k$ is the map $U(\mathfrak{b}) \to U(\mathfrak{t}) \xrightarrow{\lambda} k$ and $\lambda(h) = 3$) is an infinite dimensional representation of $\mathfrak{sl}_2$ of highest weight 3. ◀

You might argue 'just use whichever one happens to be irreducible,' but this will run you into some problems. Specifically, the representation $U(\mathfrak{sl}_2) \otimes_{U(\mathfrak{b})} k$ which kills $\mathfrak{n}$ and sends $h \in \mathfrak{t}$ to 3.5 *is* irreducible, and it's a bit awkward to use only the definition which critically changes depending on whether the associated character in $\mathfrak{t}^*$ is integral or not. So let's introduce some different terminology for the sort of $U(\mathfrak{g})-$module we've defined above:

Definition 1.3. Fix a character $\lambda \in \mathfrak{t}^*$. We set the *Verma module* M_λ to be the $\mathfrak{g}$ representation $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} k$, where the latter map is $U(\mathfrak{b}) \to U(\mathfrak{t}) \xrightarrow{\lambda} k$.

Note that Verma modules M_λ are $\mathfrak{t}-$semisimple, that is, M is a $\mathfrak{t}$-direct sum of its $\mathfrak{t}$ eigenspaces; this is a consequence of the Poincare-Birkhoff-Witt theorem. It follows, then, that all Verma modules have a unique maximal submodule, and therefore a unique simple quotient, which we will denote L_λ. (The existence of the simple quotient of the Verma M_λ is proven by arguing that the λ eigenspace is one dimensional and no proper submodule contains a vector in this λ eigenspace.)

Example 1.4. If $\mathfrak{g} = \mathfrak{sl}_2$, identifying $\mathfrak{t}^* \cong \mathbb{C}$ by evaluation at $h \in \mathfrak{sl}_2$, we obtain that $L_x = M_x$ if x is not a nonnegative integer. In contrast, the unique simple quotient of M_n is the the finite dimensional representation of $\mathfrak{sl}_2$ of highest weight n. Note this implies that $M_{-n-2} \hookrightarrow M_n$. ◀

Note that on Verma modules, $\mathfrak{n}$ acts *locally nilpotently*, that is, for all $\xi \in \mathfrak{n}$ and $v \in M$ there is some $N \in \mathbb{N}$ with $\xi^N v = 0$. These properties are not true for all $\mathfrak{g}$ reps and the representations with these properties form one of the first categories of $\mathfrak{g}$-modules that one studies after the finite dimensional ones:

Definition 1.5. The **BGG Category** $\mathcal{O}$ is the full subcategory of finitely generated $\mathfrak{g}$ for which $\mathfrak{n}$ acts locally nilpotently and $\mathfrak{t}$ acts semisimply.

Through the use of the *Harish-Chandra isomorphism*, one obtains (see Chapter 1 of *Representations of Semisimple Lie Algebras in the BGG Category* $\mathcal{O}$):

Proposition 1.6. *The category $\mathcal{O}$ is Noetherian and Artinian. In particular, Jordan Holder filtrations exist. Furthermore, all simple objects in $\mathcal{O}$ are given by L_λ for some $\lambda \in \mathfrak{t}^*$.*

This brings us to the big question of the talk (and of representation theorists in the 1970's): What is the multiplicity that L_λ appears in the composition series of M_μ for $\mu, \lambda \in \mathfrak{t}^*$? Bigger ideas will come at play later, but first we answer the easier question 'Which L_λ *can* appear as a subquotient of M_μ?'

Definition 1.7. The **Weyl group** of an algebraic group is the group $Nm(T)/T$.

It turns out that the Weyl group is finite. For example, for SL_2 you can check that the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ normalizes the torus, and the Weyl group is $S_2 \cong \mathbb{Z}/2\mathbb{Z}$ (and more generally the Weyl group of SL_n is the symmetric group S_n). The Weyl group acts on T (via conjugation), and so the Weyl group acts on $\mathfrak{t}$ and therefore $\mathfrak{t}^*$ by reflections. But note that our SL_2 example (which said that $M_{-n-2} \hookrightarrow M_n$ for $n \geq 0$) tells us that it's not the case that we want to reflect about zero, but we actually want to reflect about -1. In general, we replace -1 as the sum of the negative roots, which we denote $-\rho \in \mathfrak{t}^*$. We call the action of the Weyl group given by 'reflecitng, except if $-\rho$ was the origin instead of 0' as the **W, • action**. Then some more elementary analysis of category $\mathcal{O}$ and the Harish-Chandra map (again, see Chapter one of Humphreys for details):

Proposition 1.8. *If L_λ appears as a subquotient of M_μ in the Jordan Holder filtration, then $\lambda \in W \bullet \mu$.*

Because of this proposition, we will focus on the **block** $\mathcal{O}_0$, the full subcategory of $\mathcal{O}$ which is obtained by successive extensions of the simple modules $L_{w \bullet 0}$, viewing $0 \in \mathfrak{t}^*$ (the results from here on out can be extended to $\mathcal{O}_\lambda$ but this makes it clear as to what's going on).

Originally, motivated by the examples of $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$, people had hoped that each simple appeared exactly once as a subquotient. Shortly thereafter, someone showed that there was a subquotient of the Verma M_0 when $\mathfrak{g} = \mathfrak{sl}_4$ which had multiplicity two. Work shortly thereafter led to the following conjecture by Kazhdan-Lusztig, which used the ideas of the **Hecke algebra**, inspired by an idea from number theory and the representation theory of finite groups:

Proposition 1.9. *There is an $R := \mathbb{Z}[q^{\pm\frac{1}{2}}]$ algebra structure on the R-module $\oplus_{w \in W} R\theta_w$ which 'entirely respects the Coxeter group structure of W.' (i.e. 'entirely respects the reflection group structure of W.')*

Sadly, for the sake of time we'll be vague about how this algebra structure is defined, but the multiplication is far from standard:

Example 1.10. For $G = SL_3$, $W = S_3$ (with $(1,2)$ and $(2,3)$ as the chosen set of 'Coxeter generators'), $\theta_{(1,2)}\theta_{(2,3)} = \theta_{(1,3)}$ but $\theta_{(1,2)}\theta_{(1,3)} \neq \theta_{(2,3)}$ and $\theta^2_{(1,2)} = (q-1)\theta_{(1,2)} + q$. ◀

Ideas behind this led Kazhdan-Lusztig to make the following conjecture. They proved the first part but were not able to prove the second part at the time:

Theorem 1.11. *(Kazhdan-Lusztig Conjecture) There is another canonical, computable basis of the Hecke algebra $\{C_w\}_{w \in W}$ (now called the **Kazhdan-Lusztig basis**) which is invariant with respect to the 'involution' operator on the Hecke algebra and is normalized so that the polynomials $P_{v,w}(q^{\frac{1}{2}})$ for which $C_w = \sum_{v \leq w} P_{w,v}(q^{\frac{1}{2}})\theta_v$ have low degree. The multiplicity of the simple $L_{w \bullet 0}$ inside the composition series of M_0 are given by $P_{0,w}(1)$.*[1]

The beauty of this conjecture, when it was made, is that it gives a computable answer for the multiplicity. The annoying part of this conjecture is it gives no hint as to how to see any connection between these elements. This connection came by means of geometry:

Definition 1.12. The **flag variety** of a reductive ($\implies$ semisimple) group G with Borel subgroup B is the variety G/B.

Theorem 1.13. *(Beilinson-Bernstein Localization Theorem) The global sections functor provides an exact equivalence of categories $\Gamma : \mathcal{D} - Mod(G/B) \to (\mathfrak{g} - mod)_0$.*

This is precisely the connection to geometry that solves this problem. In particular, one can use the well known geometry of G/B to solve this problem:

Theorem 1.14. *(Bruhat Decomposition) The group G can be written as a disjoint union $\coprod_W BwB$ (and thus G/B is stratified by $|W|$-elements). If $w_0 \in W$ is the longest word in the Weyl group (eg for our setup in $S_3, w_0 = (1,3) = (1,2)(2,3)(1,2)$) then the embedding $Bw_0B \hookrightarrow G/B$ is open.*

Specifically, letting Loc denote the inverse functor, one can compute that under this equivalence, one can show that computing the multiplicity of a simple module L_w in M_0 is given by the stalk of $j_*(\underline{k})$ at a chosen point of BwB. One can also show that this precisely gives the numbers in the Kazhdan-Lusztig conjecture, thus proving it!

[1]Warning: I may be off by 'swapping w_0 with 0', this depends on conventions

Lecture 2.

Setup and Formalities

In this section, we'll give the general outline of the proof of the Beilinson-Bernstein theorem. We let G be any reductive algebraic group over any algebraically closed field of characteristic zero k. Let $\mathcal{D}(G/B)$ be the (derived) category of $\mathcal{D}$ modules on G/B, and similarly for any (not necessarily commutative) ring A we let A-Mod denote the derived category of A modules. We will use the notation $D(G/B)^\heartsuit$ and $A-\text{Mod}^\heartsuit$ for the underlying abelian categories. In the motivation section, we discussed the statement of the Beilinson-Bernstein localization theorem:

Theorem 2.1. *(Beilinson-Bernstein '81) The global sections functor yields an exact equivalence between the category of $\mathcal{D}$-modules on G/B and the category of $\mathfrak{g}$-modules of central character zero*[2].

We will show this by showing that the derived global sections functor, which we will hereafter denote by Γ, provides an equivalence $\mathcal{D}(G/B) \to U(\mathfrak{g})_0 - Mod$, where $U(\mathfrak{g})_0 := U(\mathfrak{g}) \otimes_{Z\mathfrak{g}} k$ for the quotient map $Z\mathfrak{g} \to Z\mathfrak{g}/(Z\mathfrak{g})^{>0} = k$. Passing to the corresponding abelian categories (i.e. passing to the heart of the t-structure) and using the exactness of $H^0\Gamma$, we will obtain the Beilinson-Bernstein theorem.[3]

First, we will use a standard result in category theory, called the *Barr-Beck Theorem*, which will allow us to reduce this statement about categories to a statement more about rings. We will write the theorem only in the generality we need it, or else we would have to define generally what it means to be a module over a monad:

Theorem 2.2. *(Barr-Beck for Modules) Suppose that we have a functor $R : \mathcal{D} \to A - Mod$ for some category $\mathcal{D}$ and that R admits a left adjoint $L : A - Mod \to \mathcal{D}$. Then $RL(A) \in A - Mod$ acquires a canonical algebra structure, and R acquires a lift $R^{enh} : \mathcal{D} \to RL(A) - Mod(A - Mod)$, i.e. there exists a functor $R^{enh} : \mathcal{D} \to RL(A) - Mod(A - Mod)$ such that composition with $RL(A) - Mod(A - Mod) \to A - Mod$ (the functor induced by the ring map $A \to RL(A)$) is naturally isomorphic to R. Furthermore, if R is conservative (i.e. only sends isomorphisms to isomorphisms) and preserves all geometric realizations (see below), then the functor R^{enh} is an equivalence.*

Here, a *geometric realization* is a particular type of colimit that forgetful functors preserve (and this is all we will need about this type of colimit), and all we will need about the category $RL(A) - Mod(A - Mod)$ is that if the map $A \to RL(A)$ is a surjection of (classical) rings, then $RL(A) - Mod(A - Mod) \cong RL(A) - Mod$.

In our setup, we will let $R = \Gamma : \mathcal{D}(G/B) \to U(\mathfrak{g}) - Mod$[4]. We see that R has left adjoint $\mathcal{D}_{G/B} \otimes_{U(\mathfrak{g})} -$. Note that the underlying vector space of the global sections functor factors as $\mathcal{D}(G/B) \xrightarrow{oblv} QCoh(G/B) \xrightarrow{\Gamma} Vect$, and both of these functors commute with colimits (the latter by Serre duality, which can be interpreted as saying that since G/B is proper, the global sections functor is a left adjoint and thus commutes with colimits). Thus since we have that (derived!) Γ commutes with colimits, and therefore Barr-Beck tells us that the equivalence $\mathcal{D}(G/B) \xrightarrow{\sim} (\mathfrak{g} - Mod)_0$ follows from the following three theorems:

Theorem 2.3. *The global sections functor Γ preserves the heart of the t-structures of the respective categories (in other words, $H^0(\Gamma)$ is exact).*

Theorem 2.4. *The global sections functor $\Gamma : \mathcal{D}(G/B) \to \mathfrak{g} - Mod$ is conservative.*[5]

Theorem 2.5. *There is a natural ring isomorphism $U(\mathfrak{g}) \otimes_{Z\mathfrak{g}} k \xrightarrow{\sim} \Gamma(G/B)$.*

To see that the Beilinson-Bernstein localization theorem is satisfied, we check that the two conditions of the Barr-Beck Theorem are satisfied. Conservativity is mentioned explicitly. To show that Γ commutes with geometric realizations, recall that the flag variety is proper (for example, $SL_2/B \cong \mathbb{P}^1$). This in particular implies that the derived functor $\Gamma : QCoh(G/B) \to \mathfrak{g} - Mod$ is itself a left adjoint, and therefore commutes

[2]This means that any element in $Z\mathfrak{g}$ acts by the constant term. For example, the element $h^2 + 2ef + 2fe + 33 \in Z\mathfrak{sl}_2$ acts the same as multiplication by 33.

[3]Here, we are using the fact that for any commutative ring B, we can identify the abelian categories $\{M \in B\text{-Mod}: rM = 0$ for all $r \in I\}$ and B/I-Mod.

[4]Note that, since G acts on G/B, for any $\mathcal{F} \in \mathcal{D}(G/B)$, we have an action of $\mathfrak{g}$ as a Lie algebra on the global sections of $\mathcal{F}$. Thus, we have that the global sections functor lifts to a functor $\Gamma : \mathcal{D}(G/B) \to \mathfrak{g} - Mod$.

[5]This immediately gives the formal equivalence $\mathcal{D}(G/B) \xrightarrow{\sim} \Gamma(D_{G/B})$-Mod of derived categories.

with all colimits. Similarly, we mentioned earlier that the forgetful functor $oblv : \mathcal{D}(G/B) \to QCoh(G/B)$ commutes with geometric realizations, and so we see that in particular our global sections functor preserves geometric realizations.

In the next three parts of these notes, we'll prove the above three theorems. For now, here is also some additoinal motivation for this theorem I learned from Geordie Williamson in his awesome lecture series he gave at The Simons Center in August 2019.

Specifically, when one first learns the classification of finite dimensional representations of $\mathfrak{sl}_2$, one quickly finds that these are precisely in bijection with the nonnegative integers. This quick classification might lead one to believe that all $\mathfrak{sl}_2$ modules, or at least the irreducible ones, are classifiable. However, facts about the etale fundamental group of a many times punctured $\mathbb{P}^1$ tells us that the category $\mathcal{D}(G/B) = \mathcal{D}(\mathbb{P}^1)$ is complicated.

Specifically, one can take the intermediate extension of any local system on $\mathbb{P}^1 - \{0, 1, \infty\}$ (i.e. a representation of its etale fundamental group) and obtain a distinct, irreducible $\mathcal{D}$ module on G/B, i.e., by the Beilinson-Bernstein theorem, a distinct irreducible $\mathfrak{sl}_2$ module. Thus the (abelian) category of $\mathfrak{sl}_2$ modules is at least as complicated as the category of representations of the etale fundamental group of $\mathbb{P}^1 - \{0, 1, \infty\}$, an object that is very complicated and largely regarded to be intractable!

Lecture 3.

Exactness of Global Sections

We'll now prove the exactness of Γ, i.e. prove Theorem 2.3, by following the proof of Frenkel-Gaitsgory found in [FG09]. First, though, we will need to review a slight amount about the map $q : G \to G/B$. Specifically, we claim that the map q expresses G as a (Zariski) B-torsor (or principal B-bundle) over G/B. Specifically, in this context I mean that there are Zariski open sets $U \subset G/B$ such that the fiber of the quotient map q can locally be expressed as the trivial B torsor, i.e. $U \times B \to U$.

You can see this by using the Bruhat decomposition, which in particular says that if we choose a maximal unipotent subgroup N (think 'upper triangular matrices with 1's on the diagonal') and an opposite N^- (think 'lower triangular matrices with 1's on the diagonal'), then the multiplication map $N^- \times B \to G$ is an open embedding. This shows that most points of G/B have an open subset U containing them where you can write q as $U \times B \to U$, and you can 'G-translate' to obtain such an open subset for all the other points in G/B.

Because $q : G \to G/B$ is a B-torsor, for any sheaf $\mathcal{G} \in QCoh(G/B)$, we see that the (quasicoherent) pullback is locally of the form $\mathcal{G} \boxtimes \mathcal{O}_B$ for on some open $U \subset G/B$. Furthermore, the vector space of U-sections of $\mathcal{G} \boxtimes \mathcal{O}_B$ has a right B action, and we can recover the U-sections of the sheaf, i.e. $\mathcal{G}(U)$, as the B fixed points of this action. This globalizes to obtain:

Proposition 3.1. *For any sheaf $\mathcal{G} \in QCoh(G/B)$, the pullback $q^*(\mathcal{G})$ obtains a (right) B-action, and there is an isomorphism of vector spaces $\Gamma(G/B, \mathcal{G}) \xrightarrow{\sim} \Gamma(G, q^*\mathcal{G})^B$.*

Now we'll consider what extra structure stems from the fact that we are taking global sections of $\mathcal{D}$-modules, and not just quasicoherent sheaves. First, note that the (quasicoherent) pullback functor q^* lifts to a $\mathcal{D}$ module map, which by abuse of notation we will also denote $q^* : \mathcal{D}(G/B) \to \mathcal{D}(G)$. Note also that, since we can identify $\mathfrak{g}$ as either left or right invariant vector fields on G, the global sections of any $\mathcal{F} \in \mathcal{D}(G)$ naturally acquires a $\mathfrak{g}$-bimodule structure. In other words, global sections lifts to a functor $\Gamma : \mathcal{D}(G) \to \mathfrak{g} - BiMod$. By the fully faithfulness of the functor of abelian categories $Rep(B) \to \mathfrak{b} - Mod$ (See Proposition 3.41), we obtain:

$$\Gamma(G, q^*(-))^B = Hom_{Rep(B)}(k_{triv}, \Gamma(q^*(-))) = Hom_{Mod-U\mathfrak{b}}(k_{triv}, \Gamma(q^*(-))) \tag{3.2}$$

and so we further by the tensor-Hom adjunction obtain that

$$\Gamma(G, q^*(-))^B = Hom_{Mod-U\mathfrak{g}}(k \otimes_{U\mathfrak{b}} U\mathfrak{g}, \Gamma(q^*(-))) \tag{3.3}$$

Thus since q^* is exact (since q is flat, as we saw locally, and thus globally, above) and $\Gamma(G, -)$ is exact (since G is affine), the question of exactness has been reduced to the question of the projectivity of the Verma module $M^0 := k \otimes_{U\mathfrak{b}} U\mathfrak{g}$. Sadly, the Verma module is not projective in the category of $\mathfrak{g}$-modules (for example, we have a map $M^0 \to k$, but no map $M^0 \to U\mathfrak{g}$ which lifts the augmentation map $U\mathfrak{g} \to k$ since M^0 is locally $\mathfrak{n}$ nilpotent).

On the other hand, we can look closer as to where the functor $\Gamma(q^*(-))$ actually maps. Note that, in particular, we lost some information by forgetting that the $\mathfrak{b}$-module structure actually came from a representation of the group B. The general theory of algebraic groups and their relation to Lie algebras yields:

Proposition 3.4. *An $\mathfrak{n}$-module is in the essential image of $Rep(N) \to \mathfrak{n}$-Mod if and only if $\mathfrak{n}$ acts locally nilpotently. A $\mathfrak{t}$-module is in the essential image of $Rep(T) \to \mathfrak{t}$-Mod if and only if $\mathfrak{t}$ acts semisimply with integral weights (i.e. the eigenvalues in $\mathfrak{t}^*$ come from the weight lattice).*

Thus, by this proposition (which I haven't proven, but is (II.6.c) in [Mil13]), we obtain that if some $\mathfrak{g}$ module M has $\mathfrak{b}$-module structure which comes from an algebraic group B, then M is in category $\mathcal{O}$[6] This allows us to use all the tools we know about category $\mathcal{O}$. In particular, we have as a result of the theory of central characters, which allows us to classify the blocks of category $\mathcal{O}$, we obtain:

Proposition 3.5. *The Verma module M^0 is projective in $\mathcal{O}$.*

Thus, finalizing our discussion above, we see that $\Gamma(G, q^*(-))^B = Hom_{\mathcal{O}}(M^0, \Gamma(G, q^*(-))$ is exact!

It turns out, in fact, that if $\lambda \in \mathfrak{t}^*$ has the property that $\lambda + \rho$ is dominant, then the Verma M^λ is projective by a similar central character discussion (Here, the Verma module associated to $\lambda \in \mathfrak{t}^*$ is given by $k \otimes_{U\mathfrak{b}} U\mathfrak{g}$, where the map $U\mathfrak{b} \to k$ is given by $U(\mathfrak{b}) \to U(\mathfrak{b}/\mathfrak{n}) = U(\mathfrak{t}) \xrightarrow{\lambda} k)$. Thus, once we put the correct structure on λ-twisted $\mathcal{D}$-modules on G/B, we obtain a more general version of the exactness part of the Beilinson-Bernstein localization theorem:

Theorem 3.6. *If $\lambda \in \mathfrak{t}^*$ is such that $\lambda + \rho$ is dominant, then $\mathcal{D}_\lambda(G/B) \to \mathfrak{g} - Mod$ is exact.*

Studying this notion of 'Beilinson-Bernstein for every $\lambda \in \mathfrak{t}^*$' might suggest studying this notion in families. David Ben-Zvi and David Nadler do so in their paper, which requires the use of DG categories, a certain type of infinity category.

Lecture 4.

Conservativity

We'll now prove the conservativity of Γ, i.e. prove Theorem 2.4. Note that we've already shown this functor is exact (i.e. preserves t-structures) and so it conservativity is now equivalent to the following claim:

Theorem 4.1. *If $\mathcal{F}$ is a $\mathcal{D}$ module on G/B such that $\Gamma(\mathcal{F}) = 0$, then $\mathcal{F} = 0$.*

Remark 4.2. Note this is in huge contrast to the usual global sections functor on $\mathbb{P}^1 \cong G/B$ for $G = SL_2$, which kills $\mathcal{O}(-n)$ for any positive n. Thus one interpretation of this theorem is that it is not easy for quasicoherent sheaf on G/B to have any $\mathcal{D}$ module structure at all. This relates to the fact that we only can capture those $\mathfrak{g}$-modules of central character zero as global sections of the flag variety. Certain $\mathfrak{g}$-modules of other central characters can be realized as twisted $\mathcal{D}$ modules on G/B, or, using more ∞-categorical techniques, we can describe how (the derived category of) $\mathfrak{g}$-modules in general relate to the category $\mathcal{D}(G/N)^{T,w}$, the category of weakly T-equivariant $\mathcal{D}$ modules on the basic affine space G/N. This is further discussed in [BZN12]. ◂

Remark 4.3. Note that this statement is also fundamentally a statement about abelian categories. Therefore in this section, we will use abelian categorical notation (in particular, tensor products are underived). ◂

Before we prove this theorem, we first remind of the construction and some of the results of the Borel-Weil Theorem. Specifically, recall that we can view (or define, depending on your preference) the weight lattice as $\Lambda := Hom_{\mathbb{Z}-Mod}(T, \mathbb{G}_m)$. Using a weight $\lambda \in \Lambda$, we can create a one dimensional representation $k_{-\lambda}$ of B via $B \to B/N = T$, and can further create a one dimensional line bundle $G \times^B k_{-\lambda} \to G/B$. Denote the sheaf of sections of this line bundle by $\mathcal{O}(\lambda)$.

Remark 4.4. Note the negative sign in $k_{-\lambda}$ is necessary for us to say that $\mathcal{O}(n\rho) \cong \mathcal{O}_{\mathbb{P}^1}(n)$ for $G = SL_2$. This convention differs from that in [HTT08]! ◂

[6] Some people (eg, [Hum08]) like to only include finitely generated objects in their category $\mathcal{O}$, but this doesn't affect our analysis.

The map $G \times^B k_{-\lambda} \to G/B$ is (left) G equivariant, and thus global sections of the associated bundle acquire a G representation from the fact that if σ is a global section, then so is $g\sigma(g^{-1} \cdot -) : G/B \to G \times^B k_\lambda$. We then have:

Theorem 4.5. *(Borel-Weil) If λ is dominant, the global sections of $\mathcal{O}(\lambda)$ is, as a G-representation, the dual to the representation of highest weight λ (i.e. the representation of lowest weight $w_0\lambda$).*

We set ρ to be half of the sum of the positive roots (for example, if $G = SL_2$ and we identify the weight lattice with the integers via $n \mapsto (\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \mapsto \alpha^n)$, the positive root of SL_2 is 2 and thus $\rho = 1$). Then there is a result that says the line bundle $\mathcal{O}(-\rho)$ is ample (and in fact, $\mathcal{O}(\nu)$ is ample for any ν such that is $\nu + \rho$ is antidominant).

Now, we'll prove the theorem. This proof is modeled after 11.4 of [HTT08]. Fix a $\mathcal{D}$-module $\mathcal{F}$ on G/B. The upshot of the above discussion (along with the fact that $\mathcal{O}(\lambda) \otimes \mathcal{O}(\mu) \cong \mathcal{O}(\lambda + \mu)$ for any weights λ, μ) is that there is some ('sufficiently positive') weight μ for which $\mathcal{F} \otimes \mathcal{O}(\mu)$ is generated by global sections. Choose such a μ. Then we have a surjection of sheaves $\mathcal{F} \otimes_k L^-(w_0\mu) \to \mathcal{F} \otimes_{\mathcal{O}_{G/B}} \mathcal{O}(\mu)$, where we set $L^-(w_0\mu) := \Gamma(G/B, \mathcal{O}(\mu))$ is the representation of lowest weight $w_0\mu$, i.e. the dual of the representation with highest weight μ.

Remark 4.6. Note that the object $\mathcal{F} \otimes_k L^-(\mu)$ is a sheaf, since, for example, we can view $\mathcal{F}$ as an $\mathcal{O}$ module with a commuting k action and thus tensoring with this k action retains an $\mathcal{O}_X$ action. Also, this is a surjection of sheaves, which in particular does not automatically imply a surjection on global sections. ◀

Note as a consequence of the remark above, $\Gamma(G/B, \mathcal{F} \otimes_k L^-(w_0\mu)) = \Gamma(G/B, \mathcal{F}) \otimes_k L^-(w_0\mu)$, so we have reduced the above theorem to showing:

Proposition 4.7. *The surjection $\mathcal{F} \otimes_k L^-(w_0\mu) \to \mathcal{F} \otimes_{\mathcal{O}_{G/B}} \mathcal{O}(\mu)$ splits.*

We'll prove this via analysis of how the center of the universal enveloping algebra, $Z\mathfrak{g}$, acts on both sides. Specifically, note that $\mathcal{F}$ and $\mathcal{O}(\mu)$ are, in particular, $\mathfrak{g}$-modules (the latter since $\Gamma(\mathcal{O}(\mu))$ is a G rep), and so (every section of) their tensor product $\mathcal{F} \otimes_{\mathcal{O}_{G/B}} \mathcal{O}(\mu)$ is also a $\mathfrak{g}$-module by the 'product rule.' The plan to prove this proposition will be two steps:

(1) Show that the action of $Z\mathfrak{g}$ on any section of $\mathcal{F} \otimes_{\mathcal{O}_{G/B}} \mathcal{O}(\mu)$ is locally finite, that is, for any section v, the set $Z\mathfrak{g} * v$ is finite dimensional. In particular, since any $U\mathfrak{g}$ module M is a $(U\mathfrak{g}, Z\mathfrak{g})$ bimodule, we obtain that we can write $\mathcal{F} \otimes_{\mathcal{O}_{G/B}} \mathcal{O}(\mu)$ as the direct sum of the $Z\mathfrak{g}$ generalized eigenspaces. In other words, the locally finiteness of the action buys us that $M = \oplus_\chi M^\chi$, where χ varies over the *central characters*, i.e. ring maps $Z\mathfrak{g} \xrightarrow{\chi} k$, and $M^\lambda = \{v \in M : \forall \xi \in Z\mathfrak{g}, \exists N \gg 0 s.t. (\xi - \chi(\xi))^N(v) = 0\}$.

(2) We can identify $\mathcal{F} \otimes_k L^-(w_0\mu)$ as the generalized eigenspace associated to a particular central character.

To be clear, I haven't shown (1) or (2), but the proposition will follow if we show both.

Proof of (1): Recall that the lowest weight representation has a filtration by B modules $L^-(w_0\mu) =: V^1 \supset V^2 \supset ... \supset V^r = 0$ where each quotient V^i/V^{i+1} is a one dimensional B representation such that V^1/V^2 has weight $w_0\mu$ and for $i > 1, V^i/V^{i+1}$ has some other (higher) weight. Set $\nu := w_0\mu$.

This filtration provides us a filtration on the total space of the bundle $G \times^B L^-(\nu)$. Explicitly, the filtration is $G \times^B L^-(\nu) = G \times^B V^1 \supset G \times^B V^2 \supset ... \supset G \times^B V^r = 0$. The isomorphism of G/B spaces $G/B \times L^-(\nu) \xrightarrow{\sim} G \times^B L^-(\nu)$ given by $(gB, v) \mapsto (g, gv)$ then translates this filtration to $G/B \times L^-(\nu)$ (i.e. the trivial bundle) to obtain a filtration of the sheaf of sections, $\mathcal{O}_{G/B} \otimes_k L^-(\nu) =: \mathcal{U}^1 \supset \mathcal{U}^2 \supset ... \supset \mathcal{U}^r = 0$. Note that in particular we obtain that $\mathcal{U}^1/\mathcal{U}^2$ the sheaf of sections $G \times^B V^1/V^2$, i.e., we have that $\mathcal{U}^1/\mathcal{U}^2 = \mathcal{O}(\mu)$. Similarly, for all $j \in \{1, ..., r-1\}$ we have $\mathcal{U}^j/\mathcal{U}^{j+1} = \mathcal{O}(\mu_j)$ for some $\mu_j \geq \nu$, and if $j > 1$, $\mu_j \neq \nu$.

In particular, our map as in the proposition can be viewed as the map $\mathcal{F} \otimes_k \mathcal{U}^1 \xrightarrow{id \otimes q} \mathcal{F} \otimes_{\mathcal{O}_{G/B}} \mathcal{U}^1/\mathcal{U}^2$, where q denotes the quotient map. Now note that, viewing $\mathcal{F} \otimes_k \mathcal{U}^1$ as a $\mathfrak{g}$-module by the product rule, we can determine exactly how the center $Z\mathfrak{g}$ acts.

Explicitly, we know from the identification of rings below (which doesn't depend on anything in this section) that $\xi \in Z\mathfrak{g}$ acts on $\mathcal{F} \in \mathcal{D}(G/B)$ by 'multiplication by zero' (which, in particular, shows that we have a map $U\mathfrak{g}/Z\mathfrak{g}^{>0} \to \Gamma(G/B, \mathcal{D}_{G/B})$). We also know that $(\xi - \chi_{\mu_1}(\xi))$ maps U^1 into U^2 since $U^1/U^2 \cong L^-(\mu)$ and since we know that the Harish-Chandra map is given by projecting $\xi \in Z\mathfrak{g}$ onto the $\mathfrak{t}$ components of the PBW basis.

Thus we obtain that $(\xi - \chi_{\mu_1}(\xi))(\mathcal{F} \otimes_k L^-(\nu)) \subset \mathcal{U}^2$. But we can play a similar game with U^2/U^3 to obtain even more–namely, $(\xi - \chi_{\mu_2}(\xi))(\xi - \chi_{\mu_1}(\xi))(\mathcal{F} \otimes_k L^-(\nu)) \subset \mathcal{U}^3$. You should see where this is going, and we obtain the fundamental equation that $\prod_{j=1}^{r-1}(\xi - \chi_{\mu_j}(\xi))$ acts as zero on $\mathcal{F} \otimes_k L^-(\nu)$, which in particular implies that $Z\mathfrak{g}$ acts locally finitely on $\mathcal{F} \otimes_k L^-(\nu)$!

Proof of (2): Note that the proof above that $\mathcal{F} \otimes_{\mathcal{O}_{G/B}} \mathcal{O}(\nu)$ breaks up into a direct sum of generalized eigenspaces associated to specific central characters–namely, the central characters given by the various χ_{μ_i}, and in particular, we've shown that the image of $\mathcal{F} \otimes_k L^-(\nu)$ inside $\mathcal{F} \otimes_{\mathcal{O}_{G/B}} \mathcal{O}(\mu)$ is the χ_ν (recall $\mu_1 = \mu$) generalized eigenspace, and the other sections of the sheaf lie in the χ_{μ_j} for $j \neq 1$.

This requires further study of the Harish-Chandra map $\chi : \mathfrak{t}^* \to Hom_{Ring}(Z\mathfrak{g}, k)$. Since analysis of the Harish-Chandra map is one of the foundational results on the work of category $\mathcal{O}$, we'll just quote the theorem here–see [Hum08] for details on the proofs:

Theorem 4.8. *Two weights $\nu, \lambda \in \mathfrak{t}^*$ are sent to the same central character under χ if and only if ν and λ lie in the same $W, \cdot$ orbit (read 'W-dot orbit'), equivalently, there is some $w \in W$ for which $w(\nu + \rho) = \lambda + \rho$.*

Fix some weight μ_j . If we had $\chi_{\mu_j} = \chi_\nu$, then the above theorem would tell us that, after some rearranging, $(w\rho - \rho) + (\nu - w\mu_j) = 0$. However, note that we have both $w\rho \leq \rho$ and $\nu \leq w\mu_j$ (the latter being since $w\mu_j$ is a weight of $L^-(\nu)$, the ν-lowest weight representation).

(Recall that, for two weights $\phi, \lambda \in \mathfrak{t}^*$, we say $\phi \leq \lambda$ if $\lambda - \phi$ is a positive integral root. In particular, when identifying the weight lattice for SL_2 with the integers $\mathbb{Z}$, we obtain that, for example, $0 \not\leq 1$, but $0 \leq 2$. For similar reasons, the weight $-\frac{1}{2}$ counts as a dominant weight. These don't apply a ton in this current proof, but it does comes up in the twisted version, and the Satake equivalence, and I may have saved someone reading this two hours of their life.)

From these inequalities, we then see that both $w\rho - \rho = 0$ and $w\mu_j - \nu = 0$. From the former equality (and the definition of ρ), we obtain that $w = 1$ (since ρ is regular) and therefore the second equality says that $\mu_j = \nu$, and therefore $j = 1$. Thus we may identify $\mathcal{F} \otimes_k L^-(\nu)$ with the χ_ν eigenspace of $\mathcal{F} \otimes_{\mathcal{O}_{G/B}} \mathcal{O}(\mu)$, thus proving (2) and, therefore, conservativity!

Remark: It's not as clear as it was on the last part how to generalize this proof to twisted $\mathcal{D}$-modules. Examining the proof carefully above, we see what mattered essentially was the fact that ρ had $|W|$ many objects in its orbit. We call such weights *regular*.

Theorem 4.9. *(Derived Twisted Beilinson-Bernstein Localization) If $\lambda \in \mathfrak{t}^*$ such that $\lambda + \rho$ is a regular weight, the functor of derived categories $\mathcal{D}_\lambda(G/B) \to \mathfrak{g} - Mod_{\chi_\lambda}$ is an equivalence.*

Note that our above proof only applies for regular *dominant* weights, for we used exactness in the proof of the conservatity theorem. This is the only case for which we can apply the twisted version of the Beilinson-Bernstein localization theorem to the level of abelian categories:

Theorem 4.10. *If $\lambda + \rho$ is a dominant weight, then the associated global sections functor is exact. In particular, if $\lambda + \rho$ is a regular dominant weight, the global sections functor $\mathcal{D}_\lambda(G/B) \to \mathfrak{g} - Mod_{\chi_\lambda}$ is an equivalence on the level of abelian categories.*

Lecture 5.

Algebra Isomorphism

We'll soon prove the algebra isomorphism Theorem 2.5, but first, let's summarize where we are so far. The general Barr-Beck formalism, along with the results of the last two sections, have given us an exact equivalence $\Gamma : \mathcal{D}(G/B) \xrightarrow{\sim} \Gamma(G/B, \mathcal{D}_{G/B}) - Mod$.

Remark 5.1. Contrast this situation with usual quasicoherent sheaves, where the global sections functor on a variety $QCoh(X)^\heartsuit \to \Gamma(X, \mathcal{O}_X) - Mod^\heartsuit$ is an equivalence if and only if X if an affine variety. A consequence of the Beilinson-Bernstein theorem (at least the parts that we've argued so far) is that G/B is $\mathcal{D}$-affine. Later, Beilinson-Bernstein proved that for any parabolic subgroup $P, G/P$ is $\mathcal{D}$-affine. It's known that these G/P are the only possible non-affine surfaces which are $\mathcal{D}$-affine. All known examples of non-affine $\mathcal{D}$-affine varieties are of the form G/P for some parabolic P, and it's open whether there are any more than these! ◂

Now it's time to give an explicit description of the essential image of Γ in $\mathfrak{g}$-Mod.

Theorem 5.2. *The map $U\mathfrak{g} \to \Gamma(G/B, \mathcal{D}(G/B))$ has kernel $Z\mathfrak{g}^{>0}$ and is a surjection.*

We'll actually show the stronger, twisted version in this email. Recall that N was defined as the unipotent radical of the Borel, eg, for SL_n it's upper triangular matrices with 1's on the diagonal. Note that since N is a normal subgroup of B, we in particular obtain that T acts on the right on the *basic affine space* G/N. For $SL_2, G/N \cong \mathbb{A}^2 \setminus 0$. We will then show:

Theorem 5.3. *The map $U\mathfrak{g} \to \Gamma(G/N, \mathcal{D}_{G/N})^T$ of $U\mathfrak{g}$ modules induces a map $U\mathfrak{g} \otimes_{Z\mathfrak{g}} Sym\mathfrak{t} \xrightarrow{\sim} \Gamma(G/N, \mathcal{D}_{G/N})^T$, where the map $Z\mathfrak{g}$ module structure on $Sym\mathfrak{t}$ is given by the Harish-Chandra map $\chi : Z\mathfrak{g} \to Sym\mathfrak{t}$.*

The fact that Theorem 2.5 follows from the above theorem is a consequence of quantum Hamiltonian reduction, which we will review only briefly here. Namely, one consequence of quantum Hamiltonian reduction is that it allows us, for any X with a right action of some group H, to describe differential operators on X/H (the stack quotient) in terms of differential operators on X. Naively, one might guess that $\mathcal{D}_{X/H} = \mathcal{D}_X^H$. However, this fails for $X = H$, since $\mathcal{D}_H^H = U\mathfrak{h}$. However, this example may lead to the correct guess as to how to rectify the situation:

Theorem 5.4. *We have identifications $\Gamma(\mathcal{D}_{X/H}) \cong \Gamma(\mathcal{D}_X)^H \otimes_{U\mathfrak{h}} k \cong (\Gamma(\mathcal{D}_X) \otimes_{U\mathfrak{h}} k)^H$.*

Quantum Hamiltonian reduction (i.e. the above theorem) applied to the right action of T on G/N then yields the first theorem, since we then obtain the identification $\Gamma(G/B, \mathcal{D}_{G/B}) \cong \Gamma(G/N, \mathcal{D}_{G/N})^T \otimes_{Sym\mathfrak{t}} k$ so we in turn[7] obtain $\Gamma(G/B, \mathcal{D}_{G/B}) \cong U\mathfrak{g} \otimes_{Z\mathfrak{g}} Sym\mathfrak{t} \otimes_{Sym\mathfrak{t}} k \cong (U\mathfrak{g})_0$, where recall $(U\mathfrak{g})_0 := U\mathfrak{g}/(Z\mathfrak{g}^{>0})$. Quantum Hamiltonian reduction will also give us insight as to how to reduce the second theorem to facts about representation theory, but first, let's actually construct the map!

Note that $G \times T$ acts on G/N (again, the T action is on the right), so we automatically obtain the map $U\mathfrak{g} \otimes_k Sym\mathfrak{t} \to \Gamma(G/N, \mathcal{D}_{G/N})^T$, and, via this ring map, the ring $U\mathfrak{g} \otimes_k Sym\mathfrak{t}$ naturally acts on the k submodule $M^{univ} := U\mathfrak{g} \otimes_{U\mathfrak{n}} k = U\mathfrak{g} \otimes_{U\mathfrak{b}} U\mathfrak{t}$. Note that multiplication by any element $\xi \in Z\mathfrak{g}$ yields a $\mathfrak{g}$ module endomorphism of M^{univ}. It's a theorem that the natural map $Sym(\mathfrak{t}) \to End_{\mathfrak{g}}(M^{univ})$ is an isomorphism, and once you know this, it's easy to tell what the induced map $Z\mathfrak{g} \to Sym\mathfrak{t}$ is–it's the Harish-Chandra map!

Anyway, this yields our induced map $U\mathfrak{g} \otimes_{Z\mathfrak{g}} Sym\mathfrak{t} \to \Gamma(G/N, \mathcal{D}_{G/N})^T$. Furthermore, this map is filtered, essentially because all of the objects involved are differential operators. Therefore, to check whether this map is an isomorphism, it suffices to check on the level of the associated graded. After taking associated graded, the domain of the map above is the ring of functions on the space $\mathfrak{g} \times_{\mathfrak{g}//G} \mathfrak{t}$, and the codomain of the map above is the ring $Fun(T^*(G/N))^T$, where Fun stands for global functions.

To understand the associated graded of the left hand side, we first use the usual (i.e. non-quantum) version of Hamiltonian reduction to better understand (the associated graded of) differential operators on G/N. Specifically, considering the right action of N on G by right multiplication, we obtain the moment map $\mu : T^*G \to \mathfrak{n}^*$ given by the map $(g, \zeta) \mapsto (v \mapsto \zeta(v))$, and Hamiltonian reduction says that we can identify $T^*(G/N)$ with $\mu^{-1}(0)/N$, and thus, we can further identity $T^*(G/N)/T \cong \mu^{-1}(0)/B$.

Now identify $T^*G \cong G \times \mathfrak{g}^*$ by using the left action of G on itself, thus identifying T^*G with the space of right G invariant covector fields. The moment map similarly becomes the map $G \times \mathfrak{g}^* \to \mathfrak{n}^*$ which sends $(g, \zeta) \mapsto (v \mapsto \zeta(gvg^{-1}))$. In particular, under this identification we see that $\mu^{-1}(0) = \{(g, \zeta) : \forall v \in \mathfrak{n}, \zeta(gvg^{-1}) = 0\}$, where B only acts on the G factor. Using the Killing form, we can similarly identify $G \times \mathfrak{g} \cong G \times \mathfrak{g}^*$. We thus have a chain of maps $G \times \mathfrak{g} \xrightarrow{id \times \kappa} G \times \mathfrak{g}^* \cong T^*G \xrightarrow{\mu} \mathfrak{n}^* \cong \mathfrak{g}^*/\mathfrak{b}^{-^*}$, where $\mathfrak{b}^{-^*}$ is the opposite Borel subalgebra. If you chase through all these identifications, you will see that we can identify $\mu^{-1}(0)$ with $\{(g, \xi) \in G \times \mathfrak{g} : Ad_{g^{-1}}(\xi) \in \mathfrak{b}^{-^*}\}$, with B only acting nontrivially by right multiplication on the first factor. We thus may identify $\mu^{-1}(0)/B$ with $\{(gB, \xi) \in G/B \times \mathfrak{g} : Ad_{g^{-1}}(\xi) \in \mathfrak{b}^{-^*}\}$. But further still, one easily checks that this is equivalent to the space $\tilde{\mathfrak{g}} := \{(x, \xi) \in G/B \times \mathfrak{g} : \xi \in x\}$, where we are identifying G/B with the set of all Borel subalgebras of $\mathfrak{g}$.

Here's brief overview of the space we've just identified $\mu^{-1}(0)/B$ with. This space $\tilde{\mathfrak{g}}$ is called the *universal resolution.* This is good because it in particular has a name, which means people know a lot about it (for instance, it is used to construct the 'characteristic polynomial' map $\mathfrak{g} \to \mathfrak{t}//W$.) Note that there in particular

[7]Here, we are using the Harish-Chandra map $Z\mathfrak{g} \to Sym\mathfrak{t}$. This map is defined as follows: given an element of $Z\mathfrak{g}$, the Harish-Chandra map is simply the projection onto the $Sym\mathfrak{t} \subset U\mathfrak{g}$ parts of the PBW basis. This can be defined more canonically so that it becomes clear that this is in fact a map of algebras.

is a canonical map $\tilde{\mathfrak{g}} \to \mathfrak{t}$ given by sending $(gB, \xi) \mapsto \overline{\xi} \in \mathfrak{b}^-/\mathfrak{n}^-$, using the Killing form to identify each Lie algebra with its dual. (Heuristically, this map is 'more canonical' than the more obvious seeming map to $\mathfrak{b}^-$ because a Borel subalgebra is a particular choice, whereas once you've chosen a Borel subalgebra, we automatically obtain a choice for the Cartan subalgebra $\mathfrak{t}$. This relates to our previous discussion on the 'abstract Cartan.') We now appeal to some basic facts from geometric representation theory about the universal resolution:

Proposition 5.5. *The appropriate diagram commutes such that we have an induced map* $\tilde{\mathfrak{g}} \to \mathfrak{g} \times_{\mathfrak{t}//W} \mathfrak{t}$*.*

Note that the map $\tilde{\mathfrak{g}} \to \mathfrak{g} \times_{\mathfrak{t}//W} \mathfrak{t}$ is proper since the map $\tilde{\mathfrak{g}} \to \mathfrak{g}$ is proper (it factors as a closed subvariety of $G/B \times \mathfrak{g}$ and a proper projection onto $\mathfrak{g}$) and the map $\mathfrak{t} \to \mathfrak{t}//W$ is separated (by, for instance, the Cancellation Theorem for Properness, 10.1.19). We wish to appeal to Zariski's Main Theorem for Birational Morphisms, see Exercise 29.5.B in [Vak17]:

Theorem 5.6. *Assume* $\phi : X \to Y$ *is a proper morphism of locally Noetherian schemes,* Y *is normal, and is an isomorphism over a dense open subset of* Y*. Then* ϕ *is* $\mathcal{O}$*-connected, meaning that the induced map* $\mathcal{O}_Y \to H^0\phi_*\mathcal{O}$ *is an isomorphism.*

Remark: While the stronger theorem $U\mathfrak{g} \otimes_{Z\mathfrak{g}} Sym\mathfrak{t} \xrightarrow{\sim} \Gamma(G/N, \mathcal{D}_{G/N})^T$ is true derivedly, from here on out we'll stick to proving it in the non-derived case since we already proved $\Gamma(G/B, -)$ is exact on $\mathcal{D}$-modules. The derived case amounts to showing that the higher cohomology groups of the pushforward also vanish (because $U\mathfrak{g}$ is a flat $Z\mathfrak{g}$ module). The details can be found in 7.13 of [Gai05].

To show the non-derived isomorphism, we will need to show that $\mathfrak{g} \times_{\mathfrak{t}//W} \mathfrak{t}$ is normal and that the map is an isomorphism on some dense open subset. To see the isomorphism part, we may restrict to the regular semisimple locus of $\mathfrak{t}$, i.e. restrict to the open subset $\mathfrak{t}^{rs} \subset \mathfrak{t}$ where the map $\mathfrak{t} \to \mathfrak{t}//W$ actually becomes a W torsor (for example, if $G = SL_n$, we are asking for those traceless diagonal matrices with distinct entries.) When restricted to the regular semisimple (or even the regular locus), we obtain that the map $\tilde{\mathfrak{g}}^{rs} \to \mathfrak{g}^{rs}$ is also a W torsor, and therefore the map $\tilde{\mathfrak{g}} \to \mathfrak{g} \times_{\mathfrak{t}//W} \mathfrak{t}$ is an isomorphism on the open subset $\mathfrak{g}^{rs} \times_{\mathfrak{t}^{rs}//W} \mathfrak{t}^{rs}$.

We can also use the regular semisimple locus to show that the ring $Sym(\mathfrak{g}) \otimes_{Z\mathfrak{g}} Sym\mathfrak{t}$ is regular in codimension ≤ 1. We can also argue that the ring $Sym(\mathfrak{g}) \otimes_{Z\mathfrak{g}} Sym\mathfrak{t}$ is Cohen-Macaulay (in fact, it's a locally complete intersection ring) and so in particular it satisfies the criterion S_2. By Serre's criterion for normality, we now see (at least a sketch of the fact) that $\mathfrak{g} \times_{\mathfrak{t}//W} \mathfrak{t}$ is normal. and thus we get our isomorphism!

Thus we have shown that $\mathcal{D}(G/B) \xrightarrow{\sim} (U\mathfrak{g})_0 - Mod$. But finally, note that $Z\mathfrak{g}U\mathfrak{g} = U\mathfrak{g}Z\mathfrak{g}$ is a two sided ideal of $U\mathfrak{g}$. In particular, for any ring $A \in CAlg^{\heartsuit}$ and two sided ideal I of it we can identify $A/I - Mod^{\heartsuit}$ as the full subcategory of A modules for which the I action is trivial. Thus we have finally identified $\mathcal{D}(G/B)$ with $(\mathfrak{g} - Mod)_0$ at the level of both abelian categories and derived!

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