Arrow's Impossibility Theorem
(aka The Mathematics of Voting)

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What Is a Voting System?

We'll first declare a list of *alternatives* to choose between.

A voting system, informally, should:

Take as Input:

- A *personal preference list* of the alternatives for each person.

Return as Output:

- Return a *societal preference list* of the alternatives.

Example: Who should be math club president?

- *Alternatives:* {Ryan, Shannon}
- *Voters:* The people in this Zoom call

- **First Past the Post:** Order the societal preference list by how many times the person appeared as the top preference.
- **'Weighted FPTP'**: Same as above, but the current president's vote counts for 2 votes.
- **Last Past the Post:** Order the societal preference list by how many times the person appeared as the bottom preference.
- **Dictatorship:** Declare the societal preference list is identical to Tom's personal preference list.

Properties of Voting Systems: Pareto

Definition

If, when every person puts in the *same* personal preference list, the voting system returns that list as the societal preference list, we say a voting system satisfies **the Pareto condition**.

- *Examples:* First Past the Post, ‘Weighted FPTP’, Dictatorship
- *Non-example:* Last Past the Post

Example: Who should get the Florida electoral votes?

- *Alternatives:* {George W. Bush, Al Gore, Ralph Nader}
- *Voters:* Eligible Florida residents

Candidate	Number of Votes
George W. Bush	2,912,790
Al Gore	2,912,253
Ralph Nader	97,488

Table: Florida 2000 Election Results for FPTP Voting

- **Instant Runoff Voting** - Voters rank all preferences. Declare person who got the least number of votes last on the preference list, and 'repeat'.

  Could this have changed the results of the 2000 election?

Properties of Voting Systems: IIA

Definition

We say that a voting system is **independent of irrelevant alternatives** if...

- Informally, for every pair of alternatives x, y we can know the relative position of x and y on the societal preference list just from knowing the relative position of x and y on all of the individual's preference lists.

Question

- Is there a voting system which satisfies both the pareto condition and is independent of irrelevant alternatives?
- Answer: Yes, a dictatorship!
- Okay, are there any others?

Arrow's Impossibility Theorem

Theorem (Arrow's Impossibility Theorem)

Assume that V is a voting system with more than two alternatives which satisfies the Pareto condition and is independent of irrelevant alternatives. Then V is a dictatorship.

Corollary

There are no voting systems with more than two alternatives which satisfy the Pareto condition, independence of irrelevant alternatives, and are not a dictatorship.

- Break. (Questions?)

Warm Up: No Ties

Proposition

Assume we have a voting system with more than two alternatives which satisfies Pareto and IIA. Then the voting system can produce no ties.

Proof:

If

Left Side of Room	Right Side of Room
$a > b$	$b > a$

$\mapsto a = b$,

then

Left Side of Room	Right Side of Room
$a > c > b$	$c > b > a$

$\mapsto c > b = a$,

and

Left Side of Room	Right Side of Room
$a > b > c$	$b > c > a$

$\mapsto a = b > c$.

By IIA,

Left Side of Room	Right Side of Room
$a > c$	$c > a$

$\mapsto c > a$ and $a > c$.

A contradiction. ■

Dictating Sets

Definition

We say a subset S of voters are a **dictating set** if, whenever everyone in S puts the same personal preference list into the voting system, that list is the societal preference list, regardless of what anyone else votes.

- *Example:* The set of all voters is a dictating set.
- *Note:* If S is a set with one element, then S is a dictating set if and only if the person in S is a dictator.

Definitions: Monotonicity and Forcing

Simplifying Assumption: We will assume our voting system is *monotonic:*

Definition

A voting system is **monotonic** if for all alternatives a, b, the following property holds:

If

Left Side of Room	Everyone Else
$a > b$	$b > a$

$\mapsto a > b$

and some people in the 'everyone else' part switch their vote to $a > b$ then the societal preference list still has $a > b$.

- *Idea:* Assumption allows us to focus on worst case scenario.

Definition

Given two alternatives a, b we say that a subset S of voters can **force** $a > b$ if

People in S	Everyone Else
$a > b$	?

$\mapsto a > b$.

Outline of Proof

Theorem will follow directly from these two claims:

Forcing Lemma

If a subset of voters X can force $a > b$ and we partition $X = L \sqcup M$, then for any alternative c, either L can force $a > c$ or M can force $c > b$.

Proposition

If X can force some element a over some element b, then X can force *any* element over *any* other element, i.e. X is a dictating set.

Forcing Lemma

Forcing Lemma

If a subset of voters X can force $a > b$ and we partition $X = L \sqcup M$, then for any alternative c, either L can force $a > c$ or M can force $c > b$.

Proof: We know

L	M	Everyone Else
$a > b > c$	$c > a > b$	$b > c > a$

$\mapsto a > b$ (but don't know where c lies).

- Either the output is $c > a > b$ or $a > c$.
- If the output is $c > a > b$, then M can force $c > b$.
- If the output has $a > c$, then L can force $a > c$.

Question: Can we gain any more information from this proof when $M = \emptyset$? When $L = \emptyset$?

Corollary of Forcing Lemma

Forcing Lemma For $\emptyset$

If a subset X can force $a > b$ for two distinct alternatives a, b, then X can force $a > c$ and $c > b$ for any third alternative c (meaning different from a and b).

Corollary

If a subset X can force $a > b$ for two distinct alternatives a, b, then X can force $b > a$.

- X can force $a > b \underset{\text{Forcing } a>b}{\Longrightarrow} X$ can force $a > c$.
- X can force $a > c \underset{\text{Forcing } a>c}{\Longrightarrow} X$ can force $b > c$.
- X can force $b > c \underset{\text{Forcing } b>c}{\Longrightarrow} X$ can force $b > a$.

Any Set That Can Force Something is a Dictator

Corollary/Exercise

Use the forcing lemma to show that if X can force $a > b$ then for *any* distinct alternatives c, d, X can force $c > d$.

This proves our proposition!

Proposition

Any subset of people that can force $a > b$ for some alternatives a, b is a dictating set.

Review

We showed

Proposition

Any subset of people that can force $a > b$ for some alternatives a, b is a dictating set.

and

Forcing Lemma

If a subset of voters X can force $a > b$ and we partition $X = L \sqcup M$, then for any third alternative c (meaning different from a and b), either L can force $a > c$ or M can force $c > b$.

which proves Arrow's impossibility theorem! ■