Surgery Theory Problems

Summer 2017

Problem 1. Describe the surgeries on S^3. (Possible hint: the Heegaard splitting of S^3 into two solid tori may prove useful)

Problem 2. Fix $n \geq 4$. Use surgery to prove that any finitely presented group G is the fundamental group of a closed oriented n-manifold.

Problem 3. Let M be a 4-manifold and let S be a 2-sphere in M with a trivial normal bundle. Define M' be the manifold obtained by cutting out $S^2 \times D^2$ (we may assume that $N(S)$ is diffeomorphic to $S^2 \times D^2$) and regluing it by the self-diffeomorphism τ of $S^2 \times S^1$ that rotates each 2-sphere $S^2 \times \{\theta\}$ through the angle θ. Often this is made explicit by identifying S^1 with the unit circle of $\mathbb{C}$ and S^2 with the Riemann sphere, so the map becomes:

$$\tau : S^2 \times S^1 \to S^2 \times S^1 \quad ; \quad \tau(z, \alpha) = (\alpha z, \alpha)$$

and M' is $M - \text{Int}N(S)$ glued with $S^2 \times D^2$ along the boundary using τ. This is known as the Gluck surgery (along an embedded 2-sphere with trivial normal bundle).

(a) Prove that if M is simply connected, then M' is simply connected.

(b) If M is also closed and S is nullhomologus, prove that M' has the same intersection form as M.

Problem 4. Describe the surgery on the torus $S^1 \times S^1$ that kills the fundamental group. Construct a manifold with a homotopy group that cannot be trivialized by a sequence of surgeries. Milnor describes, in *A Procedure for Killing Homotopy Groups of Differentiable Manifolds* various conditions on manifolds allowing for such sequences to exist.

Note: There were a few questions asked about blow-ups that weren't resolved.