A PROOF OF THE GINZBURG-KAZHDAN CONJECTURE: TALK NOTES

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1. Symplectic Implosion

1.1. **Introduction.** In symplectic topology, it is often of interest to study a symplectic (real) manifold $(M, \omega_{\mathbb{R}})$ with an action of a compact Lie group K. Since K is not typically abelian, it can be useful to 'abelianize' your space. Guillemen-Jeffery-Sjamaar [2] constructed a procedure to do just this: using the symplectic geometry of M, they construct a space M_{impl} with an action of a maximal torus $U(1)^\ell$ of K.

The relevant part for our talk is that this construction also admits an algebro-geometric interpretation. Specifically, recall that the complexification of a compact Lie group $G := K_{\mathbb{C}}$ admits a real symplectic form. This algebro-geometric interpretation is modelled on the fact that there is a certain (choice of) subgroup $N \leq G$ (the 'unipotent radical of some Borel') such that G/N admits a right action of a maximal (complex) torus which commutes with the G-action:

Example 1.1. If $K = \mathrm{SU}_n$, then $G \cong \mathrm{SL}_{n,\mathbb{C}}$, and one possible choice of N is the group of strictly upper triangular matrices, i.e. those upper triangular matrices whose diagonal entries are all 1.

The fact that symplectic implosion admits an algebro-geometric description is one of the main theorems of [2]:

Theorem 1.2. The symplectic implosion of the 'universal example' $T^*(K)$ is the *affine closure* $\overline{G/N}$ of G/N, i.e. the spectrum of the ring of functions on G/N. Moreover, for M as above, M_{impl} is the symplectic quotient of $M \times \overline{G/N}$ by the action of the diagonal copy of K.

1.2. **Universal Symplectic Implosion.** Here, the *affine closure* of G/N is the spectrum of its global functions. Because all symplectic implosions are entirely determined by the variety $\overline{G/N}$, the variety $\overline{G/N}$ is sometimes called the *universal symplectic implosion.* We'll discuss this space briefly.

The variety G/N is quasi-affine, and so it is an open subscheme of $\overline{G/N}$. This is a nice space, although usually singular:

Example 1.3. $\overline{\mathrm{SL}_2/N} = \mathbb{A}^2$, $\overline{\mathrm{SL}_3/N} = \mathrm{Spec}(\mathbb{C}[a,b,c,x,y,z]/(ax+by+cz))$.

On the other hand, one can use the Peter-Weyl theorem to stratify $\overline{G/N}$ by G-invariant subsets of the associated Dynkin diagram:

$$\overline{G/N} := \mathrm{Spec}(A) = \bigcup_{\theta} G/[P_\theta, P_\theta] \tag{1}$$

which, for example, gives $\overline{\mathrm{SL}_2/N} = \mathbb{A}^2 \setminus 0 \cup 0$ and $\overline{\mathrm{SL}_3/N}$ is the union of four strata, one being SL_3/N and the smallest being the 'cone point.'

2. Hyperkähler Implosion

In this talk, we'll be studying the holomorphic symplectic analogue of this story. The analogue of the universal symplectic implosion in the following setting is the *universal hyperkähler implosion*:

Definition 2.1. The *universal hyperkähler implosion* is the space $\overline{T^*(G/N)}$, the affine closure of the cotangent bundle $T^*(G/N)$.

Example 2.2. When $G = \mathrm{SL}_2$, $\overline{T^*(G/N)} = \mathbb{A}^4$. When $G = \mathrm{SL}_3$, it's a theorem [3] that $\overline{T^*(G/N)}$ can be identified with the minimal (non-zero) nilpotent orbit of SO_8.

The first work I know of on this subject was done in [1]:

Theorem 2.3. The variety $\overline{T^*(\mathrm{SL}_n/N)}$ admits a description by a certain hyperkähler reduction obtained from a quiver representation. In particular, it is stratified by hyperkähler manifolds.

However, the universal hyperkähler implosion for general reductive groups G (such as SO_n), much less is known. For example, up until this year, essentially the only known result on the geometry of the universal hyperkähler implosion was the following result of Ginzburg-Riche, who studied the space $\overline{T^*(G/N)}$ from the point of view of geometric Langlands duality:

Theorem 2.4. The scheme $\overline{T^*(G/N)}$ is a variety, i.e. the global functions on $T^*(G/N)$ is finitely generated. Moreover, this space admits a generically transitive action of the Weyl group W of G.

The proof of this theorem is extremely interesting and highly nontrivial. It shows that the global functions on $T^*(G/N)$ are not generated by anything obvious like 'the functions on G/N plus the global tangent vector fields.' The proof that the ring of functions is Noetherian uses a highly nontrivial action of the Weyl group on $\overline{T^*(G/N)}$ known as the *Gelfand-Graev* action induced from an analogous W-action on the differential operators on G/N.

Remark 2.5. We take a brief digression to illustrate how tricky the subject of finite generation is. If N above is replaced with the unipotent radical of some *parabolic* subgroup N_P, then if $G = \mathrm{SL}_n$ then the same quiver methods used to prove Theorem 2.3 shows that the ring of functions on $T^*(G/U_P)$ is finitely generated. For general G, this question is still open.

Besides this W-action and this finite generation, though, not much else was known about the geometry of $\overline{T^*(G/N)}$. In particular, there was no known stratification of $\overline{T^*(G/N)}$ by finitely holomorphic symplectic varieties (compatible with the Poisson bracket on the ring of functions). However, in trying to understand this Gelfand-Graev action, Ginzburg and Kazhdan proposed the following conjecture:

Conjecture 2.6. (Ginzburg-Kazhdan '18) The variety $\overline{T^*(G/N)}$ has symplectic singularities for every G.

The main theorem of this talk will be that this theorem is true. Before defining what it means for a variety to have symplectic singularities, we first note that this theorem will give the following consequence due to a theorem of Kaledin:

Corollary 2.7. The variety $\overline{T^*(G/N)}$ is stratified by finitely many smooth locally closed symplectic subvarieties called *symlectic leaves*.

3. Symplectic Singularities

3.1. **Symplectic Singularities via Nilpotent Cone.** The definition of symplectic singularities does not give the best insight into why it is defined the way it is.[1] Instead, we will motivate the study of symplectic duality through one of the first basic examples: the set of nilpotent matrices, referred to as the *nilpotent cone* $\mathcal{N}$. (Experts will note that there is a similar definition for any

[1]We record the definition in a footnote: we say that a normal variety X has *symplectic singularities* if the smooth locus admits a symlectic 2-form and such that for any resolution of singularities $p : Y \to X$, the induced 2-form on the pullback extends to a 2-form on Y.

More algebro-geometrically minded people may prefer the following characterization: a normal variety X has symplectic singularities if and only if if the smooth locus admits a symlectic 2-form and X is Gorenstein and has *rational singularities*, which in our case (for an affine variety X) means that some/any resolution of singularities has no higher Cěch cohomology.

semisimple Lie algebra $\mathfrak{g}$.) The Kostant-Kirillov form gives a symplectic form on $\mathcal{N}$, and moreover it is known that there is a resolution of singularities known as the *Springer resolution* $\tilde{N}$ for which the pullback of this symplectic form extends to all of $\tilde{N}$.

One of the reasons this variety is studied is its relevance to the *BGG Category* $\mathcal{O}_0^{\mathfrak{g}}$, a category of representations of the Lie algebra of G. In the 80's and 90's, representation theorists proved a bunch of 'duality' theorems involving this category. For example:

Theorem 3.1. (Soergel) There is an equivalence of categories $\mathcal{O}_0^{\mathfrak{g}} \cong \mathcal{O}_0^{{}^L\mathfrak{g}}$, where ${}^L\mathfrak{g}$ denotes the Lie algebra of the group Langlands dual to G.

This theorem can be generalized to a ton of equivalences involving Langlands dual groups in representation theory that fall under the general heading of *Koszul duality.*

Remark 3.2. This theorem isn't as interesting for $\mathfrak{sl}_n$ since ${}^L\mathfrak{sl}_n = \mathfrak{sl}_n$, and gives some motivation for why studying these varieties is important for general G.

3.2. **Symplectic Duality in General.** An program due to Braden-Licata-Proudfoot-Webster predicts that the entire story of Section 3.1 only depended on the fact that $\mathcal{N}$ is an affine variety with symplectic singularities which has a conical action of $\mathbb{G}_m$ $(= \mathbb{C}^\times)$ which we will refer to as varieties with *conical symplectic singularities.* A striking feature of this program is its breadth: a Koszul duality equivalence (and more!) is predicted (and in many cases proved) for any of the following pair of varieties:

X	$X^!$
$\mathcal{N}_G$	$\mathcal{N}_{^LG}$
Minimal nilpotent orbit for $\mathfrak{sl}_n$	$\mathbb{C}^2 /\!\!/ (\mathbb{Z}/n\mathbb{Z})$
Hypertoric variety	(Gale) dual hypertoric variety
ADE quiver varieties	Affine Grassmannian slices
$X^!$	X

In fact, these pairs are predicted to come from physics as the *mirror symmetry* of Higgs/Coulomb branches of a 3d $N = 4$ supersymmetric quantum field theory.

4. Main Theorem

Now having motivated the theory of symplectic duality, we now state our main result on the geometry of $\overline{T^*(G/N)}$. Rougly speaking, one can translate facts known from the quiver description of $\overline{T^*(\mathrm{SL}_n/N)}$ to predict what might be true about $\overline{T^*(G/N)}$, and then prove these statements using representation theory. Our main example of this principle is the following: Boming Jia [3] used the quiver description of Theorem 2.3 to show the following:

Theorem 4.1. The variety $\overline{T^*(\mathrm{SL}_n/N)}$ has symplectic singularities 'because the codimension of its singular locus is at least four.'

One can use this to prove the following main theorem of this talk, which confirms the Ginzburg-Kazhdan conjecture:

Theorem 4.2. The variety $\overline{T^*(G/N)}$ has symplectic singularities 'because the codimension of its singular locus is at least four.'

We also show that this variety is conical, so that $\overline{T^*(G/N)}$ fits into the symplectic duality program we discussed above!

References

[1] Andrew Dancer, Amihay Hanany, and Frances Kirwan. "Symplectic duality and implosions". In: *Adv. Theor. Math. Phys.* 25.6 (2021), pp. 1367–1387. ISSN: 1095-0761.

[2] Victor W. Guillemin, Lisa C. Jeffrey, and Reyer Sjamaar. "Symplectic Implosion". In: *Transformation Groups* 7 (2001), pp. 155–185.

[3] Boming Jia. *The Geometry of the affine closure of* $T^*(\mathrm{SL}_n/U)$. 2021. DOI: `10.48550/ARXIV.2112.08649`. URL: `https://arxiv.org/abs/2112.08649`.