The Cotangent Bundle of $G/U_P$ and Kostant-Whittaker Descent

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1. Introduction

The main result of this article, Theorem 1.1, generalizes an ‘implosion’ description of $\mathcal{GK}$ for the functions on the cotangent bundle of the basic affine space of a complex reductive group simultaneously to the parabolic setting and to the modular setting. Using Theorem 1.1 we also derive an implosion description for the functions on the partial Whittaker cotangent bundle of a reductive group, stated precisely in Corollary 2.8 which proves an isomorphism whose existence was conjectured by Devalapurkar [Dev, Conjecture 3.6.15].

Before stating our main theorem, we set some notation. Let $G$ denote some reductive group defined over some algebraically closed field $k$ whose characteristic does not lie in some ‘small’ set of characteristics described explicitly in Section 2.1 (for example, if $G = \text{GL}_n$ we impose no restriction on the characteristic of $k$) and let $P$ denote some parabolic subgroup of $G$. Choose some Levi factor $L$ in $P$, so that we have a semidirect product decomposition $U_P \rtimes L \simeq P$ where $U_P$ is the unipotent radical of $P$. Let $\mathfrak{g}, \mathfrak{p}, \mathfrak{l}$ and $\mathfrak{u}_P$ denote the Lie algebras of their respective groups. We set $\mathfrak{c}_L := \text{Spec}(\text{Sym}(\mathfrak{l}^L))$, and denote by $J_G$ the group scheme of universal centralizers studied in [Ngô, Section 2], which we describe more precisely in Section 2.1. The main result of this article is the following:

**Theorem 1.1.** There is an action of $J_G$ on $G \times \mathfrak{c}_L \times L$ inducing an isomorphism of algebras

$$\mathcal{O}(T^*(G/U_P)) \cong \mathcal{O}(G \times \mathfrak{c}_L \times L)^{J_G}$$

compatible with the natural actions of $G$ and $L$.

In fact, we show slightly more: we show that the right hand side of (1) naturally acquires a $\text{Sym}(\mathfrak{g} \oplus \mathfrak{l})$-algebra structure and show this isomorphism is an isomorphism of $\text{Sym}(\mathfrak{g} \oplus \mathfrak{l})$-algebras.

The ring $\mathcal{O}(T^*(G/U_P))$ has been previously studied due to its appearance in the geometric Langlands program [Mac], [Dev] as well in the study of 3d $N = 4$ supersymmetric gauge theories and their Coulomb branches [BDG⁺], [GW], [DGMZ]. Our result may be useful in studying in the conjectures made in [BDG⁺] and [DGMZ]. Moreover, as we explain further in Remark 2.9 one can use Theorem 1.1 to derive a natural candidate for the dual of the Hamiltonian $G$-variety...

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$T^*(G_c/L_c)$ in the relative Langlands duality program [BZSV], where $L_c$ is a Levi subgroup of $G_c$.

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2. Proof of the Main Theorem

2.1. Notation. We assume $G$ is a pinned reductive group over an algebraically closed field of characteristic $p \geq 0$ which satisfies [Ric] Condition (C4). In particular, we have chosen a maximal torus $T$, a choice of Borel subgroup $\mathcal{B}$ containing $T$, and a simple root vector $f_i$ in each simple root space of $\pi := \text{Lie}(\mathcal{U})$, where $\mathcal{U} := [\mathcal{B}, \mathcal{B}]$. Let $B$ be the opposite Borel to $\mathcal{B}$ which contains $T$. Let $X_*(T)$ denote the lattice of cocharacters and $Z\Phi^\vee$ denote the coroot lattice. By definition, $p$ satisfies condition (C4) if and only if $p$ is good for $G$ in the sense of [Jan] Definition 4.22] (which is automatically satisfied if $p > 5$ or $p = 0$), the quotient $X_*(T)/Z\Phi^\vee$ has no $\ell$-torsion (which in [Ric] Section 2.2] is observed to hold if the derived subgroup of $G$ is simply connected) and there exists a $G$-equivariant isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$.

2.1.1. Parabolic Subgroup Notation. Fix a subset $I \subseteq \Delta$ of the simple roots $\Delta$. Let $P \supseteq B$ denote the standard parabolic subgroup determined by $I$, or, in other words, the smallest closed subgroup scheme containing $B$ whose Lie algebra contains the $f_i$. Let $U_P$ denote the unipotent radical of $P$ and $L$ denote the quotient $P/U_P$. Our pinning gives a Levi decomposition $P \leftarrow U_P \rtimes L$, so that we may identify $L$ as a subgroup of $G$. This semidirect product decomposition in turn induces a semidirect product decomposition $U \cong U_P \rtimes U_L$ where $U$ is the unipotent radical of $B$. In particular, we have a group isomorphism $U/U_P \cong U_L$ where $U_L$ is the unipotent radical of $L$. Letting $\mathcal{P}$ denote the unique parabolic subgroup conjugate to $P$ and containing $\mathcal{B}$ and $\mathcal{U}_P$ denote its unipotent radical, we have a similar semidirect product decomposition $\mathcal{U} \leftarrow \mathcal{U}_P \rtimes \mathcal{U}_L$.

Our pinning gives a canonical identification $\mathcal{U}/[\mathcal{U}, \mathcal{U}] \cong \prod_{\Delta} G_a$, which follows using the isomorphism of [Jan] II.1.7(1)], say, and so using this identification we may define the character $\psi$ as the composite

$$\mathcal{U} \to \mathcal{U}/[\mathcal{U}, \mathcal{U}] \cong \prod_{\Delta} G_a \xrightarrow{\Sigma} G_a$$

where $\Sigma$ is the sum map. Let $\psi_P := \psi|_{\mathcal{U}_P}$ and let $\psi_L : \mathcal{U}/\mathcal{U}_P \to G_a$ denote the character $\psi - \psi_P$, which we regard as a character on $\mathcal{U}_L$.

2.1.2. Kostant Section. We choose, once and for all, a Kostant section

$$\kappa : c := \text{Spec}(\text{Sym}(\mathfrak{g}^G)) \to \mathfrak{g}^* \times_{\mathfrak{g}^*} \{d\psi_c\}$$

which splits the quotient map $\mathfrak{g}^* \to c$, which we may do by for example using our $G$-equivariant isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$ and the results of [Ric] Section 3]. We let $J_G$ denote the centralizer of this Kostant section, which is a group scheme over $c$. As
above, we let \( \mathfrak{c}_L := \text{Spec}(\text{Sym}(L)^L) \), which we always view as a scheme over \( \mathfrak{c} \) by the Chevalley restriction map. Our choice of Kostant section identifies \( J_G \) with the group scheme of universal centralizers studied in [Ngô Section 2], as explained in for example [Ric Section 3.3].

2.1.3. Notation for Whittaker Reduction. Let \( N \) denote some arbitrary unipotent group. If \( X \) is some variety equipped with a free \( N \)-action and a map \( \mu : X \to \mathfrak{n}^* \) compatible with this action, for any (additive) character \( \alpha : N \to \mathbb{C} \) we set

\[
X \sslash \alpha N := (X \times_{\mathfrak{n}^* \{ d\alpha_{\mathfrak{c}} \}}) / N.
\]

where \( d\alpha_{\mathfrak{c}} \) is the induced character on the respective Lie algebras. If \( X = T^*(Y) \) for some \( Y \) with an \( N \)-action, we also set \( T^*(Y/\alpha N) := T^*(Y) \sslash \alpha N \).

2.2. Whittaker Reduction. We recall the following elementary lemma, see also [GK Lemma 3.2.3]:

**Lemma 2.1.** If \( X \) is a scheme with a \( G \)-action and a \( G \)-equivariant map \( X \to \mathfrak{g}^* \), then there is an isomorphism \( X \sslash \psi \mathcal{U} \cong X \times_{\mathfrak{g}^*} \mathfrak{c} \).

**Proof.** By definition, the scheme \( X \sslash \psi \mathcal{U} \) is obtained by the quotient of the scheme

\[
X \times_{\mathfrak{n}^*} \{ d\psi_{\mathfrak{c}} \} \cong X \times_{\mathfrak{g}^*} (\mathfrak{g}^* \times_{\mathfrak{n}^*} \{ d\psi_{\mathfrak{c}} \})
\]

by the diagonal \( \mathcal{U} \)-action. However, the product \( (\mathfrak{g}^* \times_{\mathfrak{n}^*} \{ d\psi_{\mathfrak{c}} \}) \) is a \( \mathcal{U} \)-torsor which is trivialized by \( \kappa \) [Kos2, [Ric Proposition 3.2.1]. Therefore the quotient of this scheme is canonically isomorphic to \( X \times_{\mathfrak{g}^*} \mathfrak{c} \), as required. \( \square \)

Applying Lemma 2.1 for both the action of \( G \) and the action of the Levi, we see that Theorem 1.1 is implied by the following theorem:

**Theorem 2.2.** There is an action of \( J_G \) on \( T^*(\mathcal{U}_L^L \setminus L) \) for which the induced diagonal \( J_G \)-action induces an isomorphism of \( \text{Sym}(\mathfrak{g}) \)-algebras

\[
\mathcal{O}(T^*(G/U_P)) \cong \mathcal{O}(T^*(G/\psi \mathcal{U}) \times_{\mathfrak{c}} T^*(\mathcal{U}_L^L \setminus L))^{J_G}
\]

compatible with the actions of \( G \) and \( L \).

2.3. Kostant-Whittaker Descent. Observe that the natural action of \( G \times G \) on \( G \times \mathfrak{g}^* \) induces an action of \( G \times J_G \) on \( G \times \mathfrak{c} \). In particular, to any \( M \in \text{Rep}(J_G) \) we may define the quasicoherent sheaf

\[
(a_*(\mathcal{O} \times \mathcal{O}(\mathfrak{c})) \otimes_{\mathcal{O}(\mathfrak{c})} M)^{J_G}
\]

on \( \mathfrak{g}_{\text{reg}} \), where \( a : G \times \mathfrak{c} \to \mathfrak{g}_{\text{reg}} \) is the action map. Since \( \mathcal{O}_{G \times \mathfrak{c}} \) is evidently flat as an \( \mathcal{O}(\mathfrak{c}) \)-module, this quasicoherent sheaf canonically acquires a \( G \)-equivariant structure. In this section, we construct the following isomorphism:

**Proposition 2.3.** To any \( \mathcal{F} \in \text{QCoh}(\mathfrak{g}_{\text{reg}})^G \), there is a canonical isomorphism

\[
\mathcal{F} \cong (a_*(\mathcal{O} \otimes a_*(\mathfrak{c})) \kappa^*(\mathcal{F}))^{J_G}
\]

which is an isomorphism of rings if \( \mathcal{F} \) is a ring object in \( \text{QCoh}(\mathfrak{g}_{\text{reg}})^G \).

We do this after first proving the following lemma:

**Lemma 2.4.** The functor \( \kappa^* \) lifts to an exact monoidal equivalence of categories

\[
\kappa^* : \text{QCoh}(\mathfrak{g}_{\text{reg}})^G \cong \mathcal{O}(J_G)\text{-comod}
\]

whose inverse is given by the functor \( M \mapsto (a_*(\mathcal{O} \otimes \mathcal{O}(\mathfrak{c})) M)^{J_G} \).
The fact that such an equivalence of categories exists is standard and is proved, for example, in [Ric, Proposition 3.3.11]; we now explicitly compute its inverse. The action map $a$ is smooth and surjective [Ric, Lemma 3.3.1]. Thus, using the computation of [Ric, Proposition 3.3.11], we see that descent theory gives adjoint equivalences of abelian categories
\[ a^*: \text{QCoh}(\mathfrak{g}_{\text{reg}}^*)^G \leftrightarrow \text{QCoh}(G \times \mathfrak{c})^{G \times J_G} : a_*(-)^{J_G} \]
which thus are in particular exact. It is standard (and not difficult to check) that we have equivalences of abelian categories
\[ (e, \text{id})^*: \text{QCoh}(G \times \mathfrak{c})^{G \times J_G} : a_*(-)^{J_G} \leftrightarrow \text{QCoh}(\mathfrak{c})^{J_G} : p^* \]
where $e : \text{Spec}(k) \to G$ is the identity point and we identify $(e, \text{id})^*$ with $(-)^G$. Therefore these functors are in particular adjoint. Combining these two adjunctions and using the canonical identification of $(e \times \text{id})^* a^*$ with $\kappa^*$ we obtain our desired inverse, as any adjoint to an equivalence of categories gives an inverse. □

Proof of Proposition 2.3. Our desired isomorphism is given by the unit of the monoidal adjunction of Lemma 2.4. Since $\kappa^*$ is a monoidal equivalence of categories, both $\kappa^*$ and its right adjoint are monoidal, and so the unit map induces an isomorphism of ring objects. Explicitly, this isomorphism is given by the composite
\[ (\mathcal{O}_{G \times \mathfrak{c}} \otimes \mathcal{O}_{\mathfrak{c}}(\kappa^*(F)))^{J_G} = (\mathcal{O}_{G \times \mathfrak{c}} \otimes \mathcal{O}_{\mathfrak{c}}(\mathcal{O}_{\mathfrak{g}_{\text{reg}}(\mathfrak{c})} F))^{J_G} \sim (\mathcal{O}_{G \times \mathfrak{c}} \otimes \mathcal{O}_{\mathfrak{g}_{\text{reg}}} F)^{J_G} \sim F \]
where the first equivalence is given by the unit of (3) and the second equivalence is given is the tensor-hom adjunction given by (2). □

2.4. Restriction to the Big Cell. The open embedding
\[ \mathcal{B} := \overline{U}^P P/U_p \hookrightarrow G/U_p \]
induces an embedding of an open subscheme invariant under the $\overline{U}$-action. Thus we obtain an open embedding of the cotangent bundles $T^*(\overline{U}^P P/U_p) \hookrightarrow T^*(G/U_p)$ which respects the induced Hamiltonian $\overline{U}$-action, and so we obtain an open embedding $j : T^*(\overline{U}^P \psi P U_p) \hookrightarrow T^*(\overline{U}^P \psi G/U_p)$.

Proposition 2.5. The map $j$ is an isomorphism.

We prove this after setting some notation which will also be used later. Since multiplication induces a $\overline{U}^P$-equivariant isomorphism
\[ \overline{U}^P \times P/U_p \sim \overline{U}^P P/U_p \]
we have an isomorphism
\[ T^*(\overline{U}^P \psi P \mathcal{B}) \cong T^*(\overline{U}^P \psi P \times P/U_p) \sim T^*(P/U_p) \]
of Hamiltonian $U_L$-varieties. It is standard that $T^*(\overline{U}^P \psi \mathcal{B})$ identifies with the Kostant-Whittaker reduction of the left Hamiltonian $L$-space $T^*(\overline{U}^P \psi_L \mathcal{B})$ by $\psi_L$, and so we obtain an isomorphism
\[ h : T^*(\overline{U}^P \psi \mathcal{B}) \sim T^*(\overline{U}^P \psi_L \mathcal{B}) \]
of right Hamiltonian $L$-varieties.
Proof of Proposition 2.5 Let $\mu_L$ denote the moment map for the right Hamiltonian $L$-action on $T^*(\mathcal{U}_G/U_P)$. Since $U_P/U_P$ is closed under this right $L$-action, the moment map for the right Hamiltonian $L$ on $T^*(\mathcal{U}_G/U_P)$ is $\mu_L \circ j$. Let

$$t : T^*(\mathcal{U}_G/U_P) \to \epsilon_L := \text{Spec}(\text{Sym}(L))$$

denote the composite of $\mu_L$ with the natural map $L^* \to L$. Since any map of torsors is an isomorphism, it suffices to show that $t$ and $tj$ are $L$-torsors.

Using the map $h$ and Lemma 2.1, we see that the map $tj$ is a trivial $L$-torsor. Thus it remains to show that $t$ is an $L$-torsor. First, observe that there is a Cartesian square

$$
\begin{array}{ccc}
T^*(\mathcal{U}_G/U_P) & \rightarrow & T^*(G/U_P) \\
\downarrow & & \downarrow \\
\mathcal{C} & \rightarrow & \mathfrak{g}^* \end{array}
$$

by Lemma 2.1. Moreover, this square can be expanded to the following diagram for which all squares are Cartesian:

$$
\begin{array}{ccc}
T^*(\mathcal{U}_G/U_P) & \rightarrow & T^*(G/U_P) \\
\downarrow & & \downarrow \\
Z & \rightarrow & \mathfrak{g}_P \\
\downarrow & & \downarrow \\
\mathcal{C} & \rightarrow & \mathfrak{g}^* \\
\end{array}
$$

where $\mathfrak{g}_P$ is the (dual) parabolic Grothendieck-Springer resolution, and $Z$ denotes the Cartesian product in the bottom square. We claim that the natural map $Z \to \epsilon_L$ is an isomorphism; since the map $T^*(G/U_P) \to \mathfrak{g}_P$ is an $L$-torsor, this shows that $T^*(\mathcal{U}_G/U_P) \to \epsilon_L$ is an $L$-torsor too, as desired.

The map $\epsilon \to \mathfrak{g}^*$ factors through regular locus $\mathfrak{g}^*_\text{reg}$, and so it suffices to show that there is a Cartesian square

$$
\begin{array}{ccc}
\epsilon_L & \rightarrow & \mathfrak{g}^*_\text{reg} \\
\downarrow & & \downarrow \\
\epsilon & \rightarrow & \mathfrak{g}^*_\text{reg} \\
\end{array}
$$

where $\mathfrak{g}^*_\text{reg} := \mathfrak{g}_P \times_{\mathfrak{g}^*} \mathfrak{g}^*_\text{reg}$. However, there is a Cartesian square

$$
\begin{array}{ccc}
\mathfrak{g}^*_\text{reg} & \rightarrow & \epsilon_L \\
\downarrow & & \downarrow \\
\mathfrak{g}^* \rightarrow & \epsilon \\
\end{array}
$$

where the map $\mathfrak{g}^*_\text{reg} \to \epsilon$ is the characteristic polynomial map (of which $\kappa$ is a section) by [Kos1], [Ric, Remark 3.5.4]. This immediately implies that (4) is Cartesian, as desired. 

\[\square\]

1This proof was developed in discussions with Sanath Devalapurkar.
2.5. Recovering Functions from the Regular Locus. The goal of this section will be to prove the following proposition, which allows us to recover the functions on $T^*(G/U_P)$ from its restriction

$$T^*(G/U_P)_{\text{reg}} := T^*(G/U_P) \times_{\mathfrak{g}^*} \mathfrak{g}_{\text{reg}}^*$$

to the regular locus, and whose proof will occupy the entirety of Section 2.5.

**Proposition 2.6.** The restriction map induces an isomorphism $\mathcal{O}(T^*(G/U_P)) \xrightarrow{\sim} \mathcal{O}(T^*(G/U_P)_{\text{reg}})$.

The cotangent $T^*(G/U_P)$ is well known to be identified with the variety $(G \times (\mathfrak{g}/u_P)^*)/U_P$ and, under this identification, the open subscheme $T^*(G/U_P)_{\text{reg}}$ corresponds to the variety $(G \times (\mathfrak{g}/u_P)_{\text{reg}})^*/U_P$. Thus since $T^*(G/U_P)$ is a smooth (and, in particular, normal) variety, Proposition 2.6 follows immediately from the following lemma:

**Lemma 2.7.** The codimension of the complement of $(\mathfrak{g}/u_P)_{\text{reg}}$ in $(\mathfrak{g}/u_P)$ is at least two.

**Proof.** It suffices to show that the generic point of any irreducible component of the complement of the regular semisimple locus $(\mathfrak{g}/u_P)_{\text{reg}}^*$ is in $(\mathfrak{g}/u_P)_{\text{reg}}$. Assume there is such a nonempty irreducible component $V$; since $\mathcal{O}((\mathfrak{g}/u_P)^*)$ is a polynomial algebra, we may write $V$ as the vanishing set $V(f)$ of some function $f \in \mathcal{O}((\mathfrak{g}/u_P)^*)$. We have

$$V(f|_{(\mathfrak{g}/u)^*}) = V(f) \cap (\mathfrak{g}/u)^* \subseteq \mathfrak{g}^* \mathfrak{g}_{\text{rss}} \cap (\mathfrak{g}/u)^*$$

and the proof of 

BR Proposition 1.9.3] shows that

$$\mathfrak{g}^* \mathfrak{g}_{\text{rss}} \cap (\mathfrak{g}/u)^* = (\mathfrak{g}/u_u)^* \setminus (\mathfrak{g}/u_u)_{\text{rss}}^* \cap (\mathfrak{g}/u)^*$$

for certain $h_i \in \mathcal{O}((\mathfrak{g}/u)^*)$, where $u_u := u + u$. Thus $f|_{(\mathfrak{g}/u)^*}$ is a product of some subset of the $h_i$, and is a nonempty product since the fact that $V$ is nonempty and $\mathcal{O}_m$-invariant implies $0 \in V$ so $f(0) = 0$. Now the proof of BR Proposition 1.9.3] shows that $V(h_i) \cap (\mathfrak{g}/u)^*_{\text{reg}}$ is nonempty for any good prime $p$ and so $V \cap (\mathfrak{g}/u)^*_{\text{reg}}$ is nonempty as well. Thus the generic point of any component of the complement of $(\mathfrak{g}/u_P)^*_{\text{reg}}$ does not lie in $(\mathfrak{g}/u_P)^*_{\text{reg}}$, and so the complement of $(\mathfrak{g}/u_P)^*_{\text{reg}}$ has codimension two, as desired.

2.6. Proof of the Main Theorem. We have equivalences of ring objects in $\text{QCoh}(\mathfrak{g}_{\text{reg}}^*)^G$

$$\mathcal{O}_{T^*(G/U_P)_{\text{reg}}} \xrightarrow{\sim} (\mathcal{O}_{G \times \mathbf{C} \otimes \mathcal{O}(\mathfrak{g}^*)}((\mathcal{O}_{T^*(G/U_P)_{\text{reg}}}))^J_G \xrightarrow{\sim} (\mathcal{O}_{G \times \mathbf{C} \otimes \mathcal{O}(\mathfrak{g}^*)}((\mathcal{T}^* (\mathcal{U}_G \setminus \mathcal{L}_U))))^J_G$$

$$\xrightarrow{\text{id} \otimes j^*} (\mathcal{O}_{G \times \mathbf{C} \otimes \mathcal{O}(\mathfrak{g}^*)}((\mathcal{T}^* (\mathcal{U}_G \setminus \mathcal{L}_U))))^J_G$$

where the first isomorphism is given by Proposition 2.3, the second is given by Lemma 2.1, the third map is an isomorphism by Proposition 2.5, and the map $h$ is an isomorphism by our above analysis. Using the equivalence of Lemma 2.1, we therefore obtain an isomorphism

$$\mathcal{O}_{T^*(G/U_P)_{\text{reg}}} \cong (\mathcal{O}_{T^*(G/U_P)_{\text{reg}}} \otimes \mathcal{O}(\mathfrak{g}^*)((\mathcal{T}^* (\mathcal{U}_G \setminus \mathcal{L}_U))))^J_G$$

of quasicoherent sheaves of $\text{Sym}(\mathfrak{g} \oplus \mathbf{L})$-algebras compatible with the actions of $G$ and $L$. Taking global sections, we deduce our desired equivalence from the fact that the open embedding $\mathfrak{g}_{\text{reg}}^* \to \mathfrak{g}^*$ is qcqs and so $j_*$ preserves limits and Proposition 2.6.
2.7. Corollary on Partial Whittaker Cotangent Bundle. Let $\hat{U} := U_P U_L$, and let $\hat{\psi}_L : \hat{U} \to \mathbb{G}_a$ denote the unique character whose kernel contains $U_P$ and which extends $\psi_L$. We now record the following corollary on the functions on the partial Whittaker cotangent bundle $T^*(G/\hat{\psi}_L \hat{U})$:

**Corollary 2.8.** There is an action of $J_G$ on $J_L$ for which the induced diagonal $J_G$-action induces an isomorphism of $\text{Sym}(g)$-algebras

$$\mathcal{O}(T^*(G/\hat{\psi}_L \hat{U})) \cong \mathcal{O}(T^*(G/\hat{\psi}\hat{U}) \times_{\mathcal{C}} J_L)^{J_G}$$

compatible with the actions of $G$ and $J_L$.

**Proof.** Our isomorphism in Theorem 2.2 is an isomorphism in particular compatible with the $\text{Sym}(l)$-algebra structure and the $L$-representation. We may thus apply the Kostant-Whittaker reduction functor for $L$. Since the Kostant-Whittaker reduction functor is exact by Lemma 2.4, we obtain an isomorphism

$$\mathcal{O}(T^*(G/\hat{\psi}_L \hat{U})) \cong \mathcal{O}(T^*(G/\hat{\psi}\hat{U}) \times_{\mathcal{C}} J_L)^{J_G}$$

compatible with the $\text{Sym}(g)$-algebra and the actions of $G$ and $J_L$. Using Lemma 2.1 and the fact that the diagonal map $\mathcal{C}_L \to \mathcal{C}_L \times_{\mathcal{C}_L} \mathcal{C}_L$ is an isomorphism, we may identify $T^*(\mathcal{U}_L \backslash \mathcal{U}_{\hat{\psi}_L})$ with the centralizer of some choice of Kostant section for $L$. By, for example, [Ngô Section 2.1], [Ric Remark 3.3.10], this identifies with $J_L$, as desired.

**Remark 2.9.** For the ease of exposition, let us assume $k = \mathbb{C}$. Let $G^\vee$ denote the Langlands dual group to $G$, and let $L^\vee$ and $U^\vee$ denote the corresponding subgroups of $G^\vee$. Corollary 2.8 provides a natural candidate for the dual Hamiltonian $G^\vee$-space $M^\vee$ for the (not necessarily hyperspherical) Hamiltonian $G$-variety $T^*(G/L)$ in the relative Langlands duality program [BZSV]. Namely, $M^\vee$ is the affine closure of the quasi-affine variety $T^*(G^\vee/\hat{\psi}_L \hat{U}^\vee)$. See [Dev Conjecture 3.6.15] and the surrounding discussion in loc. cit for further discussion.

**References**


