CLASSIFICATION OF NONDEGENERATE G-CATEGORIES

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APPENDIX WITH GERMÁN STEFANICH

Abstract. We classify a 'dense open' subset of categories with an action of a reductive group, which we call nondegenerate categories, entirely in terms of the root datum of the group. As an application of our methods, we also:

(1) Upgrade an equivalence of Ginzburg and Lonergan, which identifies the category of bi-Whittaker $\mathcal{D}$-modules on a reductive group with the category of $\tilde{W}^{\text{aff}}$-equivariant sheaves on a dual Cartan subalgebra $\mathfrak{t}^*$ which descend to the coarse quotient $\mathfrak{t}^* /\!\!/ \tilde{W}^{\text{aff}}$, to a monoidal equivalence (showing that the Whittaker-Hecke category is symmetric monoidal and answering a question of Drinfeld) and
(2) Show the parabolic restriction of a very central sheaf acquires a Weyl group equivariant structure such that the associated equivariant sheaf descends to the coarse quotient $\mathfrak{t}^* /\!\!/ \tilde{W}^{\text{aff}}$, proving a modified conjecture of Ben-Zvi–Gunningham.

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Contents

1. Introduction

1.1. **Main Results.** Much of modern geometric representation theory can be interpreted as the study of groups acting on *categories* and the natural symmetries that the various invariants obtain; we will see specific examples of this in Section 1.2. Therefore it is of natural interest to study the class of all categories with an action of a split reductive group G. Our main theorem in this paper provides a 'coherent' description for a localized class of G-categories known as *nondegenerate categories.*

Definition 1.1. Assume G is simply connected. A *nondegenerate* G-category is a G-category $\mathcal{C}$ such that for every rank one parabolic P_α, the invariants $\mathcal{C}^{[P_\alpha,P_\alpha]}$ vanish.

We study some of the basic properties of nondegenerate G-categories in the companion paper [Gan23]. For example, we argue that any G-category $\mathcal{C}$ admits a functor $\mathcal{C} \to \mathcal{C}_{\text{nondeg}}$ that, informally speaking, has the same properties as the map $j^! : \mathcal{D}(X) \to \mathcal{D}(U)$ for an open subset $j : U \hookrightarrow X$. This is made precise in [Gan23, Section 2.4.5].

Our main result in this paper states that the 2-category of nondegenerate G-categories admits a coherent description as modules over sheaves on an ind-scheme $\Gamma_{\tilde{W}^{\text{aff}}}$ defined only in terms of the action of the extended affine Weyl group $\tilde{W}^{\text{aff}} := X^\bullet(T) \rtimes W$ on $\mathfrak{t}^*$. As ind-schemes, we have $\Gamma_{\tilde{W}^{\text{aff}}} \simeq \pi_1(G^\vee) \times \Gamma_{W^{\text{aff}}}$, where $G^\vee$ denotes the Langlands dual group and $\Gamma_{W^{\text{aff}}}$ denotes the union of graphs in $\mathfrak{t}^* \times \mathfrak{t}^*$ given by the W^{aff}-action on $\mathfrak{t}^*$. One can identify $\Gamma_{\tilde{W}^{\text{aff}}} \simeq \mathfrak{t}^* \times_{\mathfrak{t}^* /\!\!/ \tilde{W}^{\text{aff}}} \mathfrak{t}^*$ for a certain prestack $\mathfrak{t}^* /\!\!/ \tilde{W}^{\text{aff}}$ known as the *coarse quotient*, a main object of study of [Gan22]. This implies that one can use the convolution formalism of [GR17a, Section 5.5] to equip $\text{IndCoh}(\Gamma_{\tilde{W}^{\text{aff}}})$ with a monoidal structure. Our main result can be summarized as follows:

Theorem 1.2. There is an equivalence of 2-categories

$$G\text{-mod}_{\text{nondeg}} \simeq \text{IndCoh}(\Gamma_{\tilde{W}^{\text{aff}}})\text{-mod}$$

where the left hand side denotes the 2-category of all nondegenerate G-categories.

As we will review in Section 1.5, this result can reinterpreted as an equivalence of monoidal categories $\mathcal{D}(N\backslash G/N)^{T\times T,w}_{\text{nondeg}} \simeq \text{IndCoh}(\Gamma_{\tilde{W}^{\text{aff}}})$; see Theorem 1.14 for the full statement.

A key difference in the theory of groups G acting on categories $\mathcal{C}$ from the theory of groups acting on vector spaces is the existence of nontrivial maps between invariants $\mathcal{C}^{H_1}$ and $\mathcal{C}^{H_2}$, where each H_i is a closed subgroup of G (and, in particular, we do not require $H_1 \subseteq H_2$ or vice versa). These relations will prove a key technical tool in the proof of our theorem, and are summarized in Section 1.4. In particular, we will see in Theorem 1.9 that, for any G-category, the category $\mathcal{C}^N_{\text{nondeg}}$ admits a W-action, and that there is a fully faithful functor $\mathcal{C}^{N^-,\psi} \hookrightarrow \mathcal{C}^{N,W}_{\text{nondeg}}$.

When $\mathcal{C}$ itself is given by Whittaker $\mathcal{D}$-modules on G, one can show [Gan23, Corollary 3.4] that $\mathcal{C}^N \simeq \mathcal{C}^N_{\text{nondeg}}$, and, using this special case of Theorem 1.9, in Theorem 1.4 we derive a monoidal equivalence between the category of bi-Whittaker $\mathcal{D}$-modules on G and $\tilde{W}^{\text{aff}}$-equivariant sheaves on $\mathfrak{t}^*$ which descend to the coarse quotient, providing a monoidal upgrade of [Gin18] and [Lon18]. Using nondegeneracy, we also show that the parabolic restriciton of a very central $\mathcal{D}$-module acquires a W-equivariant structure such that the sheaf (with its equivariance) descends to the coarse quotient $\mathfrak{t}^* /\!\!/ \tilde{W}^{\text{aff}}$, see Section 1.6.

1.2. **Motivation and Survey of Known Results.** Assume we are given a finite dimensional vector space V over an algebraically closed field k, equipped with an endomorphism $T : V \to V$. A familiar paradigm in representation theory and algebraic geometry is to regard V as a module over the ring $k[x]$, where x acts by the transformation T, and to write V as a direct sum of its generalized eigenspaces V_α. Furthermore, the vector space V can be recovered from the various V_α. We may equivalently view V as a sheaf over the line, and then each V_α can be identified as the subsheaf which lives over α. This particular example gives the well known Jordan normal form of a matrix, but there are analogues of this process for any k-algebra A and any module $M \in \text{QCoh}(\text{Spec}(A))$.

We can also apply this idea to other representation theoretic contexts. For example, let $\mathfrak{g}$ be a semisimple Lie algebra, and let M be a representation of the Lie algebra. Then it is known (see eg [Hum08]) that $Z\mathfrak{g}$ is a polynomial algebra, and furthermore that we may identify $\text{Spec}(Z\mathfrak{g}) \simeq \mathfrak{t}^* /\!\!/ W$. Therefore we may spectrally decompose a given $\mathfrak{g}$-representation by viewing it as a sheaf on the space $\mathfrak{t}^* /\!\!/ W$.

We will discuss analogues for these results one categorical level higher. Specifically, our notion of vector space will be replaced with that of a category. The analogue of an algebraic group acting on a vector space is a *group acting on a category.* For example, if G acts on a variety X, then the category $\mathcal{D}(X)$, the category of $\mathcal{D}$-modules on X, obtains a canonical G-action. Similarly, we can obtain a G-action on the category $\mathfrak{g}$-mod.

Analogous to the case of a group acting on a vector space, we can define the invariants of a group acting on a category. For example, one can understand representations of the Lie algebra $\mathfrak{g}$ of a semisimple algebraic group G via the invariants $\text{Rep}(G) \simeq \mathfrak{g}\text{-mod}^G$, or the associated zero block of the BGG category $\mathcal{O}$, which can be viewed (via the Beilinson-Bernstein localization theorem) as objects of $\mathcal{D}(G/B)^N$. Similarly, one can also study the other blocks of category $\mathcal{O}$ via the *twisted invariants* $\mathcal{D}(G/_{\lambda}B)^N$ for $\lambda \in \mathfrak{t}^*$.

Certain twisted invariants play a special role in geometric representation theory. Specifically, for a reductive group G acting on a category $\mathcal{C}$, one can take the *Whittaker invariants* $\mathcal{C}^{N^-,\psi}$. This category can be interpreted as the generically twisted N^- invariants of $\mathcal{C}$–the specific definition is given below. Often, Whittaker subcategories can be easier to understand than the usual N-invariants. For example, we have seen above that one may identify $\mathcal{D}(G/B)^N$ contains all of the information of the BGG category $\mathcal{O}_0$, whereas one can use the ideas of [BBM04] discussed below to show that $\mathcal{D}(G/B)^{N^-,\psi} \simeq \text{Vect}$.

The Whittaker invariants of a category have often been used to 'bootstrap' information about the original category, see, for example, [AB09] or [BY13]. In fact, our results below can be viewed as an attempt to generalize the work done by [BY13] at generalized central character 0 to the setting of varying central character. One can formally argue that the category of *bi*-Whittaker invariants of $\mathcal{D}(G)$, denoted $\mathcal{H}_{\psi} := \mathcal{D}(N^-_{\psi}\backslash G/_{-\psi}N^-)$, acts on the Whittaker invariants of any category with a G-action. It is therefore of interest to determine an explicit description for $\mathcal{H}_{\psi}$. This was identified in terms of sheaves on $\mathfrak{t}^*$ which are equivariant with respect to the extended affine Weyl group $\tilde{W}^{\text{aff}} := X^{\bullet}(T) \rtimes W$ for the Langlands dual group, where $X^{\bullet}(T)$ is the character lattice $\text{Hom}_{\text{AlgGp}}(T, \mathbb{G}_m)$:

Theorem 1.3. ([Lon18], [Gin18]) There is an equivalence identifying the abelian category of bi-Whittaker $\mathcal{D}$-modules on G with the abelian category of $\tilde{W}^{\text{aff}}$-equivariant quasicoherent sheaves on $\mathfrak{t}^*$ which descend to the coarse quotient $\mathfrak{t}^* /\!\!/ \tilde{W}^{\text{aff}}$.

Ginzburg [Gin18] and Ben-Zvi–Gunningham [BG17, Section 1.2] also recorded the expectation that a derived, monoidal variant of Theorem 1.3 should hold (see Section 2.1 for our exact categorical conventions). To state this precisely, we first recall the notion of the *Mellin transform*, a symmetric monoidal, W-equivariant equivalence $\text{FMuk} : \text{IndCoh}(\mathfrak{t}^*/X^{\bullet}(T)) \xrightarrow{\sim} \mathcal{D}(T)$, where the notation follows [Lur18a]. Here, we use ind-coherent sheaves rather than quasi-coherent sheaves since our $\mathcal{D}$-modules are right $\mathcal{D}$-modules in the sense of [GR], although since T is smooth, there is also a similar equivalence for left $\mathcal{D}$-modules $\text{QCoh}(\mathfrak{t}^*/X^{\bullet}(T)) \xrightarrow{\sim} \mathcal{D}^{\ell}(T)$. With this, we can now state the derived, monoidal version of Theorem 1.3:

Theorem 1.4. There is a monoidal, t-exact, fully faithful functor $\widetilde{\mathrm{Av}}_* : \mathcal{H}_\psi \hookrightarrow \mathcal{D}(T)^W$. Under the Mellin transform, this functor induces monoidal equivalence F' such that the following diagram commutes

$$\begin{array}{ccc} \mathrm{IndCoh}(\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}) & \xrightarrow{\pi^!} & \mathrm{IndCoh}(\mathfrak{t}^*/\tilde{W}^{\mathrm{aff}}) \\ \downarrow{\scriptstyle F'} & & \downarrow{\scriptstyle \mathrm{FMuk}} \\ \mathcal{H}_\psi & \xrightarrow{\widetilde{\mathrm{Av}}_*} & \mathcal{D}(T)^W \end{array}$$

and such that $F'[\dim(\mathfrak{t}^*)]$ is t-exact.

Remark 1.5. The heart of any t-exact functor of DG categories (or, more generally, triangulated categories) equipped with t-structures is an exact functor of abelian categories, see [BBD82, Proposition 1.3.17]. Therefore, taking the heart of the equivalence in Theorem 1.4, our methods show that there is an *exact* equivalence of the abelian categories in Theorem 1.3.

Remark 1.6. The t-exactness of $\widetilde{\mathrm{Av}}_*$ is essentially due to Ginzburg [Gin18, Theorem 1.5.4]. Moreover, as we will see below, the composite functor

$$\mathcal{H}_\psi \xrightarrow{\widetilde{\mathrm{Av}}_*} \mathcal{D}(T)^W \xrightarrow{\mathrm{oblv}^W} \mathcal{D}(T)$$

can be identified, up to cohomological shift, with an averaging functor Av_*^N, where oblv^W denotes the forgetful functor.

Of course, our proof of Theorem 1.4 is different than the proofs of [Gin18] or [Lon18]. For example, the ideas in [Lon18] pass through the geometric Satake equivalence, whereas we do not. We view our proof as closer in spirit to the proof of [Gin18]; for example, both use versions of the *Gelfand-Graev action*, see [Gan23, Section 3]. However, the idea to use the groupoid $\Gamma_{\tilde{W}^{\mathrm{aff}}}$ is taken from [Lon18] and [BG17].

1.3. **Symmetric Monoidality of Whittaker-Hecke Category.** Much of our proof of Theorem 1.4 is phrased in the language of categorical representation theory. This can provide another conceptual explanation for Theorem 1.4 which makes certain aspects of this equivalence follow from general, categorical principles. For example, in [Gin18], Ginzburg, following Drinfeld, noted that a derived version of Theorem 1.3 would have the following consequence, proved by Ben-Zvi and Gunningham shortly after the first edition of [Gin18] was published:

Corollary 1.7. [BG17, Corollary 6.15] The convolution monoidal stucture on $\mathcal{H}_\psi$ can be upgraded to a symmetric monoidal structure.

The proof of Corollary 1.7 in [BG17] is somewhat indirect. Specifically, the authors prove that the *cohomologically sheared* (or *asymptotic*) version of the Whittaker-Hecke category $\mathcal{H}_\psi^\hbar$ is symmetric monoidal using the derived, loop rotation equivariant geometric Satake theorem of [BF08], and argue that one can obtain a symmetric monoidal structure on $\mathcal{H}_\psi$ by unshearing (see [BG17, Section 5.3]) a graded lift of $\mathcal{H}_\psi^\hbar$ provided by a mixed version of the derived, loop rotation equivariant geometric Satake theorem of [BF08], which is not currently available in the literature.[1]

The fully faithfulness of Theorem 1.4 provides an alternate proof of the symmetric monoidality of $\mathcal{H}_\psi$ which is more direct. Specifically, because we will see that the functor $\widetilde{\mathrm{Av}}_*$ is monoidal in Proposition 3.4, the fully faithfulness of $\widetilde{\mathrm{Av}}_*$ immediately implies Corollary 1.7, since we can identify $\mathcal{H}_\psi$ as a monoidal subcategory of a symmetric monoidal category.

1.4. **Generalization to Nondegenerate G-categories.** The principles of categorical representation theory will also allow us to prove the following generalization of Theorem 1.4 to all nondegenerate G-categories (discussed above in Section 1.1), see Theorem 1.9. Specifically, note that we may interpret the symmetric monoidality of Theorem 1.4 as a statement regarding spectrally decomposing categories with a G-action. For example, Theorem 1.4 says that if $\mathcal{C}$ is a category with a G-action, then for each $[\lambda] \in \mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}$, we may consider the eigencategories of its Whittaker invariants $(\mathcal{C}^{N^-,\psi})_{[\lambda]}$. However, some categories do not admit Whittaker invariants. For example, one can show that $\mathrm{Vect}^{N^-,\psi} \simeq 0$, see [Gan23, Example 2.41].

On the other hand, work of [BGO20] (which we will summarize below in Theorem 1.13) shows that if $\mathcal{C}$ is any G-category, the subcategory $\mathcal{C}^N$ (with its natural symmetries) determines $\mathcal{C}$. It is therefore of interest to relate the N-invariants of a category to the Whittaker invariants. To do this, we recall the following well known result, which we provide a proof for the sake of completeness in [Gan23]:

Proposition 1.8. The restriction functor provides a canonical right T-equivariant equivalence of categories

$$\mathcal{D}(G/N)^{N^-,\psi} \xrightarrow{\sim} \mathcal{D}(N^-B/N)^{N^-,\psi} \simeq \mathcal{D}(T).$$

Therefore, we may reinterpret the statement of Theorem 1.4 in the language of groups acting on categories. Specifically, Theorem 1.4 in fact says that for the left G-category $\mathcal{C} := \mathcal{D}(G)^{N^-,-\psi}$, the averaging functor $\mathrm{Av}_*^N : \mathcal{C}^{N^-,\psi} \to \mathcal{C}^N$, after applying cohomological shift, lifts to a fully faithful, t-exact functor:

$$\mathcal{C}^{N^-,\psi} \xhookrightarrow{\widetilde{\mathrm{Av}}_*} \mathcal{C}^{N,W}. \tag{1}$$

Now let $\mathcal{C}$ be any G-category. Since $\mathcal{C}^N$ determines $\mathcal{C}$, one may ask whether a similar technique can be applied. Unfortunately, for example, in the universal case $\mathcal{C} = \mathcal{D}(G)$, the category $\mathcal{C}^N$ is not expected to admit

[1] For some progress in this direction, see the results announced in [HL22].

a natural W-action. However, as we have shown in [Gan23, Corollary 3.37], W *does* act on any nondegenerate G-category $\mathcal{C}$, and moreover for such $\mathcal{C}$ we will show that the analogue of Theorem 1.4 holds:

Theorem 1.9. For any nondegenerate G-category $\mathcal{C}$, the category $\mathcal{C}^N$ acquires a canonical W-action and there is an induced, fully faithful functor $\widetilde{\mathrm{Av}}_* : \mathcal{C}^{N^-,\psi} \hookrightarrow \mathcal{C}^{N,W}$.

Example 1.10. Let $\mathcal{C} = \mathcal{D}(G)^{N^-,-\psi}$ with its canonical left G-action. Then $\mathcal{C}$ is nondegenerate, see [Gan23, Corollary 3.4]. Furthermore, by Proposition 1.8 we have $C^N \simeq \mathcal{D}(T)$ and a result of Ginzburg's ([Gin18, Proposition 5.5.2]) states that this isomorphism is W-equivariant. Thus a special case of Theorem 1.9 gives the fully faithfulness statement in Theorem 1.4.

1.5. **A Universal Nondegenerate G-category.** Recall that if H is any algebraic group acting on a category $\mathcal{C}$, we may also define its *weak invariants*. This is defined by forgetting the action of the category $\mathcal{D}(H)$ down to an action of $\mathrm{IndCoh}(H)$ and taking invariants of $\mathcal{C}$ as an $\mathrm{IndCoh}(H)$-module category. The notion of weak invaraints is specific to groups acting on categories (as opposed to vector spaces). Moreover, for any discrete group F the data of a weak action is equivalent to a strong action since the forgetful functor $\mathrm{oblv} : \mathcal{D}(F) \to \mathrm{IndCoh}(F)$ is an equivalence.

Example 1.11. The category $\mathcal{D}(G)^{G,w} \simeq \mathfrak{g}\text{-mod}$, while $\mathcal{D}(G)^G \simeq \mathrm{Vect}$. We also note that the category $\mathcal{D}(G)$ obtains two commuting G-actions (one from the left action of G on itself and one from the right). Therefore, we may define the category $\mathcal{D}(G)^{G\times G,w}$, and this category identifies with the *Harish-Chandra category* HC_G the category of $U\mathfrak{g}$-bimodules with an integrable diagonal action. Note we also see from this example a natural way to interpret the G-action on $\mathfrak{g}$-mod.

Example 1.12. The category $\mathfrak{g}$-mod acquires a G-action, and so, in particular, the category $\mathfrak{g}\text{-mod}^N$ acquires a $T \cong B/N$ action. We can identify the category $\mathfrak{g}\text{-mod}^{N,(T,w)}$ with the *universal category* $\mathcal{O}$, see [KS]. We survey and study the connections to the BGG category $\mathcal{O}$ in much more detail in [Gan23]. In particular, we show there that the left adjoint to the functor $\widetilde{\mathrm{Av}}_*$ at a fixed central character can be identified with an enhanced version of Soergel's functor $\mathbb{V}$. Thus, as explained in more detail in [Gan23, Section 1], this gives one interpretation of the left adjoint of $\widetilde{\mathrm{Av}}_*$ in the universal case–it is an analogue of Soergel's $\mathbb{V}$ which does not require a fixed character.

The following theorem then states that a category $\mathcal{C}$ with a G-action can be recovered from $\mathcal{C}^{N,(T,w)}$ with its natural symmetries.

Theorem 1.13. [BGO20, Theorem 1.2] The monoidal categories $\mathcal{D}(G)$, $\mathcal{D}(N\backslash G/N)$, and $\mathcal{D}(N\backslash G/N)^{T\times T,w}$ are all Morita equivalent.

Therefore, to understand results on G-categories, it suffices to understand the monoidal category $\mathcal{D}(N\backslash G/N)^{T\times T,w}$. In particular, via application of

Theorem 1.3 of [BGO20], we may similarly understand nondegenerate G-categories via understanding the localized monoidal category $\mathcal{D}(N\backslash G/N)^{T\times T,w}_{\text{nondeg}}$.

We are now in a position to recast Theorem 1.2 as an equivalence of monoidal categories:

Theorem 1.14. There are monoidal equivalences of categories

$$\mathcal{D}(N\backslash G/N)^{T\times T,w}_{\text{nondeg}} \simeq \text{IndCoh}(\Gamma_{\tilde{W}^{\text{aff}}}) \simeq \text{IndCoh}(\mathfrak{t}^* \times_{\mathfrak{t}^*/\!\!/\tilde{W}^{\text{aff}}} \mathfrak{t}^*)$$

$$\mathcal{D}(N\backslash G/N)_{\text{nondeg}} \simeq \text{IndCoh}(\mathfrak{t}^*/X^\bullet(T) \times_{\mathfrak{t}^*/\!\!/\tilde{W}^{\text{aff}}} \mathfrak{t}^*/X^\bullet(T))$$

which are t-exact up to cohomological shift.

In particular, the formalism of [BG17, Theorem 1.1] applies[2] and we obtain an $\mathbb{E}_2$ functor $\text{IndCoh}(\mathfrak{t}^* /\!\!/ \tilde{W}^{\text{aff}}) \to \mathcal{Z}(\mathcal{D}(G)_{\text{nondeg}}) \simeq \mathcal{D}(G)^G_{\text{nondeg}}$, where given a monoidal category $\mathcal{A}$, $\mathcal{Z}(\mathcal{A})$ denotes its *center* $\mathcal{Z}(\mathcal{A}) := \underline{\text{End}}_{\mathcal{A}\times\mathcal{A}}(\mathcal{A})$. Using this, we may consider the eigencategories for any nondegenerate G-category over the category $\text{IndCoh}(\mathfrak{t}^* /\!\!/ \tilde{W}^{\text{aff}})$, see [BG17, Section 2.8.2].

Remark 1.15. The statement of Theorem 1.14 can be interpreted at the level of abelian categories as follows, which we state for G adjoint type for the ease of exposition. Let $\Gamma_{W^{\text{aff}}}$ be the union of the graphs of the affine Weyl group W^{aff}. Then $\Gamma_{W^{\text{aff}}}$ is an ind-scheme, and so, in particular, every compact object in $\text{IndCoh}(\Gamma_{W^{\text{aff}}})$ can be realized as the pushforward $i^{\text{IndCoh}}_{S,*}(\mathcal{F}_S)$ for $i_S : \Gamma_S \to \Gamma_{W^{\text{aff}}}$ the closed embedding of the union of some finite collection graphs of the affine Weyl group, and $\mathcal{F}_S$ an object of the abelian category of coherent sheaves on Γ_S [GR17d, Chapter 3, Section 1]. Therefore, every object of $\mathcal{D}(N\backslash G/N)^{T\times T,w,\heartsuit}$ admits a quotient which can be viewed as a filtered colimit of such sheaves.

Remark 1.16. We can also interpret nondegeneracy as a localization of 2-categories

$$\mathcal{D}(G)\text{-mod} \to \mathcal{D}(G)\text{-mod}_{\text{nondeg}}.$$

This perspective may prove useful in the local geometric Langlands correspondence, which studies twisted representations of the *loop group*. Our localization can be interpreted as an upgraded version of the functor $\mathcal{C} \mapsto \text{Whit}(\mathcal{C})$. The functor Whit is of importance to the local geometric Langlands program, see [Ras18]. However, one does not need knowledge of this program for the results below.

Remark 1.17. This result, along with Theorem 1.2, admits an interpretation in the theory of 2 ind-coherent sheaves, in upcoming work of Arinkin-Gaitsgory and di Fiore-Stefanich [DS]. In this vein, an informal interpretation of Theorem 1.2 is that we can identify a generic part of G-categories as free of rank one over $\text{IndCoh}(\mathfrak{t}^* /\!\!/ \tilde{W}^{\text{aff}})$, and furthermore we have complete understanding of the singular support behavior which can occur.

[2]For adjoint G, this particular example for the category $\text{IndCoh}(\Gamma_{\tilde{W}^{\text{aff}}})$ is, in fact, given in [BG17, Section 2.7.3]. The new input is here is providing a description of $\text{IndCoh}(\Gamma_{\tilde{W}^{\text{aff}}})$ in terms of $\mathcal{D}$-modules on G, see Theorem 1.14.

Remark 1.18. In [BG17], a $\mathbb{E}_2$-functor $\text{Ngô}_\hbar$ from the cohomologically sheared Whittaker-Hecke category $\mathcal{H}_\psi^\hbar$ to the cohomologically sheared category $\mathcal{D}_\hbar(G/G)$ is constructed using the derived, loop rotation equivariant geometric Satake of [BF08]. Using this, the authors also sketch an argument that there is an $\mathbb{E}_2$-functor $\text{Ngô} : \mathcal{H}_\psi \to \mathcal{D}(G/G)$. Assuming the existence of such an $\mathbb{E}_2$-functor, we would obtain that *all* G-categories diagonalize over $\mathfrak{t}^* /\!/ \tilde{W}^{\text{aff}}$. The idea that such categories with a strong G-action should diagonalize over $\mathfrak{t}^* /\!/ \tilde{W}^{\text{aff}}$ is implicitly used in the proofs below, and was inspired by [BG17].

1.6. **Equivariance on Very Central $\mathcal{D}$-modules on G.** Using the ideas of categorical representation theory, we can also provide some evidence for a recent conjecture of Ben-Zvi and Gunningham on the essential image of enhanced parabolic restriction, which we state explicilty below after recalling some preliminaries.

1.6.1. *The Horocycle Functor and Parabolic Restriction.* Consider the category

$$\mathcal{Z}(\mathcal{D}(G)) := \underline{\text{End}}_{G\times G}(\mathcal{D}(G)) \simeq \mathcal{D}(G)^G.$$

Here, as with all invariants in this subsection, G is acting via the adjoint action. This category is canonically the center of all categories with a G-action. Associated to it is a functor known as *parabolic restriction* $\text{Res} : \mathcal{D}(G)^G \to \mathcal{D}(T)^T$. For an excellent survey on parabolic restriction in many of its guises in representation theory, see [KS]. We will define parabolic restriction in terms of a related functor, known as the *horocycle functor* hc, which is defined as the composite:

$$\mathcal{D}(G)^G \xrightarrow{\text{oblv}_B^G} \mathcal{D}(G)^B \xrightarrow{\text{Av}_*^{N\times N}} \mathcal{D}(N\backslash G/N)^T.$$

Let $i : N\backslash B/N \hookrightarrow N\backslash G/N$ denote the closed embedding.

Definition 1.19. The *parabolic restriction* functor is the composite

$$\mathcal{D}(G)^G \xrightarrow{\text{hc}} \mathcal{D}(N\backslash G/N)^T \xrightarrow{i^!} \mathcal{D}(T)^T.$$

It was proved that parabolic restriction is t-exact in [BY]. In particular, parabolic restriction induces an exact functor of abelian categories

$$\text{Res} : \mathcal{D}(G)^{G,\heartsuit} \to \mathcal{D}(T)^{T,\heartsuit}.$$

These abelian categories can often be easier to work with than their corresponding derived counterparts. For example, a standard argument (see, for example [Ras20a, Section 10.3]) shows the forgetful functor identifies $\mathcal{D}(G)^{G,\heartsuit}$ as a full abelian subcategory of $\mathcal{D}(G)^\heartsuit$ for G any connected algebraic group. On the other hand, $\mathcal{D}(G)^G$ is not the derived category of its heart for any nontrivial reductive G, see [Lur17, Proposition 1.3.3.7, Dual Version] for the particular property which fails.

Let Ind denote the left adjoint to parabolic restriction, known as *parabolic induction*. At the level of abelian categories, it was shown in [Che20, Section 3.2] that if $\mathcal{F} \in \mathcal{D}(T)^{W,\heartsuit} \simeq \mathcal{D}(T)^{T\rtimes W,\heartsuit}$, then the sheaf

$$\mathrm{Ind}(\mathrm{oblv}^W(\mathcal{F})) \in \mathcal{D}(G)^{G,\heartsuit}$$

acquires a canonical W-representation functorial in $\mathcal{F}$. Using this, it is standard to show that one can lift parabolic restriction to a functor

$$\mathrm{WRes} : \mathcal{D}(G)^{G,\heartsuit} \to \mathcal{D}(T)^{W,\heartsuit}$$

which Ginzburg computed explicitly and showed identifies $\mathcal{D}(T)^{W,\heartsuit}$ with a quotient category of $\mathcal{D}(G)^{G,\heartsuit}$ in [Gin22, Theorem 4.4].

1.6.2. *Very Central $\mathcal{D}$-Modules.* While parabolic restriction in general has many interesting properties, it is *not* monoidal in general. However, a standard argument (see (7) below) gives that the horocycle functor *is* monoidal. This suggests that a distinguished role is played by those sheaves $\mathcal{F} \in \mathcal{D}(G)^{G,\heartsuit}$ for which one can recover $\mathrm{hc}(\mathcal{F})$ from $\mathrm{Res}(\mathcal{F})$, which, following [BG17], we call *very central*:

Definition 1.20. We say a sheaf $\mathcal{F} \in \mathcal{D}(G)^{G,\heartsuit}$ is *very central* if $\mathrm{oblv}^T \circ \mathrm{hc}(\mathcal{F}) \in \mathcal{D}(N\backslash G/N)$ is supported on $N\backslash B/N$.

We let $\mathcal{V}$ denote the category of very central $\mathcal{D}$-modules, which is an abelian category by the t-exactness of parabolic restriction. These $\mathcal{D}$-modules on G have recently appeared in the étale setting in works of Chen. Specifically, in [Che22], the author argues that the acyclicity of ρ-Bessel sheaves follows from the very centrality of certain sheaves obtained from enhanced parabolic induction on the étale analogue of sheaves in $\mathcal{D}(T)^W$ which descend to the coarse quotient. This very centrality is proved in [Che20].

In [BG17], the authors conjecture that very central $\mathcal{D}$-modules are precisely those given by the Ngô functor (discussed in Remark 1.18) at the level of abelian categories.[3] We state the following formulation of the conjecture here:

Conjecture 1.21. [BG17, Conjecture 2.14(2)] The restricted functor of abelian categories

$$\mathrm{WRes} : \mathcal{V} \to \mathcal{D}(T)^{W,\heartsuit}$$

has essential image given by those sheaves descending to the coarse quotient $\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}$.

In Section 5.1, we provide some evidence for Conjecture 1.21. Specifically, we show:

[3]This also justifies the term 'very central,' since the abelian category of very central $\mathcal{D}$-modules is expected to be a *symmetric* monoidal subcategory of an abelian category which is only braided monoidal.

Theorem 1.22. If $\mathcal{F} \in \mathcal{D}(G)^{G,\heartsuit}$ is very central, then there is a W-equivariant structure on $\text{Res}(\mathcal{F})$ such that

$$\text{oblv}^T(\text{Res}(\mathcal{F})) \in \mathcal{D}(T)^{W,\heartsuit}$$

descends to the coarse quotient.

We note that, while nondegeneracy and Whittaker invariants are used heavily in the proof of Theorem 1.22, the statement of Theorem 1.22 uses neither.

1.7. **Outline of Paper.** In Section 2, we review some conventions and prove a result on comonadicity which will be used later. In Section 3, using the foundations on nondegenerate G-categories developed in the companion paper [Gan23], we prove Theorem 1.4 and Theorem 1.9. We then prove Theorem 1.14 in Section 4 by first proving a non-monoidal variant and then equipping this equivalence with a monoidal structure. In Section 5, we then prove Theorem 1.22. This paper also contains one appendix, Appendix A, which is written jointly with Germán Stefanich and upgrades the classical Mellin transform to symmetric monoidal equivalence of DG categories.

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2. Preliminaries

2.1. **Conventions.** We use the conventions of [GR17b] regarding *DG categories*, or k-linear stable ∞-categories. In particular, for any scheme or more generally any prestack $\mathcal{Y}$, categories such as $\text{QCoh}(\mathcal{Y})$ are defined as DG categories. We spell out other conventions we use in more detail in the companion paper [Gan23, Section 2].

2.2. **Comonadicity.**

2.2.1. *Barr-Beck-Lurie.* In the proof of Theorem 1.14, we will use the Barr-Beck-Lurie theorem. We will recall the result here for the reader's convenience–this is summarized in much more depth and proved in [Lur17, Theorem 4.7.3.5]. Given a functor of infinity categories $L : \mathcal{C} \to \mathcal{D}$ which admits a right adjoint $R : \mathcal{D} \to \mathcal{C}$, we can obtain a comonad in $\mathcal{D}$ which we denote LR. The functor L canonically lifts to a functor $L^{\text{enh}} : \mathcal{C} \to LR\text{-comod}(\mathcal{D})$.

Remark 2.1. If $L : \mathcal{C} \leftrightarrows \mathcal{D} : R$ is any adjoint pair of functors, we will always reserve the superscript 'enh' for the corresponding lift $L^{\text{enh}} : \mathcal{C} \to LR\text{-comod}(\mathcal{D})$. Furthermore, if $\mathcal{D} \simeq A\text{-mod}$ for some algebra A and L and R are continuous functors of DG categories, then $LR(A)$ is a coalgebra, and $LR\text{-comod}(\mathcal{D}) \simeq LR(A)\text{-comod}$. We will abuse notation and also denote the composite functor $\mathcal{C} \to LR(A)\text{-comod}$ by L^{enh}.

Definition 2.2. We say that L is *comonadic* if L^{enh} is an equivalence.

Theorem 2.3. (Barr-Beck-Lurie Theorem for Comonads) The following are equivalent:

- The functor L is comonadic.
- The functor L is conservative, and moreover for any L-split cosimplicial object $C^\bullet$ of $\mathcal{C}$, the totalization of $C^\bullet$ exists[4] in $\mathcal{C}$ and moreover the canonical map $L(\text{Tot}(C^\bullet)) \to \text{Tot}(L(C^\bullet))$ is an equivalence.

2.2.2. *A Comonadicity Condition.* In this section, we state and prove a condition for the comonadicity of functors, see Corollary 2.6. The results of this subsection are modifications of ideas contained in the proof of [Ras20b, Proposition 3.7.1].

Proposition 2.4. If $L : \mathcal{C} \to \mathcal{D}$ is any functor of DG categories equipped with t-structures such that:

(1) The t-structures on $\mathcal{C}$ and $\mathcal{D}$ are right-complete
(2) L is t-exact and
(3) L is conservative on $\mathcal{C}^\heartsuit$

then the induced functor $L : \mathcal{C}^{\geq 0} \to \mathcal{D}^{\geq 0}$ commutes with arbitrary totalizations.

We first begin with a standard lemma on cosimplicial sets, whose proof can be found, for example, in the third paragraph of the proof of [Ras20b, Proposition 3.7.1]:

Lemma 2.5. For any DG category $\mathcal{C}$ equipped with a right-complete t-structure and any cosimplicial object $\mathcal{F}^\bullet$ of $\mathcal{C}$ such that $\mathcal{F}^i \in \mathcal{C}^{\geq 0}$ for all i, the totalization $\text{Tot}(\mathcal{F}^\bullet)$ exists and we have the identity

$$\tau^{\leq n}(\text{Tot}(\mathcal{F}^\bullet)) \simeq \tau^{\leq n}(\text{Tot}^{\leq n+1}(\mathcal{F}^\bullet))$$

[4]Observe that if $\mathcal{C}$ and $\mathcal{D}$ are DG categories then these totalizations always exist. This is because, by definition, any DG category is cocomplete and thus, by the conventions as in [GR17b, Section 1.5.1.5], all DG categories are presentable; therefore DG categories are closed under totalizations by [Lur09, Corollary 5.5.2.4].

where $\text{Tot}^{\leq n+1}(\mathcal{F}^\bullet)$ denotes the partial totalization, i.e. the limit over $\Delta_{\leq n+1}$.

Proof of Proposition 2.4. We have

$$L(\text{Tot}(\mathcal{F}^\bullet)) \xleftarrow{\sim} L(\text{colim}_n(\tau^{\leq n}(\text{Tot}(\mathcal{F}^\bullet))))$$
$$\simeq \text{colim}_n L(\tau^{\leq n}(\text{Tot}(\mathcal{F}^\bullet))) \simeq \text{colim}_n L(\tau^{\leq n}(\text{Tot}^{\leq n+1}(\mathcal{F}^\bullet)))$$

where the first step uses the right-completeness of the t-structure of $\mathcal{D}$, the second uses the fact that all functors of DG categories are continuous, and the third uses Lemma 2.5. We may continue this chain of equivalences to obtain

$$L(\text{Tot}(\mathcal{F}^\bullet)) \simeq \text{colim}_n(\tau^{\leq n}(\text{Tot}^{\leq n+1}(L\mathcal{F}^\bullet))) \simeq \text{colim}_n(\tau^{\leq n}(\text{Tot}(L\mathcal{F}^\bullet)))$$

where the first step follows from the fact that L is t-exact and commutes with finite limits and the second equivalence follows from Lemma 2.5. In particular, by the right-completeness of the t-structure of $\mathcal{D}$, we see that L preserves these totalizations, as desired. □

Corollary 2.6. Let $\mathcal{C}, \mathcal{D}$ be DG categories equipped t-structures for which the t-structure on $\mathcal{C}$ and $\mathcal{D}$ are right-complete, and assume that $L : \mathcal{C} \to \mathcal{D}$ is a t-exact functor which admits a right adjoint R. Then if L is conservative on $\mathcal{C}^\heartsuit$, the restricted functors $\mathcal{C}^{\geq 0} \to \mathcal{D}^{\geq 0}$ and $\mathcal{C}^+ \to \mathcal{D}^+$ are comonadic.

Proof. By induction on cohomological amplitude, the t-exactness of L implies that L sends no nonzero object of finite cohomological amplitude to zero. By right completeness of the t-structure on $\mathcal{C}$ and the fact that L commutes with colimits therefore implies that L sends no nonzero object of C^+ to zero, and therefore both restrictions $\mathcal{C}^{\geq 0} \to \mathcal{D}^{\geq 0}$ and $\mathcal{C}^+ \to \mathcal{D}^+$ of L are conservative.

Next, notice that if $C^\bullet$ is an L-split cosimplicial object of $\mathcal{C}^{\geq 0}$ then by Lemma 2.5 its totalization exists in $\mathcal{C}$ and by Proposition 2.4 L commutes with these totalizations. Thus by Theorem 2.3 we see that $\mathcal{C}^{\geq 0} \to \mathcal{D}^{\geq 0}$ is comonadic, and an identical argument shows $\mathcal{C}^{\geq -i} \to \mathcal{D}^{\geq -i}$ is comonadic for any integer i. This, in turn, implies that the functor $\mathcal{C}^+ \to LR\text{-comod}(\mathcal{D}^+)$ is fully faithful and essentially surjective, and so it is an equivalence of categories as desired. □

3. Proofs of Theorem 1.4 and Theorem 1.9

In this section, we prove Theorem 1.4 and Theorem 1.9.

3.1. **Reminder on Descent to the Coarse Quotient.**

In this section, we summarize the contents of [Gan22, Section 4]. Specifically, recall the existence of the *coarse quotient* $\mathfrak{t}^* /\!\!/ \tilde{W}^{\text{aff}}$. This space admits a quotient map $\overline{s} : \mathfrak{t}^*/\tilde{W}^{\text{aff}} \to \mathfrak{t}^* /\!\!/ \tilde{W}^{\text{aff}}$ for which the pullback $s^!$ is fully faithful, and if an object of $\text{IndCoh}(\mathfrak{t}^*)^{\tilde{W}^{\text{aff}}}$ lies in the essential image of this functor, we say it *descends to the coarse quotient.* There are some equivalent conditions on a given sheaf descending to the coarse quotient. The ones that will be used here are the following:

Proposition 3.1. A sheaf $\mathcal{F} \in \mathrm{IndCoh}(\mathfrak{t}^*)^{\tilde{W}^{\mathrm{aff}}}$ descends to the coarse quotient $\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}$ if and only if it satisfies one of the following equivalent conditions:

(1) For every field-valued point $x \in \mathfrak{t}^*(K)$, the canonical W_x^{aff}-representation on $\overline{x}^!(\mathrm{oblv}_{W_x^{\mathrm{aff}}}^{\tilde{W}^{\mathrm{aff}}}(\mathcal{F}))$ is trivial.

(2) The object $\mathrm{oblv}_{\langle s\rangle}^{\tilde{W}^{\mathrm{aff}}}(\mathcal{F}) \in \mathrm{IndCoh}(\mathfrak{t}^*/\langle s\rangle)$ descends to the coarse quotient $\mathfrak{t}^* /\!\!/ \langle s\rangle$ for every simple reflection $s \in W$.

(3) For every simple coroot γ with associated simple reflection s of the (finite) Weyl group W and associated closed subgroup scheme $\mathbb{G}_m^{\gamma} \hookrightarrow T$, the functor

$$\mathcal{D}(T)^W \xrightarrow{\mathrm{oblv}} \mathcal{D}(T)^{\langle s\rangle} \xrightarrow{\mathrm{Av}_*^{\mathbb{G}_m^{\gamma}}} \mathcal{D}(T/\mathbb{G}_m^{\gamma})^{\langle s\rangle} \simeq \mathcal{D}(T/\mathbb{G}_m^{\gamma}) \otimes \mathrm{Rep}(\langle s\rangle)$$

maps $M(\mathcal{F})$ into the subcategory $\mathcal{D}(T/\mathbb{G}_m^{\gamma}) \simeq \mathcal{D}(T/\mathbb{G}_m^{\gamma}) \otimes \mathrm{Vect}_{\mathrm{triv}} \hookrightarrow \mathcal{D}(T/\mathbb{G}_m^{\gamma}) \otimes \mathrm{Rep}(\langle s\rangle)$, where $\mathbb{G}_m^{\gamma}$ is the rank 1 subgroup scheme of T associated to γ, and $\mathrm{Vect}_{\mathrm{triv}}$ is the full subcategory generated by the trivial representation of the order two group $\langle s\rangle$.

Proof. The equivalence of an $\mathcal{F}$ descending to the coarse quotient, (1), and (2) are given in [Gan22]. We now show that the full subcategories given by (2) and (3) of Proposition 3.1 are equivalent. To this end, fix a simple reflection s, and note that if F denotes the composite functor

$$\mathcal{D}(T)^{\langle s\rangle} \xrightarrow{\mathrm{Av}_*^{\mathbb{G}_m^{\gamma}}} \mathcal{D}(T/\mathbb{G}_m^{\gamma})^{\langle s\rangle} \simeq \mathcal{D}(T/\mathbb{G}_m^{\gamma}) \otimes \mathrm{Rep}(\langle s\rangle)$$

then for any $\mathcal{F} \in \mathcal{D}(T)^W$, $F(\mathcal{F})$ lies in the subcategory $\mathcal{D}(T/\mathbb{G}_m^{\gamma}) \simeq \mathcal{D}(T/\mathbb{G}_m^{\gamma}) \otimes \mathrm{Vect}_{\mathrm{triv}}$ if and only if the object $\mathrm{Av}_*^{T/\mathbb{G}_m^{\gamma},w} F(\mathcal{F})$ lies in the full subcategory $\mathcal{D}(T/\mathbb{G}_m^{\gamma})^{T/\mathbb{G}_m^{\gamma},w} \simeq \mathcal{D}(T/\mathbb{G}_m^{\gamma})^{T/\mathbb{G}_m^{\gamma},w} \otimes \mathrm{Vect}_{\mathrm{triv}}$, by the conservativity of weak averaging, see [Gai15]. Furthermore, if we assume s has associated coroot γ such that s reflects across the hyperplane $\mathfrak{t}^*_{\gamma=0} := \{\gamma = 0\} \hookrightarrow \mathfrak{t}^*$, the Mellin transform allows us to identify the functor $\mathrm{Av}_*^{T/\mathbb{G}_m^{\gamma},w} F$ with the composite:

$$\mathrm{IndCoh}(\mathfrak{t}^*/X^{\bullet}(T))^{\langle s\rangle} \to \mathrm{IndCoh}(\mathfrak{t}^*_{\gamma\in\mathbb{Z}}/X')^{\langle s\rangle}$$

$$\simeq \mathrm{IndCoh}(\mathfrak{t}^*_{\gamma\in\mathbb{Z}}/X') \otimes \mathrm{Rep}(\langle s\rangle) \xrightarrow{\mathrm{oblv}^{X'} \otimes \mathrm{id}} \mathrm{IndCoh}(\mathfrak{t}^*_{\gamma\in\mathbb{Z}}) \otimes \mathrm{Rep}(\langle s\rangle)$$

where X' is the lattice of weights generated by the fundamental weights distinct from the fundamental weight associated to γ. Therefore, any sheaf satisfying (2) immediately satisfies (3), and any sheaf satisfying (3) also satisfies (2). □

As in [Gan22], we can define a t-structure on $\mathrm{IndCoh}(\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}})$ by declaring $\mathrm{IndCoh}(\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}})^{\leq 0}$ to be the full ordinary ∞-subcategory closed under colimits and containing $\overline{s}_*^{\mathrm{IndCoh}}(\mathcal{O}_{\mathfrak{t}^*})$. We also can similarly define a

t-structure on $\mathrm{IndCoh}(\mathfrak{t}^*/\tilde{W}^{\mathrm{aff}})$ (respectively, $\mathrm{IndCoh}(\mathfrak{t}^*/X^\bullet(T))$) by declaring $\mathrm{IndCoh}(\mathfrak{t}^*/\tilde{W}^{\mathrm{aff}})^{\leq 0}$ to be the full ordinary ∞-subcategory closed under colimits and containing the respective IndCoh pushforward given by the quotient map of the structure sheaf $\mathcal{O}_{\mathfrak{t}^*}$. Let $\Gamma_{\tilde{W}^{\mathrm{aff}}}$ denote the balanced product $\tilde{W}^{\mathrm{aff}} \overset{W^{\mathrm{aff}}}{\times} \Gamma_{W^{\mathrm{aff}}}$, where $\Gamma_{W^{\mathrm{aff}}}$ denotes the union of graphs of the affine Weyl group. We recall some results of [Gan22] which will be used below.

Theorem 3.2. The following diagram is Cartesian

$$\begin{array}{ccc} \Gamma_{\tilde{W}^{\mathrm{aff}}} & \xrightarrow{\mathrm{s}} & \mathfrak{t}^* \\ \big\downarrow{\scriptstyle \mathrm{t}} & & \big\downarrow{\scriptstyle \overline{s}} \\ \mathfrak{t}^* & \xrightarrow{\overline{s}} & \mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}} \end{array}$$

where s, respectively t, denote the source and target maps, and $\overline{s}$ denotes the quotient map. The base change theorem holds for this Cartesian diagram, and the functors $s_*^{\mathrm{IndCoh}}, s^!, t_*^{\mathrm{IndCoh}}, t^!, \overline{s}_*^{\mathrm{IndCoh}}, \overline{s}^!$ are all t-exact.

Proposition 3.3. Analogous results to Theorem 3.2 also hold if we replace $\mathfrak{t}^* /\!\!/ W^{\mathrm{aff}}$ with $\mathfrak{t}^*/W^{\mathrm{aff}}$ or $\mathfrak{t}^*/X^\bullet(T)$. For example,

$$\begin{array}{ccc} \tilde{W}^{\mathrm{aff}} & \xrightarrow{\mathrm{act}} & \mathfrak{t}^* \\ \big\downarrow{\scriptstyle \mathrm{proj}} & & \big\downarrow{\scriptstyle q} \\ \mathfrak{t}^* & \xrightarrow{q} & \mathfrak{t}^*/\tilde{W}^{\mathrm{aff}} \end{array}$$

is Cartesian and base change holds. Moreover, the pullback and pushforward functors given by this diagram are t-exact.

3.2. **Monoidality of Averaging Functor.** In this section, we prove the following proposition:

Proposition 3.4. The composite functor $\mathcal{H}_\psi \xrightarrow{\tilde{\mathrm{Av}}_*^N} \mathcal{D}(N\backslash G/_{-\psi}N^-)^W \simeq \mathcal{D}(T)^W$ is monoidal.

Proof. The functoriality of Av_*^N gives that the following diagram canonically commutes:

$$\begin{array}{ccc} \underline{\mathrm{End}}_G(\mathcal{D}(G/_{-\psi}N^-)) & \xrightarrow{(-)^N} & \underline{\mathrm{End}}_{T\rtimes W}(\mathcal{D}(N\backslash G/_{-\psi}N^-)) \\ \big\downarrow{\scriptstyle \mathrm{ev}_{\delta_{N^-,\psi}}} & & \big\downarrow{\scriptstyle \mathrm{ev}_{\mathrm{Av}_*^N(\delta_{N^-,\psi})}} \\ \mathcal{H}_\psi & \xrightarrow{\mathrm{Av}_*^N \mathrm{oblv}^{N^-,\psi}} & \mathcal{D}(N\backslash G/_{-\psi}N^-)^W \end{array} \tag{2}$$

where the vertical arrows are the evaluation maps and the top arrow is the functor induced by W-equivariance, see [Gan23, Proposition 3.23]. Similarly, note if I denotes the isomorphism of Proposition 1.8 (which we recall is W-equivariant again by [Gan23, Proposition 3.23]) then we have a canonical identification of the following diagram

$$\begin{array}{ccc} \underline{\text{End}}_{T\rtimes W}(\mathcal{D}(N\backslash G/_{-\psi}N^-)) & \xrightarrow{I\circ - \circ I^{-1}} & \underline{\text{End}}_{T\rtimes W}(\mathcal{D}(T)) \\ \downarrow{\scriptstyle \text{ev}_{\text{Av}_*^N(\delta_{N^-,\psi})}} & & \downarrow{\scriptstyle \text{ev}_{\delta_1}} \\ \mathcal{D}(N\backslash G/_{-\psi}N^-)^W & \xrightarrow{I} & \mathcal{D}(T)^W \end{array} \tag{3}$$

since $I(\text{Av}_*^N(\delta_{N^-,\psi})) \simeq \delta_1 \in \mathcal{D}(T)$. Now, note that the functors given by the top horizontal arrows of (2) and (3) are monoidal, the left vertical arrow of (2) is a monoidal equivalence, and the right vertical arrow of (3) is a monoidal equivalence. Thus, since these diagrams commute, we see that $I\text{Av}_*^N$ is a composite of monoidal functors. □

3.3. **Proof of Theorem 1.9.** In this subsection, we prove Theorem 1.9. Let $\mathcal{C}$ be a G-category. Then, using the fact that invariants and coinvariants agree (see [Gai20, Corollary 3.1.5]) and that tensor products commute with colimits, we may identify Av_*^N with the functor

$$\mathcal{C}^{N^-,\psi} \simeq \mathcal{C}_{N^-,\psi} \simeq \mathcal{D}(G)^{N^-,\psi} \otimes_G \mathcal{C} \xrightarrow{\text{Av}_*^N \otimes_G \text{id}_{\mathcal{C}}} \mathcal{D}(G)^{N,W}_{\text{nondeg}} \otimes_G \mathcal{C}$$
$$\simeq \mathcal{D}(G)_{N,W,\text{nondeg}} \otimes_G \mathcal{C} \simeq \mathcal{C}_{N,W,\text{nondeg}} \simeq \mathcal{C}^{N,W}_{\text{nondeg}}$$

and similarly for the adjoint $\text{Av}_!^{\psi}$. We therefore obtain that the fully faithfulness Av_*^N follows by proving the general universal case:

Theorem 3.5. The functor of G-categories $\text{Av}_*^N : \mathcal{D}(G)^{N^-,\psi} \to \mathcal{D}(G)^N$ lifts to a fully faithful functor of G-categories $\text{Av}_*^N : \mathcal{D}(G)^{N^-,\psi} \hookrightarrow \mathcal{D}(G)^{N,W}_{\text{nondeg}}$. Moreover, the functor $\text{Av}_*^N[\dim(N)]$ lifts to a fully faithful t-exact functor of G-categories $\widetilde{\text{Av}}_* : \mathcal{D}(G)^{N^-,\psi} \hookrightarrow \mathcal{D}(G)^{N,W}_{\text{nondeg}}$.

Proof. Note that all functors of DG categories (or any stable ∞-categories) are by definition exact, so they commute with cohomological shifts. Therefore to construct the lift of $\text{Av}_*^N[\dim(N)]$ it suffices to construct the lift of Av_*^N, where the t-exactness of $\widetilde{\text{Av}}_*$ follows since the forgetful functor oblv^W reflects the t-structure.

To construct the lift of Av_*^N, note that the G-functor Av_*^N is given by an integral kernel in $\mathcal{D}(N\backslash G)^{N^-,-\psi}_{\text{nondeg}}$, and, under the equivalences $\mathcal{D}(N\backslash G)^{N^-,-\psi}_{\text{nondeg}} \simeq \mathcal{D}(N\backslash G)^{N^-,-\psi} \simeq \mathcal{D}(T)$ given by [Gan23, Proposition 3.14] and Proposition 1.8 respectively, this kernel is given by $\delta_1[-\dim(N)] \in \mathcal{D}(T)$ by Proposition 3.4. Furthermore, these equivalences are canonically W-equivaraint by [Gin18, Proposition 5.5.2] (see also [Gan23, Proposition 3.23]) and therefore the integral kernel can be canonically equipped with W-equivariant structure since $* \hookrightarrow T$ is W-equivariant.

We wish to show this lift is fully faithful. By [Gan23, Theorem 1.5], it suffices to show that the resulting functor on invariants $\mathrm{Vect} \simeq \mathcal{D}(G/_{\lambda}B)^{N^-,\psi} \to \mathcal{D}(G/_{\lambda}B)^{N,W}_{\mathrm{nondeg}}$ is fully faithful. By base change ([Gan23, Proposition 2.7]) it suffices to assume that λ is a k-point. By continuity of Av_*^N, we may show the counit of the adjunction is an isomorphism on the one dimensional vector space $k \in \mathrm{Vect}$. By [Gan23, Proposition 4.6](1), this object is a cohomological shift of the direct sum of the indecomposable antidominant injectives $\underline{I}$, or, equivalently by self duality, the direct sum of the antidominant projectives $\underline{P}$. Soergel's endomorphismensatz [Soe90] gives $W \overset{W_{[\lambda]}}{\times} C_\lambda \xrightarrow{\sim} \underline{\mathrm{End}}_{\mathcal{D}(G/B_\lambda)^N_{\mathrm{nondeg}}}(\underline{P})$. Therefore, since $T \rtimes W$ acts on $\mathcal{D}(G/_{\lambda}B)^N_{\mathrm{nondeg}}$ we have that by [Gan23, Proposition 4.5] that this equivalence of classical algebras is W-equivariant. Therefore we see

$$\underline{\mathrm{End}}_{\mathcal{D}(G/B_\lambda)^{N,W}_{\mathrm{nondeg}}}(\underline{P}_\lambda) \simeq \underline{\mathrm{End}}_{\mathcal{D}(G/B_\lambda)^{N}_{\mathrm{nondeg}}}(\underline{P}_\lambda)^W \simeq (W \overset{W_{[\lambda]}}{\times} C_\lambda)^W \simeq k$$

where the second to last equivalence follows since $\underline{P}_\lambda \simeq \mathrm{Av}_*^N(k)$ lies in the nondegenerate subcategory, and the last equivalence follows since C_λ^W can be identified with the regular W-representation by Soergel's endomorphismensatz. Therefore, we see our functor is fully faithful. ☐

3.4. **Proofs of Theorem 1.4 and Theorem 1.3 from Theorem 1.9.** In this subsection, we verify the essential image of our shifted and lifted functor $\widetilde{\mathrm{Av}}_*$ and complete the proofs of Theorem 1.4 and Theorem 1.3. We first make the following computation on the essential image:

Proposition 3.6. Fix some simple root α. The composite given by

$$\mathcal{D}(G)^{N^-,\psi} \xrightarrow{\mathrm{Av}_*^N} \mathcal{D}(G/N)^{\langle s_\alpha \rangle}_{\mathrm{nondeg}} \xrightarrow{\mathrm{Av}_*^{\mathbb{G}_m^\alpha}} \mathcal{D}(G/N)^{\mathbb{G}_m^\alpha \rtimes \langle s_\alpha \rangle}_{\mathrm{nondeg}} \simeq \mathcal{D}(G/N)^{\mathbb{G}_m^\alpha}_{\mathrm{nondeg}} \otimes \mathrm{Rep}(\langle s_\alpha \rangle)$$

where the final equivalence is given by [Gan23, Corollary 3.45], factors through the subcategory labelled by the trivial representation.

Proof. Let A denote the composite functor. By [BGO20], it suffices to show that

$$A(\delta_{N^-,\psi}) \in \mathcal{D}(G/N)^{(N^-,\psi),\mathbb{G}_m^\alpha}_{\mathrm{nondeg}} \otimes \mathrm{Rep}(\langle s_\alpha \rangle)$$

lies in the full (G-)subcategory labelled by the trivial representation. However, by direct computation or Proposition 3.4 below, we have that the sheaf $\mathrm{Av}_*^N(\delta_{N^-,\psi})$ can be identified with $\delta_1 \in \mathcal{D}(T)$ by Proposition 1.8. Furthermore, by the W-equivariance of the equivalence of Proposition 1.8 ([Gin18, Proposition 5.5.2]) we see that the given W-equivariance on $\mathrm{Av}_*^N(\delta_{N^-,\psi})$ can be identified with the W-equivariance on δ_1 given by the W-equivariant closed embedding $* \hookrightarrow T$. However, for the equivariant sheaf $\delta_1 \in \mathcal{D}(T)^W$, we see that $\text{Av}_*^{\mathbb{G}_m^\alpha}(\delta_1)$ acquires a trivial $\langle s_\alpha \rangle$-representation. Therefore, the $\langle s_\alpha\rangle$ action is trivial and so the same holds for $A(\delta_{N^-,\psi}) \simeq \text{Av}_*^{\mathbb{G}_m^\alpha}\text{Av}_*^N(\delta_{N^-,\psi})$. □

Corollary 3.7. The functor $\text{Av}_*^N : \mathcal{H}_\psi \to \mathcal{D}(T)^W$ factors through the full subcategory of objects of $\mathcal{D}(T)^W$ which descend to the coarse quotient under the Mellin transform.

Proof. By the final point of Proposition 3.1, it suffices to show that if $\mathcal{F} \in \mathcal{H}_\psi$ then the canonical $\langle s_\alpha\rangle$-representation on $\text{Av}_*^{\mathbb{G}_m^\alpha}\text{Av}_*^N(\mathcal{F})$ is trivial. However, this directly follows from taking (N^-,ψ)-invariants of the composite functor of Proposition 3.6. □

Lemma 3.8. Fix some field-valued point λ. We have a canonical isomorphism of functors:

$$\begin{array}{ccc} \mathcal{D}(B_\lambda\backslash G/N)^{T,w}_{\text{nondeg}} & \xrightarrow{\text{Av}_!^{-\psi}\text{oblv}^{N,(T,w)}} & \mathcal{D}(B_\lambda\backslash G/_{-\psi}N^-) \simeq \text{Vect} \\ \uparrow{\scriptstyle \text{Av}_*^{B_\lambda}\text{oblv}^{N^-,\psi}} & & \uparrow{\scriptstyle \text{Av}_*^{B_\lambda}\text{oblv}^{N^-,-\psi}} \\ \mathcal{D}(N^-_\psi\backslash G/N)^{T,w} & \xrightarrow{\text{Av}_!^{-\psi}\text{oblv}^{N,(T,w)}} & \mathcal{H}_\psi \end{array}$$

Proof. In the above diagram, the horizontal arrows are averaging with respect to the right action, and the vertical arrows are averaging with respect to the left action. Therefore since all four functors are maps of G-categories, the diagram canonically commutes. □

In the diagram of Lemma 3.8, we claim that the associated right adjoints to the horizontal arrows are functors of $\tilde{W}^{\text{aff}}$-categories, where we take the $\tilde{W}^{\text{aff}}$-action to be trivial on the categories of the right side of the diagram. To see this, note that via the Mellin transform, we may identify the right adjoint to the bottom functor as the composite of a W-equivariant functor and, via the Mellin transform, the forgetful functor $\text{oblv}^{X^\bullet(T)}$. In particular, the fact that $\tilde{W}^{\text{aff}}$ is placid allows us to apply [Gai20, Lemma D.4.4] to show that the adjoint is also $\tilde{W}^{\text{aff}}$-equivariant and induces a functor on coinvariants. Since invariance is coinvariance for infinite discrete groups ([Gan23, Proposition 2.7]) we therefore see:

Lemma 3.9. Fix some field-valued $\lambda \in \mathfrak{t}^*$. Then the following diagram canonically commutes:

$$\begin{array}{ccc}
\mathcal{D}(B_\lambda \backslash G/N)^W_{\text{nondeg}} & \xrightarrow{\operatorname{Av}_!^{-\psi}} & \mathcal{D}(B_\lambda \backslash G/_{-\psi}N^-) \simeq \operatorname{Vect} \\
{\scriptstyle \operatorname{Av}_*^{B_\lambda}\operatorname{oblv}^{N^-,\psi}}\uparrow & & \uparrow{\scriptstyle \operatorname{Av}_*^{B_\lambda}\operatorname{oblv}^{N^-,\psi}} \\
\mathcal{D}(N^-_\psi \backslash G/N)^W & \xrightarrow{\operatorname{Av}_!^{-\psi}} & \mathcal{H}_\psi
\end{array}$$

Proof of Theorem 1.4. We have seen in Theorem 3.5 that the functor $\operatorname{Av}_*^N : \mathcal{H}_\psi \to \mathcal{D}(G/N)^{N^-,\psi,W} \xrightarrow{\sim} \mathcal{D}(T)^W$ is fully faithful, and is t-exact up to cohomological shift by [Gin18, Theorem 1.5.4]. Furthermore, we have shown that this functor factors through the full subcategory of sheaves descending to the coarse quotient in Corollary 3.7. Therefore it remains to show that the adjoint $\operatorname{Av}_!^{\psi}$ is conservative on this the subcategory of sheaves descending to the coarse quotient. Let $\mathcal{F} \in \mathcal{D}(T)^W$ be a sheaf which descends to the coarse quotient. By considering the fully faithful embedding of the zero category into the full G-subcategory $\mathcal{D}(G/N)$ generated by $\operatorname{oblv}^{N^-,\psi}(\mathcal{F})$, by [Gan23, Corollary 2.19] we see that there exists some field-valued point λ such that $\operatorname{Av}_*^{B_\lambda}\operatorname{oblv}^{N^-,\psi}(\mathcal{F})$ does not vanish. By applying the categorical extension of scalars ([Gan23, Section 2.4.1]) it suffices to assume λ is a k-point.

Now note that the following diagram commtues

$$\begin{array}{ccc}
\mathcal{D}(B_\lambda \backslash G)^{N^-,-\psi} & \xrightarrow{\sim} & \operatorname{IndCoh}(*) \\
{\scriptstyle \operatorname{Av}_!^{-\psi}[-\dim(N)]}\uparrow & & \uparrow{\scriptstyle (\alpha_*^{\operatorname{IndCoh}}(-))^W} \\
\mathcal{D}(B_\lambda \backslash G/N)^W_{\text{nondeg}} & \xrightarrow{\tilde{\mathbb{V}}} & \operatorname{IndCoh}(\operatorname{Spec}(C_\lambda) \overset{W_\lambda^{\text{aff}}}{\times} W)^W
\end{array}$$

since again we may identify the images the left adjoints via the image of $k \in \operatorname{Vect}$. Thus since by assumption $\mathcal{F} \in \mathcal{D}(T)^W$ descends to the coarse quotient, by Proposition 3.1(1) that the sheaf $\operatorname{Av}_!^{-\psi}\operatorname{Av}_*^{B_\lambda}\operatorname{oblv}^{N_\ell^-,\psi}(\mathcal{F})$ does not vanish. Thus by Lemma 3.9, we see that $\operatorname{Av}_!^{\psi}(\mathcal{F})$ does not vanish. $\square$

Finally, note that to derive the exact equivalence of abelian categories in Theorem 1.3 from Theorem 1.4, as in Remark 1.5 it suffices to show that each functor in Theorem 1.4 is t-exact. By [Gin18, Theorem 1.5.4], $\operatorname{Av}_*^N[\dim(N)]$ is t-exact, and $\widetilde{\operatorname{Av}}_*$ is t-exact since oblv^W reflects the t-structure. Furthermore, the pullback map $\phi^!$ is t-exact, see [Gan22, Proposition 4.18]. Therefore it remains to show the following, completing the proof of an exact equivalence of abelian categories as in Theorem 1.3:

Proposition 3.10. The shifted Mellin transform FMuk$[d]$ is t-exact, where $d := \dim(\mathfrak{t}^*)$.

Proof. As we mention above, this follows directly from our definition of the Mellin transform. The Mellin transform also admits a Fourier-Mukai description and we include an alternate proof using this definition. It is standard that the functors

$$\text{IndCoh}(T_{dR}) \xrightarrow{\phi^!} \text{IndCoh}(T) \xrightarrow{\Gamma^{\text{IndCoh}}} \text{Vect}$$

correspond, under the associated Fourier-Mukai transformations, to the functors

$$\text{IndCoh}(\mathfrak{t}^*/X^\bullet(T)) \xrightarrow{\Gamma^{\text{IndCoh}}[-d]/X^\bullet(T)} \text{IndCoh}(*/X^\bullet(T)) \xrightarrow{c^!} \text{Vect}$$

where $c : * \to */X^\bullet(T)$ is the quotient map, see the proof of [Lau96, Théorème 6.3.3(ii)], whose proof also applies to IndCoh in the DG categorical context.

Let F denote the composite $\Gamma^{\text{IndCoh}}\phi^!$ and let G denote the composite $\Gamma^{\text{IndCoh}}[-d]/X^\bullet(T) \circ c^!$. Then by this observation we see that there is a canonical identification exhibiting that the following diagram commutes:

$$\begin{array}{ccc} \text{IndCoh}(\mathfrak{t}^*/X^\bullet(T)) & \xrightarrow{G} & \text{Vect} \\ \downarrow{\scriptstyle \text{FMuk}} & & \downarrow{\scriptstyle \text{id}} \\ \mathcal{D}(T) := \text{IndCoh}(T_{dR}) & \xrightarrow{F} & \text{Vect} \end{array}$$

since the Fourier-Mukai transform for the trivial group is the identity. Because the functors F, $G[-d]$, and id are t-exact, we see that FMuk$[d]$ is t-exact as well. □

4. Proof of Theorem 1.14

In this section, we prove Theorem 1.14. We first identify the two categories as DG categories in Section 4.1. Then, in Section 4.2, after some preliminary categoral recollections we prove this equivalence can be equipped with a monoidal structure by relating both categories of Theorem 1.14 to the category $\underline{\text{End}}_{\mathcal{H}_\psi}(\text{IndCoh}(\mathfrak{t}^*))$.

4.1. **Identification of Theorem 1.14 as DG Categories.** Let s_*^{IndCoh} : $\text{IndCoh}(\mathfrak{t}^* \times_{\mathfrak{t}^*/\!\!/\tilde{W}^{\text{aff}}} \mathfrak{t}^*) \to \text{IndCoh}(\mathfrak{t}^*)$ be the pushforward associated to the projection map $s : \mathfrak{t}^* \times_{\mathfrak{t}^*/\!\!/\tilde{W}^{\text{aff}}} \mathfrak{t}^* \to \mathfrak{t}^*$, and, as before, we denote the canonical functor $\mathcal{D}(N\backslash G/N)^{(T\times T,\text{ w})}_{\text{nondeg}} \to \text{IndCoh}(\mathfrak{t}^*)$ given by the composite $\text{Av}_!^\psi \text{oblv}^N$, Proposition 1.8, and the Mellin transform also by $\text{Av}_!^\psi$. We first state the eventually coconnective version of Theorem 1.14.

Theorem 4.1. We have the following:

(1) The functor $s_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\mathfrak{t}^* \times_{\mathfrak{t}^* /\!/ \tilde{W}^{\mathrm{aff}}} \mathfrak{t}^*)^+ \to \mathrm{IndCoh}(\mathfrak{t}^*)^+$ is comonadic.
(2) The functor $\mathrm{Av}_!^{\psi} : \mathcal{D}(N\backslash G/N)_{\mathrm{nondeg}}^{(T\times T,\ \mathrm{w}),\ +} \to \mathrm{IndCoh}(\mathfrak{t}^*)^+$ is comonadic.
(3) The coalgebras given by s_*^{IndCoh} and $\mathrm{Av}_!^{\psi}$ are canonically isomorphic.

We will prove Theorem 4.1 in Section 4.1.2 and Section 4.1.4, and show how it implies the non-monoidal version of Theorem 1.14 in Section 4.1.5. To prove Theorem 4.1, we will use Corollary 2.6, which requires us to argue that the t-structure on $\mathrm{IndCoh}(\mathfrak{t}^* \times_{\mathfrak{t}^* /\!/ \tilde{W}^{\mathrm{aff}}} \mathfrak{t}^*)^+$ is right-complete. We will argue more generally that the t-structure on $\mathrm{IndCoh}(\mathcal{X})$ is right-complete for any ind-scheme $\mathcal{X}$ in Section 4.1.1.

4.1.1. *Right-Completeness of t-Structure of Ind-Coherent Sheaves on Ind-Scheme.* Let $\mathcal{X}$ denote any ind-scheme. Recall the standard t-structure on $\mathrm{IndCoh}(\mathcal{X})$ characterized by the property that $\mathrm{IndCoh}(\mathcal{X})^{\geq 0}$ contains precisely those $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})$ for which $i^!(\mathcal{F}) \in \mathrm{IndCoh}(X)^{\geq 0}$ for any closed subscheme $X \overset{i}{\hookrightarrow} \mathcal{X}$, and which is compatible with filtered colimits [GR17a, Chapter 4, Section 1.2].

Proposition 4.2. The t-structure on $\mathrm{IndCoh}(\mathcal{X})$ is right-complete.

Proof. Given some $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})$, let $\phi^{\mathcal{F}} : \mathcal{F} \to \mathrm{colim}_n \tau^{\leq n}\mathcal{F}$ denote the canonical map, and let $\mathcal{K}$ denote the fiber of $\phi^{\mathcal{F}}$. Since the t-structure on $\mathrm{IndCoh}(\mathcal{X})$ is compatible with filtered colimits, we see that $\tau^{\leq n_0}(\phi^{\mathcal{F}})$ is an equivalence for all $n_0 \in \mathbb{Z}$, and that, in particular, $\mathcal{K} \in \mathrm{IndCoh}(\mathcal{X})^{\geq 0}$.

We first prove this in the case where $\mathcal{X}$ is itself a scheme X. In this case, we have a t-exact equivalence $\Psi_X : \mathrm{IndCoh}(X)^{\geq 0} \xrightarrow{\sim} \mathrm{QCoh}(X)^{\geq 0}$. Because Ψ_X is exact, we see that $\Psi_X(\mathcal{K}) \simeq \mathrm{fib}(\Psi_X(\phi^{\mathcal{F}}))$. Because Ψ is continuous and t-exact, we obtain a canonical identification $\Psi_X(\phi^{\mathcal{F}}) \simeq \phi^{\Psi_X(\mathcal{F})}$. However, $\phi^{\Psi_X(\mathcal{F})}$ is an equivalence since the t-structure on $\mathrm{QCoh}(X)$ is right-complete [GR17a, Chapter 3, Corollary 1.5.7]. Thus $\Psi_X(\mathcal{K}) \simeq 0$, and since Ψ_X is in particular conservative on $\mathrm{IndCoh}(X)^{\geq 0}$, we see that $\mathcal{K} \simeq 0$ in this case.

We now assume the result of Proposition 4.2 for schemes. For any closed subscheme $i : X \hookrightarrow \mathcal{X}$, we therefore obtain equivalences

$$i^!(\mathcal{K}) \xrightarrow{\phi^{i^!(\mathcal{F})}} \mathrm{colim}_n \tau^{\leq n} i^!(\mathcal{K}) \simeq \mathrm{colim}_n \tau^{\leq n} i^!(\tau^{\leq n}\mathcal{K}) \simeq \mathrm{colim}_n \tau^{\leq n} i^!(0)$$

where the first map is an equivalence t-structure on $\mathrm{IndCoh}(X)$ is right-complete, the second step uses the fact that $i^!$ is right t-exact, and the third equivalence is a direct consequence of the above fact that $\tau^{\leq n}(\phi^{\mathcal{F}})$ is an equivalence for all n. □

4.1.2. *Conservativity of IndCoh Side for Union of Graphs.* By [Gan23, Corollary 2.44] $L_{\mathcal{D}} = \mathrm{Av}_!^{\psi}[-\dim(N)]$ is t-exact, and $L_I = \pi_*^{\mathrm{IndCoh}}$ is t-exact because it is ind-affine [GR17d, Chapter 3, Lemma 1.4.9]. Therefore, by Corollary 2.6, to prove points (1) and (2) of Theorem 4.1 it suffices to verify that both functors are conservative, since the t-structure on both categories are right-complete by Proposition 4.2 and [Gan23, Proposition 3.18]. The functor $\mathrm{Av}_!^{\psi}$ is conservative on the eventually coconnective subcategory by construction, so we prove the analogous conservativity for π_*^{IndCoh} in this subsection.

We first prove a preliminary result regarding a locally almost finite type ind-scheme[5] $\mathcal{X}$ with colimit presentation ${}^{cl}\mathcal{X} = \mathrm{colim}_{\alpha} X_{\alpha}$ and associated closed embeddings $i_{\alpha} : X_{\alpha} \hookrightarrow X$. This allows us to place a t-structure on the category $\mathrm{IndCoh}(\mathcal{X})$, see Chapter 3 of [GR17d].

Corollary 4.3. Let $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})^{\heartsuit}$. Then there exists some closed subscheme $X := X_{\alpha} \overset{i}{\hookrightarrow} \mathcal{X}$ such that $H^0(i^!(\mathcal{F}))$ is nonzero.

Proof. Pick a nonzero $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})^{\heartsuit}$. By [GR17d, Chapter 3, Corollary 1.2.7], there exists some closed subscheme $X_{\alpha} \overset{i}{\hookrightarrow} \mathcal{X}$ and some $\mathcal{G} \in \mathrm{Coh}(X)^{\heartsuit}$ such that the map $i_{S,*}^{\mathrm{IndCoh}}(\mathcal{G}) \to \mathcal{F}$ is nonzero, so that the space $\mathrm{Hom}_{\mathrm{IndCoh}(\mathcal{X})}(i_{S,*}^{\mathrm{IndCoh}}(\mathcal{G}), \mathcal{F})$ is a discrete space (as both objects lie in the heart of a t-structure) with more than one point. Therefore, by adjunction, the same holds for $\mathrm{Hom}_{\mathrm{IndCoh}(X_{\alpha})}(\mathcal{G}, i^!\mathcal{F})$. However, since i_*^{IndCoh} is t-exact, its right adjoint is left t-exact, and so we see that this implies that there exists a nonzero map $\mathcal{G} \to \tau^{\leq 0} i^!\mathcal{F} \simeq H^0(i^!\mathcal{F})$ which obviously implies our claim. □

Corollary 4.4. Assume $q : \mathcal{X} \to Y$ is a map from an ind-affine scheme $\mathcal{X}$ to a scheme Y. Then q_*^{IndCoh} is conservative on $\mathrm{IndCoh}(\mathcal{X})^{\heartsuit}$.

Proof. Pick $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})^{\heartsuit}$. Note that, by Corollary 4.3, there exists some closed subscheme $i : X \hookrightarrow \mathcal{X}$ for which $H^0(i^!(\mathcal{F}))$ is nonzero, and, by ind-affineness, we may assume X is affine. Let π denote the composite $q \circ i : X \to Y$. Then we have that $H^0(\pi_*^{\mathrm{IndCoh}} i^!(\mathcal{F})) \simeq H^0(q_*^{\mathrm{IndCoh}} i_*^{\mathrm{IndCoh}} i^!(\mathcal{F}))$ is a subobject of $H^0(q_*^{\mathrm{IndCoh}}(\mathcal{F}))$, as q_*^{IndCoh} is ind-affine and thus is t-exact by [GR17d, Chapter 3, Lemma 1.4.9]. However, we see that π_*^{IndCoh} is conservative (it is the pushforward of an affine morphism) and so $H^0(\pi_*^{\mathrm{IndCoh}} i^!(\mathcal{F}))$ is nonzero, and therefore so too is $H^0(q_*^{\mathrm{IndCoh}}(\mathcal{F}))$. □

Of course, as a special case of this, we obtain our desired conservativity:

Corollary 4.5. The functor $\mathrm{IndCoh}(\mathfrak{t}^* \times_{\mathfrak{t}^*/\!\!/\tilde{W}^{\mathrm{aff}}} \mathfrak{t}^*) \to \mathrm{IndCoh}(\mathfrak{t}^*)$ is conservative when restricted to the full subcategory $\mathrm{IndCoh}(\mathfrak{t}^* \times_{\mathfrak{t}^*/\!\!/\tilde{W}^{\mathrm{aff}}} \mathfrak{t}^*)^{\heartsuit}$.

We now record a consequence of Corollary 4.4 for later use.

Corollary 4.6. The functor $\overline{s}_*^{\mathrm{IndCoh}}$ is conservative.

Proof. It suffices to show that the functor $\overline{s}^!\overline{s}_*^{\mathrm{IndCoh}}$ is conservative. By base change (Theorem 3.2), it suffices to show that $t_*^{\mathrm{IndCoh}} s^!$ is conservative on $\mathrm{IndCoh}(\mathfrak{t}^*)$. For a nonzero $\mathcal{F} \in \mathrm{IndCoh}(\mathfrak{t}^*)$, there exists some i for which

[5] In the notation of [GR17d], $\mathcal{X} \in \mathrm{indSch}_{\mathrm{laft}}$.

$H^i(\mathcal{F})$ is nonzero. Since, by Theorem 3.2, $t_*^{\mathrm{IndCoh}} s^!$ is t-exact, we see that $H^i(t_*^{\mathrm{IndCoh}} s^! \mathcal{F}) \cong t_*^{\mathrm{IndCoh}} s^! H^i(\mathcal{F})$ and so we may assume $\mathcal{F} \in \mathrm{IndCoh}(\mathfrak{t}^*)^{\heartsuit}$. For such an $\mathcal{F}$, we have $s^!(\mathcal{F}) \in \mathrm{IndCoh}(\mathfrak{t}^* \times_{\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}} \mathfrak{t}^*)^{\heartsuit}$ by Theorem 3.2 and is nonzero, since for example $\Delta^! s^!(\mathcal{F}) \simeq \mathcal{F}$ for Δ the diagonal map. Thus by Corollary 4.4 we see that $t_*^{\mathrm{IndCoh}} s^!(\mathcal{F})$ is nonzero, as required. □

Remark 4.7. Using Theorem 1.4, one can derive the conservativity and t-exactness of the pushforward map $(\mathfrak{t}^*/X^\bullet(T) \to \mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}})_*^{\mathrm{IndCoh}}$.

4.1.3. *Conservativity for Quotient Pushforward.* Let

$$\mathring{\phi} : \mathfrak{t}^*/X^\bullet(T) \times_{\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}} \mathfrak{t}^*/X^\bullet(T)$$

denote the projection map to the left factor. To use Corollary 2.6, we require the following proposition:

Proposition 4.8. Let $\mathcal{F} \in \mathrm{IndCoh}(\mathfrak{t}^*/X^\bullet(T) \times_{\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}} \mathfrak{t}^*/X^\bullet(T))^+$ denote some nonzero object. Then $\mathring{\phi}_*^{\mathrm{IndCoh}}(\mathcal{F})$ does not vanish.

Proof. By the conservativity of the forgetful functor, we have that $\mathrm{oblv}^{X^\bullet(T) \times X^\bullet(T)}(\mathcal{F}) \in \mathrm{IndCoh}(\Gamma_{\tilde{W}^{\mathrm{aff}}})$ is also nonzero, and so since $\Gamma_{\tilde{W}^{\mathrm{aff}}}$ is an ind-scheme we see that there exists some K-point $\lambda_0 \in \Gamma_{\tilde{W}^{\mathrm{aff}}}(K)$ for which $\lambda_0^!(\mathcal{F})$ is nonzero. In particular, for some K-point λ of $\mathfrak{t}^*/X^\bullet(T)$,the pullback of $\mathcal{F}$ via the morphism

$$\mathrm{Spec}(K) \times_{\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}} \mathfrak{t}^*/X^\bullet(T) \to \mathfrak{t}^*/X^\bullet(T) \times_{\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}} \mathfrak{t}^*/X^\bullet(T)$$

does not vanish. Now, note that the following diagram is Cartesian:

$$\begin{array}{ccc} \mathrm{Spec}(K) \times_{\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}} \mathfrak{t}^*/X^\bullet(T) & \xrightarrow{\lambda \times \mathrm{id}} & \mathfrak{t}^*/X^\bullet(T) \times_{\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}} \mathfrak{t}^*/X^\bullet(T) \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \mathring{\phi}} \\ \mathrm{Spec}(K) & \xrightarrow{x} & \mathfrak{t}^*/X^\bullet(T) \end{array}$$

where the vertical arrows are the projection maps onto the left factor.

In particular, we see that the pullback $(\lambda \times \mathrm{id})^!$ is left t-exact as well, since we may check this after pulling back to $\mathrm{Spec}(K) \times_{\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}} \mathfrak{t}^*$ and, by functoriality of !-pullback, we may identify this with the composite of the functor which forgets the $X^\bullet(T) \times X^\bullet(T)$ equivariance and then pullbacks back by the map $\mathrm{Spec}(K) \times_{\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}} \mathfrak{t}^* \to \Gamma_{\tilde{W}^{\mathrm{aff}}}$, both of which are left t-exact. Therefore, we have that

$$\phi_*^{\mathrm{IndCoh}}((\lambda \times \mathrm{id})^!(\mathcal{F}))$$

does not vanish since the pushforward map to a point from $\mathrm{Spec}(K) \times_{\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}} \mathfrak{t}^*/X^\bullet(T)$ can be identified with the pushforward to a point of an Artinian scheme, and this pushforward does not vanish on nonzero eventually coconnective objects. By base change, we see that $x^! \mathring{\phi}_*^{\mathrm{IndCoh}}(\mathcal{F})$ does not vanish, and therefore in particular $\mathring{\phi}_*^{\mathrm{IndCoh}}(\mathcal{F})$ does not vanish. □

4.1.4. *Identification of Coalgebras.* Now we carry out the explicit identification of the coalgebras given by Barr-Beck and Theorem 4.1. Note that, in the notation of Theorem 3.2, we have:

(4)

$$t_*^{\mathrm{IndCoh}}(t^!(\omega_{\mathfrak{t}^*})) \simeq \overline{s}_* s^! \overline{s}^!(\omega_{\mathfrak{t}^*/\!/\tilde{W}^{\mathrm{aff}}}) \simeq \overline{s}^! \overline{s}_*^{\mathrm{IndCoh}} \overline{s}^!(\omega_{\mathfrak{t}^*/\!/\tilde{W}^{\mathrm{aff}}}) \simeq \mathrm{Av}_*^{N_r,(T_r,w)} \mathrm{Av}_!^{N_\ell^-,\psi} \mathrm{Av}_*^{N_\ell,(T_\ell,w)}(\delta_\psi)$$

where the subscripts ℓ and r refer to the left and right averaging, and the last step follows from Theorem 1.4. Continuing this chain of equivalences, and using the fact that left and right averaging canonically commute, we obtain:

(5)

$$t_*^{\mathrm{IndCoh}}(t^!(\omega_{\mathfrak{t}^*})) \simeq \mathrm{Av}_!^{N_\ell^-,\psi} \mathrm{Av}_*^{N_\ell,(T_\ell,w)} \mathrm{Av}_*^{N_r,(T_r,w)}(\delta_\psi) \simeq \mathrm{Av}_!^{N_\ell^-,\psi} \mathrm{Av}_*^{N_\ell,(T_\ell,w)}(\delta_1^{T,w})$$

where $\delta_1^{T,w} \in \mathcal{D}(T)^{T,w}$ refers to the skyscraper sheaf at the identity with the trivial T-representation structure, i.e. the essential image of $i_*^{\mathrm{IndCoh}}(k_{triv})$ under the functor $\mathrm{Rep}(T) \simeq \mathrm{Vect}^{T,w} \to \mathcal{D}(T)^{T,w}$. Similar analysis shows that this is an isomorphism of coalgebras, where, since the composite functors in (4) and (5) are all t-exact (using [Gan23, Corollary 2.44], [Gin18, Theorem 1.5.4], and Theorem 3.2), this is a property of this identfication and not additional structure. An identical argument gives the identification of coalgebras for other equivalence in Theorem 1.14.

Remark 4.9. The structure of the comonad $\mathrm{Av}_*^N \mathrm{Av}_!^\psi$ was previously known on the full subcategory of B-bimonodromic objects of $\mathcal{D}(N\backslash G/N)_{\mathrm{nondeg}}^\heartsuit$, see [Bez16, Section 5.1].

4.1.5. *Identification of Compact Objects.* In [Gan23, Corollary 3.42], we showed that $\mathcal{D}(N\backslash G/N)_{\mathrm{nondeg}}^{(T\times T,\,\mathrm{w})}$ has a canonical set of compact generators labeled by $\tilde{W}^{\mathrm{aff}}$ given by the set $\{\delta_{\mathcal{D}} w : w \in \tilde{W}^{\mathrm{aff}}\}$, where $\delta_{\mathcal{D}}$ denotes the monoidal unit of $\mathcal{D}(N\backslash G/N)_{\mathrm{nondeg}}^{(T\times T,\,\mathrm{w})}$. We obtain a similar description for $\mathrm{IndCoh}(\mathfrak{t}^* \times_{\mathfrak{t}^*/\!/\tilde{W}^{\mathrm{aff}}} \mathfrak{t}^*)$ and use it to complete the proof of the non-monoidal version of Theorem 1.14:

Proposition 4.10. The category $\mathrm{IndCoh}(\mathfrak{t}^* \times_{\mathfrak{t}^*/\!/\tilde{W}^{\mathrm{aff}}} \mathfrak{t}^*)$ has a canonical set of compact generators given by $\{\delta w : w \in \tilde{W}^{\mathrm{aff}}\}$, where $\delta := i_*^{\mathrm{IndCoh}}(\omega_{\mathfrak{t}^*})$ and $i : \mathfrak{t}^* \hookrightarrow \mathfrak{t}^* \times_{\mathfrak{t}^*/\!/\tilde{W}^{\mathrm{aff}}} \mathfrak{t}^*$ the diagonal map, so that δ is the monoidal unit.

Proof. These objects are compact since the IndCoh pushforward by a closed embedding is a left adjoint with a continuous right adjoint, and thus preserves compact objects. We now show this set generates; fix a nonzero $\mathcal{F} \in \mathrm{IndCoh}(\mathfrak{t}^* \times_{\mathfrak{t}^*/\!/\tilde{W}^{\mathrm{aff}}} \mathfrak{t}^*)$. There exists some finite subset $S \subseteq \tilde{W}^{\mathrm{aff}}$ so that $i_S^!(\mathcal{F})$ is nonzero. Note also that that for each finite $S \subseteq \tilde{W}^{\mathrm{aff}}$, the map $\coprod_{w\in S} \mathfrak{t}^* \to \Gamma_S$ is surjective at the level of geometric points, and so in particular, by [GR17a, Proposition 6.2.2], there exists some $w \in \tilde{W}^{\mathrm{aff}}$ such that $i_w^!(\mathcal{F})$ is nonzero. Therefore $\underline{\mathrm{Hom}}(\omega_{\mathfrak{t}^*}, i_w^!(\mathcal{F})) \simeq \underline{\mathrm{Hom}}(i_{w,*}^{\mathrm{IndCoh}}(\omega_{\mathfrak{t}^*}), \mathcal{F}) \simeq \underline{\mathrm{Hom}}(\delta w, \mathcal{F})$ is nonzero. ☐

Proof of Non-Monoidal Version of Theorem 1.14. We have shown in Proposition 4.10 and [Gan23, Corollary 3.42] that both categories $\text{IndCoh}(\mathfrak{t}^* \times_{\mathfrak{t}^*/\!\!/\tilde{W}^{\text{aff}}} \mathfrak{t}^*)$ and $\mathcal{D}(N\backslash G/N)^{T\times T,w}$ are generated by the objects δw, where δ is the monoidal unit in the respective category and $w \in \tilde{W}^{\text{aff}}$. Each of the functors $t_*^{\text{IndCoh,enh}}$ and $\text{Av}_!^{\psi,\text{enh}}$ sends the monoidal unit to equivalent comodules, as we have seen in Section 4.1.4. Therefore, because the functors $t_*^{\text{IndCoh,enh}}$ and $\text{Av}_!^{\psi,\text{enh}}$ are both $\tilde{W}^{\text{aff}}$-equivariant, we see that these functors identify a collection of compact generators. Thus these functors identify the (not necessarily cocomplete) subcategories of the compact objects of $\text{IndCoh}(\mathfrak{t}^* \times_{\mathfrak{t}^*/\!\!/\tilde{W}^{\text{aff}}} \mathfrak{t}^*)$ and the compact objects of $\mathcal{D}(N\backslash G/N)^{(T\times T,\text{ w})}_{\text{nondeg}}$, since both categories may be identified as the objects Karoubi-generated by the compact generators [DG, Corollary 1.4.6]. Both categories are compactly generated and thus may be identified with the ind-completion of their compact objects [DG, Corollary 1.3.4], and therefore are equivalent as DG categories.

Similarly, we have seen in [Gan23, Corollary 3.41] that the category $\mathcal{D}(N\backslash G/N)_{\text{nondeg}}$ has a set of compact generators labelled by W. An identical description holds for the category

$$\text{IndCoh}(\mathfrak{t}^*/X^\bullet(T) \times_{\mathfrak{t}^*/\!\!/\tilde{W}^{\text{aff}}} \mathfrak{t}^*/X^\bullet(T))$$

since, for example, the IndCoh pushforward from $\Gamma_{\tilde{W}^{\text{aff}}}$ (which admits a continuous right adjoint by ind-properness) sends the $\tilde{W}^{\text{aff}}$-compact generators of $\text{IndCoh}(\Gamma_{\tilde{W}^{\text{aff}}})$ to the W objects generating the category. Therefore, we see that the non-monoidal variant of the other equivalence in Theorem 1.14 holds. □

4.2. **Monoidality.** Let $\mathcal{E}$ denote the category $\underline{\text{End}}_{\text{IndCoh}(\mathfrak{t}^*/\!\!/\tilde{W}^{\text{aff}})}(\text{IndCoh}(\mathfrak{t}^*))$ (or, equivalently by Theorem 1.4, $\underline{\text{End}}_{\mathcal{H}_\psi}(\mathcal{D}(N_\psi^-\backslash G/N)^{T,w})$. Note that we have a canonical, monoidal functor

$$F_I : \text{IndCoh}(\mathfrak{t}^* \times_{\mathfrak{t}^*/\!\!/\tilde{W}^{\text{aff}}} \mathfrak{t}^*) \to \mathcal{E}$$

given by the formalism of convolution [GR17a, Chapter 5.5]. We similarly have a monoidal functor $F_{\mathcal{D}}$ given by the composite

$$\mathcal{D}(N\backslash G/N)^{(T\times T,\text{ w})}_{\text{nondeg}} \simeq \underline{\text{End}}_G(\mathcal{D}(G/N)^{T,w}_{\text{nondeg}}) \to \underline{\text{End}}_{\mathcal{H}_\psi}(\mathcal{D}(N_\psi^-\backslash G/N)^{T,w}) \simeq \mathcal{E}$$

where the left equivalence is given by the fact that invariance is coinvariance ([Gai20, Appendix B], [Gai20, Corollary 3.1.5]) the right arrow is given by tensoring with the $\mathcal{D}(G), \mathcal{H}_\psi$ bimodule $\mathcal{D}(N_\psi^-\backslash G)$, and the right equivalence is given again by Theorem 1.4.

We will give our equivalence of categories

$$\text{IndCoh}(\mathfrak{t}^* \times_{\mathfrak{t}^*/\!\!/\tilde{W}^{\text{aff}}} \mathfrak{t}^*) \xrightarrow{\sim} \mathcal{D}(N\backslash G/N)^{(T\times T,\text{ w})}_{\text{nondeg}}$$

a monoidal structure by relating both categories to the category $\mathcal{E}$. We first give some general results on computing limits, and then in Section 4.2.3 we show that the evaluation functor $\mathcal{E} \to \text{IndCoh}(\mathfrak{t}^*)$ is comonadic. Using this, we show how to equip the equivalence of Theorem 1.14 with a monoidal structure in Section 4.2.4.

4.2.1. *Reminders on Lax Limits.* We summarize the following useful proposition on computing limits in a limit of categories in DGCat, see [AG18, Section 4.1] for more information. Assume $I \to \text{DGCat}$ is a diagram of categories for some index ∞-category I, which we denote $i \mapsto \mathcal{C}_i$, and let $\lim \mathcal{C}_i$ denote a limit category. We recall the notion of a *lax-limit category* $\text{lax-lim}\, \mathcal{C}_i$, which is defined using co-Cartesian fibrations and, in particular, whose objects consist of objects $\mathcal{F}_i \in \mathcal{C}_i$ for each $i \in I$ and for every map $i_1 \xrightarrow{\alpha} i_2$ in I, the corresponding map $\Phi_\alpha(\mathcal{F}_{i_1}) \to \mathcal{F}_{i_2}$.

Proposition 4.11. Assume $I \to \text{DGCat}, i \mapsto \mathcal{C}_i$ is defined as above.

- [AG18, Section 4.1.1] There is a natural, fully faithful functor $\lim \mathcal{C}_i \hookrightarrow \text{lax-lim}\, \mathcal{C}_i$, and an object is in the essential image if and only if the associated maps $\Phi_\alpha(\mathcal{F}_{i_1}) \to \mathcal{F}_{i_2}$ are equivalences for all α.
- [AG18, Section 4.1.8] For each $i \in I$, the natural evaluation functor $\text{ev}_i : \text{lax-lim}\, \mathcal{C}_i \to \mathcal{C}_i$ admits a left adjoint, and in particular commutes with limits.

Corollary 4.12. Assume we are given a diagram $J \to \lim {}_i\mathcal{C}_i$, which we write $j \mapsto \mathcal{F}_{j,i} \in \mathcal{C}_i$, such that for each j and for each map $i_1 \xrightarrow{\alpha} i_2$ in I, the corresponding map $\Phi_\alpha(\mathcal{F}_{j,i_1}) \to \mathcal{F}_{j,i_2}$ is an equivalence. Then the corresponding limit is computed termwise.

Proof. The condition that each corresponding map $\Phi_\alpha(\mathcal{F}_{j,i_1}) \to \mathcal{F}_{j,i_2}$ is an equivalence implies that the limit over our J-shaped diagram, computed in the category $\text{lax-lim}\, \mathcal{C}_i$, lies in the category $\lim \mathcal{C}_i$. Since the evaluation functor is a right adjoint, it commutes with limits, thus giving our claim. □

4.2.2. *Nilpotent Towers and Effective Limits.* In this section, we recall the DG-analogue of ideas of Akhil Mathew (see, for example, [Mat18, Subsection 2.3]) which will be used later. For this subsection, fix two DG categories $\mathcal{C}, \mathcal{D}$.

Definition 4.13. Assume we are given a *tower* in $\mathcal{C}$, or, equivalently, a sequence $... \to \mathcal{F}^2 \to \mathcal{F}^1 \to F^0$ in $\mathcal{C}$. We say this tower is *weakly nilpotent* if for all $n \in \mathbb{N}^{\geq 0}$ there exists an N such that for all $m \geq N$, the natural map $\mathcal{F}^{m+n} \to \mathcal{F}^n$ is nullhomotopic.

Definition 4.14. Let $\mathcal{C}$ be some DG category or, more generally, any stable ∞-category, and fix some $\mathcal{F} \in \mathcal{C}$.

(1) Let $\mathcal{F}_\bullet := (... \to \mathcal{F}_1 \to \mathcal{F}_0)$ be a tower in $\mathcal{C}$, and let $\underline{\mathcal{F}}$ denote the constant tower. We say the map of towers $\underline{\mathcal{F}} \to \mathcal{F}_\bullet$ forms an *effective limit* (or, more informally, the maps $\mathcal{F} \to \mathcal{F}_\bullet$ *form an effective limit*) if the tower $n \mapsto \text{cofib}(\mathcal{F} \to \mathcal{F}_n)$ is weakly nilpotent.

(2) Let $S^\bullet$ denote some cosimplicial object of a category $\mathcal{C}$ and temporarily denote by $\mathcal{F}^\bullet$ the constant cosimplicial object. We say the map of cosimplicial objects $\mathcal{F}^\bullet \to S^\bullet$ (or, more informally, the maps $\mathcal{F} \to S^\bullet$) *form an effective limit* if the maps $\mathcal{F} \to \mathrm{Tot}^{\leq n}(S^\bullet)$ form an effective limit.

Remark 4.15. By definition, a tower in $\mathcal{C}$ is an object of the $(\infty, 1)$-category of functors $\mathrm{Fun}(\mathbb{Z}^{op}_{\geq 0}, \mathcal{C})$. Since colimits in functor categories are computed termwise, the cokernel of a map of towers is the tower of cokernels. Note also that if the tower $n \mapsto \mathrm{cofib}(\mathcal{F} \to \mathcal{F}_n)$ is weakly nilpotent, then its limit is zero.

We therefore see that, if the tower $n \mapsto \mathrm{cofib}(\mathcal{F} \to \mathcal{F}_n)$ is weakly nilpotent, the canonical map $\mathcal{F} \simeq \lim(\underline{\mathcal{F}}) \to \lim_n \mathcal{F}_n$ is an equivalence, so the term 'effective limit' is justified. By abuse of notation, we sometimes say that the maps $\mathcal{F} \to \mathcal{F}^i$ form an effective limit.

We now record a basic property of effective limits, see [Mat18, Proposition 2.20]:

Lemma 4.16. Let $F : \mathcal{C} \to \mathcal{D}$ is some exact functor of stable ∞-categories (which is always satisfied if F is a map in $\mathrm{DGCat}^k_{\mathrm{cont}}$), and let $\mathcal{F} \to \mathcal{F}^i$ be a compatible family of maps as in Definition 4.14. Then if the maps $\mathcal{F} \to \mathcal{F}^i$ form an effective limit in $\mathcal{C}$, then the maps $F(\mathcal{F}) \to F(\mathcal{F}^i)$ form an effective limit in $\mathcal{D}$.

Proof. By definition of effective limit, the tower of cofibers given by $C_i := \mathrm{cofib}(\mathcal{F} \to \mathcal{F}_i)$ is weakly nilpotent. By the definition of exactness, F commutes with finite colimits, so that $F(C_i) \simeq \mathrm{cofib}(F(\mathcal{F}) \to F(\mathcal{F}_i))$. Therefore, since F preserves the class of maps which are equivalent to the 0 map, our claim follows, since exact functors preserve the zero object. $\square$

4.2.3. *An Intermediate Comonadic Category.* Observe we have a canonical functor given by the composite

$$\underline{\mathrm{End}}_{\mathcal{H}_\psi}(\mathrm{IndCoh}(\mathfrak{t}^*)) \xrightarrow{P_{\overline{s}^!}} \underline{\mathrm{Hom}}_{\mathcal{H}_\psi}(\mathcal{H}_\psi, \mathrm{IndCoh}(\mathfrak{t}^*)) \simeq \mathrm{IndCoh}(\mathfrak{t}^*)$$

where $P_{\overline{s}^!}$ is the functor which precomposes a functor with $\overline{s}^!$. Since this functor is given on objects via the formula $F \mapsto F(\omega_{\mathfrak{t}^*})$, we refer to it as the *evaluation functor* and denote it by E. The functor $P_{\overline{s}^!}$ has a continuous right adjoint $P_{\overline{s}_*^{\mathrm{IndCoh}}}$, and therefore E has a right adjoint which we denote by E^R. The remainder of this subsection will be devoted to the proof of the following Proposition:

Proposition 4.17. The functor

$$E^{\mathrm{enh}} : \underline{\mathrm{End}}_{\mathcal{H}_\psi}(\mathrm{IndCoh}(\mathfrak{t}^*)) \to EE^R\text{-comod}(\mathrm{IndCoh}(\mathfrak{t}^*))$$

is fully faithful.

We prove Proposition 4.17 after showing the following lemma and deducing a corollary from it. Let $\mathring{\phi} : \mathfrak{t}^*/X^\bullet(T) \to \mathfrak{t}^* /\!\!/ \tilde{W}^{\text{aff}}$ denote the quotient map.

Lemma 4.18. The maps $\text{id}_{\text{IndCoh}(\mathfrak{t}^*)} \to (\overline{s}^! \overline{s}_*^{\text{IndCoh}})^{\bullet+1}$ and $\text{id}_{\mathcal{D}(T)} \to (\mathring{\phi}^! \mathring{\phi}_*^{\text{IndCoh}})^{\bullet+1}$ form an effective limit.

Proof. Using the identification $\text{ev}_{\omega_{\mathfrak{t}^*}} : \underline{\text{End}}(\text{IndCoh}(\mathfrak{t}^*)) \xrightarrow{\sim} \text{IndCoh}(\mathfrak{t}^* \times \mathfrak{t}^*)$ this claim is equivalent to the claim that the maps $\omega_{\mathfrak{t}^*} \to (\overline{s}^! \overline{s}_*^{\text{IndCoh}})^{\bullet+1}(\omega_{\mathfrak{t}^*})$ form an effective limit. Let $c_n : \omega_{\mathfrak{t}^*} \to \text{Tot}^{\leq n}(\overline{s}^! \overline{s}_*^{\text{IndCoh}})^{\bullet+1}(\omega_{\mathfrak{t}^*})$ denote the canonical map for each n. We claim that for each n, we have that $\tau^{\leq n-1}c_n$ is an equivalence. To see this, note that by the conservativity of $\overline{s}_*^{\text{IndCoh}}$ (Corollary 4.6) it suffices to show that $\overline{s}_*^{\text{IndCoh}}\tau^{\leq n-1}c_n$ is an equivalence. The t-exactness of $\overline{s}_*^{\text{IndCoh}}$ (Theorem 3.2) allows us to identify this map with $\tau^{\leq n-1}\overline{s}_*^{\text{IndCoh}}c_n$. Since $\overline{s}_*^{\text{IndCoh}}$ is exact, it commutes with finite limits, so we may furthermore identify $\tau^{\leq n-1}\overline{s}_*^{\text{IndCoh}}c_n$ with the map

$$\tau^{\leq n-1}\overline{s}_*^{\text{IndCoh}}\omega_{\mathfrak{t}^*} \to \tau^{\leq n-1}\text{Tot}^{\leq n}\overline{s}_*^{\text{IndCoh}}(\overline{s}^! \overline{s}_*^{\text{IndCoh}})^{\bullet+1}(\omega_{\mathfrak{t}^*})$$

and by Lemma 2.5 we may further identify this map with the canonical map

$$\tau^{\leq n-1}\overline{s}_*^{\text{IndCoh}}\omega_{\mathfrak{t}^*} \to \tau^{\leq n-1}\text{Tot}(\overline{s}_*^{\text{IndCoh}}(\overline{s}^! \overline{s}_*^{\text{IndCoh}})^{\bullet+1}(\omega_{\mathfrak{t}^*}))$$

using the fact that $(\overline{s}^! \overline{s}_*^{\text{IndCoh}})^{j+1}(\omega_{\mathfrak{t}^*})$ lies in the heart for every $j \in \mathbb{Z}^{\geq 0}$, see Theorem 3.2. However, this map is an equivalence since the cosimplicial object $\overline{s}_*^{\text{IndCoh}}(\overline{s}^! \overline{s}_*^{\text{IndCoh}})^{\bullet+1}$ is split by $\overline{s}_*^{\text{IndCoh}}$. We therefore see that $\tau^{\leq n-1}c_n$ is an equivalence.

We also have that $\text{Tot}^{\leq n}(\overline{s}^! \overline{s}_*^{\text{IndCoh}})^{\bullet+1}(\omega_{\mathfrak{t}^*})$ is a totalization of objects in the heart of a category equivalent to A-mod for some classical ring A, again using the exactness of Theorem 3.2. We thus see see $\text{Tot}^{\leq n}(\overline{s}^! \overline{s}_*^{\text{IndCoh}})^{\bullet+1}(\omega_{\mathfrak{t}^*})$ lies in cohomological degree $[0, n]$. Therefore, if K^n denotes the cofiber of the map c_n, this cofiber is concentrated in degree n since $\tau^{\leq n-1}c_n$ is an equivalence. In particular, we may choose $N \gg 0$ so that the space $\text{Hom}_{\text{IndCoh}(\mathfrak{t}^* \times \mathfrak{t}^*)}(K^{N+n}, K^n)$ is connected (by the finite cohomological dimension of the t-structure on $\text{IndCoh}(\mathfrak{t}^* \times \mathfrak{t}^*)$), so the maps from the identity to the tower of partial totalizations of our cosimplicial object form an effective limit by definition. Since we have obtained the analogous conservativity and t-exactness of the pushforward $(\mathfrak{t}^*/X^\bullet(T) \to \mathfrak{t}^* /\!\!/ \tilde{W}^{\text{aff}})_*^{\text{IndCoh}}$, an identical argument holds for the latter functor as well, since $\mathcal{D}(T)$ also has finite cohomological dimension. □

Corollary 4.19. For any $i \in \mathbb{Z}^{\geq 1}$, the maps $\text{id} \to (\text{id}_{\mathcal{H}_\psi^{\otimes i-1}} \otimes \overline{s}^! \overline{s}_*^{\text{IndCoh}})^{\bullet+1}$ and the maps $\text{id} \to (\text{id}_{\mathcal{H}_\psi^{\otimes i-1}} \otimes \mathring{\phi}^! \mathring{\phi}_*^{\text{IndCoh}})^{\bullet+1}$ form an effective limit.

Proof. We show the first claim, the second follows from an identical argument. First note that the functor

$$\text{id}_{\mathcal{H}_\psi^{\otimes i}} \otimes - : \underline{\text{End}}(\text{IndCoh}(\mathfrak{t}^*)) \to \underline{\text{End}}(\mathcal{H}_\psi^{\otimes i-1} \otimes \text{IndCoh}(\mathfrak{t}^*))$$

is exact since it is continuous [Lur17, Proposition 1.1.4.1]. Thus this functor preserves effective limits by Lemma 4.16, so we see that the maps

$$\mathrm{id} \to \mathrm{id}_{\mathcal{H}_\psi^{i-1}} \otimes \mathrm{Tot}^{\leq n}(\overline{s}^!\overline{s}_*^{\mathrm{IndCoh}})^{\bullet+1}$$

form an effective limit. Our claim then follows from the fact that the functor $\mathrm{id}_{\mathcal{H}_\psi^{i-1}} \otimes -$ is exact and thus commutes with finite limits. □

Proof of Proposition 4.17. It suffices to show that for any $F \in \underline{\mathrm{End}}_{\mathcal{H}_\psi}(\mathrm{IndCoh}(\mathfrak{t}^*))$ that the canonical map $F \to \lim_\Delta(EE^R)^{\bullet+1}(F))$ is an isomorphism as this will show that the unit map associated to the left adjoint E^{enh} is an isomorphism. Since the evaluation functor is conservative, it suffices to show that the canonical map $E(F) \to E(\lim_\Delta(EE^R)^{\bullet+1}(F)))$ is an equivalence. Since $(EE^R)^{\bullet+1}(F)$ is canonically an E-split totalization, the map $E(F) \to \lim_\Delta(E(EE^R)^{\bullet+1}(F))$ is an equivalence. Therefore it suffices to prove that the canonical map

$$E\lim_{\Delta}((EE^R)^{\bullet+1}(F)) \to \lim_{\Delta}(E(EE^R)^{\bullet+1}(F))$$

is an equivalence.

We identify $\underline{\mathrm{End}}_{\mathcal{H}_\psi}(\mathrm{IndCoh}(\mathfrak{t}^*))$ as the limit

$$\lim_\Delta(\underline{\mathrm{Hom}}(\mathcal{H}_\psi^\bullet \otimes \mathrm{IndCoh}(\mathfrak{t}^*), \mathrm{IndCoh}(\mathfrak{t}^*))).$$

Let I denote the inclusion of this category into the lax-limit category

$$\text{lax-lim}_\Delta\underline{\mathrm{Hom}}(\mathcal{H}_\psi^\bullet \otimes \mathrm{IndCoh}(\mathfrak{t}^*), \mathrm{IndCoh}(\mathfrak{t}^*)).$$

We compute the limit $\lim_\Delta I(F(\overline{s}^!\overline{s}_*^{\mathrm{IndCoh}})^{\bullet+1})$ and show it lies in the subcategory $\lim_\Delta\underline{\mathrm{Hom}}(\mathcal{H}_\psi^\bullet \otimes \mathrm{IndCoh}(\mathfrak{t}^*), \mathrm{IndCoh}(\mathfrak{t}^*))$. Fix some map in Δ, say $i_1 \xrightarrow{\alpha} i_2$, and let

$$\Phi_\alpha : \underline{\mathrm{Hom}}(\mathcal{H}_\psi^{i_1} \otimes \mathrm{IndCoh}(\mathfrak{t}^*), \mathrm{IndCoh}(\mathfrak{t}^*)) \to \underline{\mathrm{Hom}}(\mathcal{H}_\psi^{i_2} \otimes \mathrm{IndCoh}(\mathfrak{t}^*), \mathrm{IndCoh}(\mathfrak{t}^*))$$

denote the canonical map obtained by pullback. Since this map is exact and the maps $\mathrm{id} \to (\mathrm{id}_{\mathcal{H}_\psi} \otimes \overline{s}^!\overline{s}_*^{\mathrm{IndCoh}})^{\bullet+1}$ form an effective limit (Corollary 4.19), by Lemma 4.16 we obtain a canonical equivalence

$$\Phi_\alpha \lim_\Delta IF_{\bullet,i_1} \xrightarrow{\sim} \lim_\Delta \Phi_\alpha IF_{\bullet,i_1}$$

where we denote $IF_{\bullet,i_1}$ the object $\mathrm{ev}_{i_1}(IF_{\bullet,i_1})$. We obtain an equivalence

$$\lim_\Delta \Phi_\alpha IF_{\bullet,i_1} \xrightarrow{\sim} \lim_\Delta IF_{\bullet,i_2}$$

since $F_{\bullet,i_1}$ lies in the limit category. Thus by Proposition 4.11 we see that the canonical map

$$I(\lim_\Delta F(\overline{s}^!\overline{s}_*^{\mathrm{IndCoh}})^{\bullet+1}) \to \lim_\Delta I(F(\overline{s}^!\overline{s}_*^{\mathrm{IndCoh}})^{\bullet+1})$$

is an equivalence. In particular, this limit is computed termwise by Proposition 4.11(2), as desired. □

An analogous argument gives:

Corollary 4.20. The functor $E_0^{\text{enh}} : \underline{\text{End}}_{\mathcal{H}_\psi}(\mathcal{D}(T)) \to E_0 E_0^R\text{-comod}(\mathcal{D}(T))$ is fully faithful.

4.2.4. *Identification of Monoidal Categories.* Since, tautologically, $E \circ F_I \simeq t_*^{\text{IndCoh}}$ and $E \circ F_{\mathcal{D}} \simeq \text{Av}_!^{\psi}$, we obtain the following commutative diagram:

$$\begin{array}{ccccc} I(\mathfrak{t}^* \times_{\mathfrak{t}^* /\!\!/ \tilde{W}^{\text{aff}}} \mathfrak{t}^*)^+ & \xrightarrow{F_I} & \underline{\text{End}}_{\mathcal{H}_\psi}(I(\mathfrak{t}^*)) & \xleftarrow{F_{\mathcal{D}}} & \mathcal{D}(N\backslash G/N)_{\text{nondeg}}^{(T\times T,\text{ w}),\,+} \\ \downarrow{\scriptstyle t_*^{\text{IndCoh,enh}}} & & \downarrow{\scriptstyle E^{\text{enh}}} & & \downarrow{\scriptstyle \text{Av}_!^{\psi,\text{enh}}} \\ t_*t^!\text{-comod}(I(\mathfrak{t}^*)^+) & \longrightarrow & EE^R\text{-comod}(I(\mathfrak{t}^*)) & \longleftarrow & \text{Av}_!^{\psi}\text{Av}_*^{N,(T,w)}\text{-comod}(I(\mathfrak{t}^*)^+) \end{array}$$

where for notational shorthand we use the symbol I to denote IndCoh. Moreover, by the computation Section 4.1.4, all of the comonads appearing in the lower row of this diagram are naturally isomorphic, and so both of the maps contained in the bottom row of this diagram are fully faithful. Moreover, the fully faithfulness of $t_*^{\text{IndCoh,enh}}$ and $\text{Av}_!^{\psi,\text{enh}}$ follow directly from the comonadicity of the non-enhanced functors of Theorem 4.1. Finally, in Proposition 4.17 we have seen that E^{enh} is fully faithful. We therefore see that F_I and $F_{\mathcal{D}}$ are fully faithful when restricted to the respective eventually coconnective subcategories; a completely analogous argument using Corollary 4.20 shows that the monoidal functors

$$\text{IndCoh}(\mathfrak{t}^*/X^\bullet(T)\times_{\mathfrak{t}^* /\!\!/ W^{\text{aff}}}\mathfrak{t}^*/X^\bullet(T))^+ \to \text{End}_{\mathcal{H}_\psi}(\mathcal{D}(T)) \leftarrow \mathcal{D}(N\backslash G/N)^+_{\text{nondeg}}$$

are fully faithful. Moreover, we claim that all of these eventually coconnective subcategories are monoidal, which we now prove:

Proposition 4.21. The convolution structure on the categories

$$\text{IndCoh}(\mathfrak{t}^* \times_{\mathfrak{t}^* /\!\!/ \tilde{W}^{\text{aff}}} \mathfrak{t}^*),\ \text{IndCoh}(\mathfrak{t}^*/X^\bullet(T) \times_{\mathfrak{t}^* /\!\!/ \tilde{W}^{\text{aff}}} \mathfrak{t}^*/X^\bullet(T))$$

$$\mathcal{D}(N\backslash G/N)_{\text{nondeg}}^{(T\times T,\text{ w})},\text{ and } \mathcal{D}(N\backslash G/N)_{\text{nondeg}}$$

preserve the respective eventually coconnective subcategories.

Proof. Convolution is given by the pullback by a closed embedding and the pushforward of $(\text{id}, \overline{s}, \text{id}) : \mathfrak{t}^* \times_{\mathfrak{t}^* /\!\!/ \tilde{W}^{\text{aff}}} \mathfrak{t}^* \times_{\mathfrak{t}^* /\!\!/ \tilde{W}^{\text{aff}}} \mathfrak{t}^* \to \mathfrak{t}^* \times_{\mathfrak{t}^* /\!\!/ \tilde{W}^{\text{aff}}} \mathfrak{t}^* /\!\!/ \tilde{W}^{\text{aff}} \times_{\mathfrak{t}^* /\!\!/ \tilde{W}^{\text{aff}}} \mathfrak{t}^*$. We have seen that the latter is exact by Theorem 3.2, and the former is exact since the pullback by a closed embedding is a right adjoint to a t-exact functor and therefore left t-exact. Thus convolution preserves the eventually coconnective subcategory $\text{IndCoh}(\mathfrak{t}^* \times_{\mathfrak{t}^* /\!\!/ \tilde{W}^{\text{aff}}} \mathfrak{t}^*)^+$. Since forgetting $X^\bullet(T)$ reflects the t-structure, we see by base change that the convolution monoidality on $\text{IndCoh}(\mathfrak{t}^*/X^\bullet(T) \times_{\mathfrak{t}^* /\!\!/ \tilde{W}^{\text{aff}}} \mathfrak{t}^*/X^\bullet(T))$ preserves the eventually coconnective subcategories as well.

For $\mathcal{D}(N\backslash G/N)_{\text{nondeg}}^{(T\times T,\text{ w})}$ and $\mathcal{D}(N\backslash G/N)_{\text{nondeg}}$, we note that the convolution is given by the composite of a forgetful functor and an averaging functor, which are t-exact and left t-exact (again as the right adjoint of a right t-exact functor is left t-exact) respectively, so the claim follows. □

We have therefore constructed a monoidal equivalence of the respective eventually coconnective subcategories. Since the convolution structures commute with colimits, all compact objects are eventually coconnective, and the four categories of Theorem 1.14 are compactly generated, we in particular can equip our earlier 'non-monoidal' equivalences with a monoidal structure, thus completing the proof of monoidality and thus the proof of Theorem 1.14. □

Remark 4.22. We expect similar methods yield equivalences of categories

$$\mathfrak{g}\text{-mod}^N_{\text{nondeg}} \simeq \text{IndCoh}(\mathfrak{t}^* /\!\!/ W \times_{\mathfrak{t}^*/\!\!/ W^{\text{aff}}} \mathfrak{t}^*/X^\bullet(T))$$

compatible with the T action and the $\text{IndCoh}(\mathfrak{t}^* /\!\!/ W)$ action, and a monoidal equivalence

$$\mathcal{HC}_{\text{nondeg}} := \underline{\text{End}}_G(\mathcal{D}(G)^{G,w}_{\text{nondeg}}) \simeq \text{IndCoh}(\mathfrak{t}^* /\!\!/ W \times_{\mathfrak{t}^*/\!\!/ W^{\text{aff}}} \mathfrak{t}^* /\!\!/ W).$$

5. The Nondegenerate Horocycle Functor

In this section, we construct a nondegenerate variant of the horocycle functor and show in Section 5.2 that it can be used to equip $\text{Res}(\mathcal{F})$ with a W-equivariance which descends to the coarse quotient $\mathfrak{t}^* /\!\!/ \tilde{W}^{\text{aff}}$ for $\mathcal{F} \in \mathcal{D}(G)^{G,\heartsuit}$ very central.

5.1. **The Nondegenerate Horocycle Functor.**

We now construct a functor on $G \times G$ categories $\mathcal{C}$ which for $\mathcal{C} = \mathcal{D}(G)$ recovers the usual horocycle functor. To define such a functor, it suffices to define it in the 'universal case' $\mathcal{C} = \mathcal{D}(G \times G)$. We consider the category $\mathcal{D}(G \times G)$ as a right $G \times G$ category and let Ψ denote the composite functor

$$\mathcal{D}(G \times G)^{\Delta_G} \xrightarrow{\text{oblv}^{\Delta_G}_{\Delta_B}} \mathcal{D}(G \times G)^{\Delta_B} \xrightarrow{\text{Av}_*^{N \times N}} \mathcal{D}(G/N \times G/N)^{\Delta_T}$$

where the group Δ_G denotes the diagonal copy of G and the rightmost functor is induced by the averaging functor. Let $J^!$ denote the quotient functor $\mathcal{D}(G/N \times G/N) \to \mathcal{D}(G/N \times G/N)_{\text{nondeg}}$ which projects onto the nondegenerate subcategory, again taken with respect to the right action. Since this nondegenerate category is closed under the action of $T \times T$, we may equivalently view $J^!$ as a functor $\mathcal{D}(G/N \times G/N)^{\Delta_T} \to \mathcal{D}(G/N \times G/N)^{\Delta_T}_{\text{nondeg}}$.

Theorem 5.1. The functor $J^!\Psi : \mathcal{D}(G \times G)^{\Delta_G} \to \mathcal{D}(G/N \times G/N)^{\Delta_T}_{\text{nondeg}}$ lifts to a functor of $G \times G$ categories:

$$\tilde{\Psi} : \mathcal{D}(G \times G)^{\Delta_G} \to \mathcal{D}(G/N \times G/N)^{\Delta_{T \rtimes W}}_{\text{nondeg}}.$$

Furthermore, fixing a simple coroot α, we have the following:

(1) The action of the Klein four group $\langle s_\alpha \times s_\alpha \rangle$ on $\mathcal{D}(G/N \times G/N)^{\Delta_T \mathbb{G}_m^\alpha}_{\text{nondeg}}$ is trivial.

(2) The composite

(6)
$$\mathcal{D}(G\times G)^{\Delta_G} \xrightarrow{\mathrm{Av}_*^{\mathbb{G}_m^\alpha}\mathrm{oblv}^W_{\langle s_\alpha\rangle}\tilde{\Psi}} \mathcal{D}(G/N\times G/N)^{\Delta_T\mathbb{G}_m^\alpha,\langle s_\alpha\rangle}_{\text{nondeg}} \simeq \mathcal{D}(G/N\times G/N)^{\Delta_T\mathbb{G}_m^\alpha}_{\text{nondeg}}\otimes \mathrm{Rep}\langle s_\alpha\rangle$$

where the second equivalence is given by (2), lies entirely in the summand indexed by the trivial representation.

(3) If $\mathcal{F}\in\mathcal{D}(G)^G \simeq \mathcal{D}(G\times G)^{\Delta_G^\ell\times\Delta_G^r}$ is very central, then the sheaf

$$\tilde{\Psi}(\mathcal{F})\in\mathcal{D}(N\backslash G/N)^{\Delta_{T\rtimes W}}_{\text{nondeg}} \simeq \mathcal{D}((G/N\times G/N)/T)^{\Delta_G,\Delta_W}_{\text{nondeg}}$$

has the property that the canonical $\langle s_\alpha\rangle$-representation on $\mathrm{Av}_*^{\mathbb{G}_m^\alpha}\tilde{\Psi}(\mathcal{F})$ is trivial.

The final point of Theorem 5.1 should be compared to the condition of descending to the coarse quotient given by the final point of Proposition 3.1. Note that we may identify $\mathrm{hc} := \Psi^{\Delta_G^\ell}$.

Proof. Note that the functor Ψ itself is G-equivariant and so it suffices to construct a lift of $\mathrm{hc}(\delta_{\Delta_G})$. However, hc is monoidal. This fact as well known, and can be seen by identifying the monoidal functor

$$\mathcal{D}(G)^G \simeq \underline{\mathrm{End}}_{G\times G}(\mathcal{D}(G)) \xrightarrow{\mathrm{Av}_*^N} \underline{\mathrm{End}}_{G\times T}(\mathcal{D}(G/N)) \simeq \mathcal{D}(N\backslash G/N)^T \quad (7)$$

with hc. Therefore, we see that we may identify $\Psi(\delta_{\Delta_G})$ (with its left Δ_G-equivariance) in the category $\mathcal{D}(N\backslash G/N)^T$ with the monoidal unit equipped with its canonical T-equivariance. In particular, $J^!\Psi(\delta_{\Delta_G})$ may be identified with $J^!(\delta_1)$, where $\delta_1\in\mathcal{D}(N\backslash G/N)^T$ is the monoidal unit. Under the equivalence Theorem 1.14, the sheaf $J^!(\delta_1)$ corresponds to the pushforward $\Delta_*^{\mathrm{IndCoh}}(\omega_{\mathfrak{t}^*}/X^\bullet(T))$. In particular, this sheaf is equivariant with respect to the diagonal W-action. Thus we see that $J^!\Psi(\delta_{\Delta_G})\in\mathcal{D}(G/N\times G/N)^{T,\Delta_G}_{\text{nondeg}} \simeq \mathcal{D}(N\backslash G/N)^T_{\text{nondeg}}$ may be equipped with a canonical W-equivariant structure, showing (1). Point (2) follows directly from [Gan23, Corollary 3.45]

To show (3), note that we have identifications

$$\mathrm{oblv}^W_{\langle s_\alpha\rangle}J^!\Psi(\delta_{\Delta_G}) \simeq \mathrm{oblv}^W_{\langle s_\alpha\rangle}J^!\mathrm{Av}_*^{\mathbb{G}_m^\alpha}\Psi(\delta_{\Delta_G}) \quad (8)$$

since the quotient functor is $T\times T$-equivariant. Under the Mellin transform, the sheaf $\mathrm{Av}_*^{\mathbb{G}_m^\alpha}\Psi(\delta_{\Delta_G})\in\mathcal{D}(T/\mathbb{G}_m^\alpha)$ corresponds to the monoidal unit. In particular, (8) gives that the integral kernel of the $G\times G$-equivariant functor given by (6) lies in the full $G\times G$-subcategory indexed by the trivial representation, establishing (3). Finally, (4) is a special case of (3). □

Remark 5.2. Note that the integral kernel of the composite $ch\circ hc$ on the level of nondegenerate categories canonically acquires a W-representation structure, and the sheaf $ch\circ hc(\delta)$ is known to be the Springer sheaf. This sheaf has endomorphisms which may be identified with the group ring of W–for a recent survey of this, see [Ben+21]. Therefore we expect that, at least on the level of nondegenerate categories, the functor $\tilde{hc} : \mathcal{D}(G)^G_{\text{nondeg}} \to \mathcal{D}(N\backslash G/N)^{T\rtimes W}_{\text{nondeg}}$ is fully faithful.

Of course, one did not need to pass to nondegenerate categories to obtain that the composite $ch \circ hc$ is given by convolution with the Springer sheaf. Therefore, one might hope that the functor hc factors through some subcategory $\mathcal{D} \hookrightarrow \mathcal{D}(N\backslash G/N)^T$ which acquires a W-action, giving rise to a fully faithful functor $\tilde{hc} : \mathcal{D}(G)^G \hookrightarrow \mathcal{D}^W$. We do not yet know what to make of this.

5.2. **Construction of W-Equivariance for Parabolic Restriction of Very Central Sheaves.** We now use the computations on the nondegenerate horocycle functor above to prove Theorem 1.22.

Proof of Theorem 1.22. Because T is connected, the forgetful functor induces an exact equivalence of abelian categories

$$\text{oblv}^T : \mathcal{D}(T)^{T\rtimes W,\heartsuit} \xrightarrow{\sim} \mathcal{D}(T)^{W,\heartsuit}$$

and so, in particular, to prove Theorem 1.22, it suffices to exhibit the required W-equivariant structure on $\text{oblv}^T\text{Res}(\mathcal{F})$. Since $\mathcal{F}$ is very central, the canonical map $i_{*,dR}\text{Res}(\mathcal{F}) \to \text{hc}(\mathcal{F})$ is an isomorphism by definition. In particular, the canonical map

$$J^! i_{*,dR}\text{oblv}^T\text{Res}(\mathcal{F}) \to J^!\text{oblv}^T\text{hc}(\mathcal{F}) \tag{9}$$

is also an isomorphism. Then, by taking the left diagonal G invariants of the functor $\tilde{\Psi}$ of Theorem 5.1, we see that we may equip $J^!\text{oblv}^T\text{hc}(\mathcal{F})$ with a W-equivariant structure, and thus we may also equip $J^! i_{*,dR}\text{oblv}^T\text{Res}(\mathcal{F})$ with a W-equivariant structure. Furthermore, the functor

$$\mathcal{D}(N\backslash G/N)_{\text{nondeg}} \xrightarrow{\text{Av}_!^{\psi}} \mathcal{D}(N^-_{\psi}\backslash G/N)$$

given by *left* Whittaker averaging is W-equivariant, where W acts on the domain diagonally and the codomain via the usual W-action.[6] We lightly abuse notation and denote the composite of $\text{Av}_!^{\psi}$ with the W-equivariant equivalence of Proposition 1.8 by $\text{Av}_!^{\psi}$. Since this functor is a map of W-categories, we obtain a W-equivariant structure on

$$\text{Av}_!^{\psi}(J^! i_{*,dR}\text{oblv}^T\text{Res}(\mathcal{F})) \simeq \text{Av}_!^{\psi}(i_{*,dR}\text{oblv}^T\text{Res}(\mathcal{F})) \simeq \text{oblv}^T\text{Res}(\mathcal{F}) \tag{10}$$

where the first equivalence is given by the definition of nondegeneracy and the second is given by direct computation.

We now show that the sheaf $\mathcal{R} := \text{oblv}^T(\text{Res}(\mathcal{F}))$ descends to the coarse quotient when equipped with the W-equivariance above. To see this, by Proposition 3.1(3) it suffices to show that, for every simple coroot γ, the

[6]Under Theorem 1.14, this corresponds to the fact that the projection map $\mathfrak{t}^*/X^\bullet(T) \times_{\mathfrak{t}^*/\!\!/\tilde{W}^{\text{aff}}} \mathfrak{t}^*/X^\bullet(T) \to \mathfrak{t}^*/X^\bullet(T)$ is W-equivariant, where W acts diagonally on the product and acts by the standard way on $\mathfrak{t}^*/X^\bullet(T)$.

$\langle s_\gamma\rangle$-representation on $\mathrm{Av}_*^{\mathbb{G}_m^\gamma}(\mathcal{R})$ is trivial, where $\mathbb{G}_m^\gamma$ acts on $\mathcal{D}(N_\psi^-\backslash G/N)$ the right. However, we see that we have W-equivariant equivalences

$$\mathrm{Av}_*^{\mathbb{G}_m^\gamma}(\mathcal{R}) \simeq \mathrm{Av}_*^{\mathbb{G}_m^\gamma}\mathrm{Av}_!^\psi(J^! i_{*,dR}(\mathcal{R})) \simeq \mathrm{Av}_!^\psi\mathrm{Av}_*^{\mathbb{G}_m^\gamma}(J^! i_{*,dR}(\mathcal{R})) \simeq \mathrm{Av}_!^\psi\mathrm{Av}_*^{\mathbb{G}_m^\gamma}(\mathrm{oblv}^T J^!\mathrm{hc}(\mathcal{F}))$$

where the first equivalence follows from (10), the second equivalence follows from the fact that $\mathrm{Av}_!^\psi$ is right T-equivariant, and the third equivalence follows from (9) and the fact that $J^!$ is $T\times T$-equivariant. We therefore see that we have a W-equivariant equivalence

$$\mathrm{Av}_*^{\mathbb{G}_m^\gamma}(\mathcal{R}) \simeq \mathrm{Av}_!^\psi\mathrm{oblv}^T\mathrm{Av}_*^{\mathbb{G}_m^\gamma}(J^!\mathrm{hc}(\mathcal{F})) \tag{11}$$

by base changing along the Cartesian diagram of quotient maps

$$\begin{array}{ccc} X & \longrightarrow & X/\mathbb{G}_m^\gamma \\ \downarrow & & \downarrow \\ X/\Delta_T & \longrightarrow & X/\Delta_T\mathbb{G}_m^\gamma \end{array}$$

where $X := G/N \times G/N$ Now, by Theorem 5.1(3) we see that the $\langle s_\gamma\rangle$-representation on $\mathrm{Av}_!^\psi\mathrm{oblv}^T\mathrm{Av}_*^{\mathbb{G}_m^\gamma}(J^!\mathrm{hc}(\mathcal{F}))$ is trivial. Thus by (11) we obtain the $\langle s_\gamma\rangle$-representation on $\mathrm{Av}_*^{\mathbb{G}_m^\gamma}(\mathcal{R})$ is trivial, as desired. ☐

Appendix A. Mellin Transform (With Germán Stefanich)

In this appendix, written jointly with Germán Stefanich, we discuss the foundations of the Mellin transform in the higher categorical setting we use above. In Appendix A.1, we give the construction of the Mellin transform in the higher categorical setting, and we exhibit a functoriality of the Mellin transform we use above in Appendix A.2. Finally, in Appendix A.3, we upgrade this Mellin transform to an equivalence of symmetric monoidal DG categories.

A.1. Derivation of Mellin Transform.

Notice that the following diagram is Cartesian

$$\begin{array}{ccc} X^\bullet(T)\times\mathfrak{t}^* & \xrightarrow{\mathrm{act}} & \mathfrak{t}^* \\ \downarrow{\scriptstyle \mathrm{proj}} & & \downarrow{\scriptstyle q} \\ \mathfrak{t}^* & \xrightarrow{q} & \mathfrak{t}^*/X^\bullet(T) \end{array} \tag{12}$$

which shows that the map q is ind-proper. Therefore, the functor q_*^{IndCoh} (defined via an identical procedure below [Gan22, Corollary 4.14]) is left adjoint to $q^!$. Moreover, since the functor $q^!$ is conservative, the sheaf $q_*^{\mathrm{IndCoh}}(\mathfrak{t}^*)$ is a compact generator of $\mathrm{IndCoh}(\mathfrak{t}^*/X^\bullet(T))$ and therefore gives an equivalence of categories

$$\mathrm{IndCoh}(\mathfrak{t}^*/X^\bullet(T)) \simeq \underline{\mathrm{End}}_{\mathrm{IndCoh}(\mathfrak{t}^*/X^\bullet(T))}(q_*^{\mathrm{IndCoh}}(\mathfrak{t}^*))\text{-mod}. \tag{13}$$

Notice that, as a graded vector space, by base changing along (12), we may identify

$$\underline{\mathrm{End}}_{\mathrm{IndCoh}(\mathfrak{t}^*/X^\bullet(T))}(q_*^{\mathrm{IndCoh}}(\mathfrak{t}^*)) \simeq \underline{\mathrm{Hom}}_{\mathrm{IndCoh}(\mathfrak{t}^*)}(\omega_{\mathfrak{t}^*}, \mathrm{proj}_*^{\mathrm{IndCoh}}(\omega_{X^\bullet(T)\times\mathfrak{t}^*}))$$
$$\simeq \oplus_{X^\bullet(T)}\underline{\mathrm{End}}_{\mathrm{IndCoh}(\mathfrak{t}^*)}(\omega_{\mathfrak{t}^*}) \simeq \oplus_{X^\bullet(T)}\mathrm{Sym}(\mathfrak{t})$$

which is in particular a discrete vector space concentrated in a single cohomological degree. One may therefore check that this equivalence of graded vector spaces upgrades to an equivalence of k-algebras

$$\underline{\mathrm{End}}_{\mathrm{IndCoh}(\mathfrak{t}^*/X^\bullet(T))}(q_*^{\mathrm{IndCoh}}(\mathfrak{t}^*)) \simeq \Gamma(\mathcal{D}_T)$$

and so the equivalence given by (13) yields equivalences

$$\mathrm{IndCoh}(\mathfrak{t}^*/X^\bullet(T)) \simeq \Gamma(\mathcal{D}_T)\text{-mod} \simeq \mathcal{D}(T) \tag{14}$$

whose composite we denote by FMuk_T and refer to as the *Mellin transform.* When the associated torus T is clear from context, we may also denote this transformation by FMuk. Equipping $\mathrm{IndCoh}(\mathfrak{t}^*/X^\bullet(T))$ with a t-structure as in [Gan22], we have that $q_*^{\mathrm{IndCoh}}(\omega_{\mathfrak{t}^*})$ is concentrated in cohomological degree $-\dim(T)$ and so the Mellin transform is t-exact up to cohomological shift.

A.2. **Functoriality of Mellin Transform.** We will use the following fact:

Proposition A.1. Assume $1 \to S \to T \xrightarrow{\phi} T/S \to 1$ is a short exact sequence of split algebraic tori. Then we have a canonical isomorphism of functors

$$\begin{array}{ccc} \mathcal{D}(T) & \xrightarrow{\phi_{*,dR}} & \mathcal{D}(T/S) \\ \big\downarrow{\scriptstyle \mathrm{FMuk}_T} & & \big\downarrow{\scriptstyle \mathrm{FMuk}_{T/S}} \\ \mathrm{IndCoh}(\mathfrak{t}^*/X^\bullet(T)) & \xrightarrow{\iota^!} & \mathrm{IndCoh}((\mathfrak{t}/\mathfrak{s})^*/X^\bullet(T/S)) \end{array} \tag{15}$$

where $\iota : (\mathfrak{t}/\mathfrak{s})^*/X^\bullet(T/S) \to \mathfrak{t}^*/X^\bullet(T)$ is the induced map given by inclusion.

Proof. The above short exact sequence splits, and therefore by induction we may assume that $S = \mathbb{G}_m$. In this case, $\phi_{*,dR}(\mathcal{D}_T)$ is equivalently given by the complex

$$0 \to \Gamma(\mathcal{D}_T) \xrightarrow{\partial_S} \Gamma(\mathcal{D}_T) \to 0$$

so that we may identify $\phi_{*,dR}(\mathcal{D}_T)$ as a direct sum of $\mathbb{Z}$-many copies of $\Gamma(\mathcal{D}_{T/S})$. We claim also that $(\mathrm{FMuk}_{T/S}^{-1} \circ \iota^! \circ \mathrm{FMuk}_T)(\mathcal{D}_T)$ maps to an isomorphic $\Gamma(\mathcal{D}_T)$-module; this can be checked explicitly since $\phi_{*,dR}(\mathcal{D}_T)$

lies in the heart of a t-structure. To see this, notice that the following diagram is Cartesian

$$
\begin{array}{ccc}
\mathfrak{t}^* \times_{\mathfrak{s}^*} X^\bullet(S) & \xrightarrow{q} & (\mathfrak{t}^* \times_{\mathfrak{s}^*} X^\bullet(S))/X^\bullet(T) \\
\downarrow & & \downarrow \\
\mathfrak{t}^* & \xrightarrow{q} & \mathfrak{t}^*/X^\bullet(T)
\end{array}
\tag{16}
$$

where the unlabelled vertical arrows are induced by inclusion. Notice also that we have a canonical isomorphism $\mathfrak{t}^* \times_{\mathfrak{s}^*} X^\bullet(S) \simeq (\mathfrak{t}/\mathfrak{s})^* \times X^\bullet(S)$ which is equivariant with respect to the action of $X^\bullet(T) \simeq X^\bullet(S) \times X^\bullet(T/S)$. This in particular shows that we have a Cartesian diagram

$$
\begin{array}{ccc}
(\mathfrak{t}/\mathfrak{s})^* \times X^\bullet(S) & \longrightarrow & (\mathfrak{t}/\mathfrak{s})^*/X^\bullet(T/S) \\
\downarrow & & \downarrow \\
\mathfrak{t}^* & \xrightarrow{q} & \mathfrak{t}^*/X^\bullet(T)
\end{array}
\tag{17}
$$

and by base changing along this Cartesian diagram we obtain our desired claim. □

A.3. **Symmetric Monoidality.** The Mellin transform of (14) provides an equivalence of categories $\mathcal{D}(T) \simeq \mathrm{IndCoh}(\mathfrak{t}^*/X^\bullet(T))$ which is t-exact up to a shift. This induces an equivalence of abelian categories, which moreover is well known to be symmetric monoidal–in the abelian categorical setting, this can be checked explicitly. However, in the higher categorical context, equipping a functor between symmetric monoidal ∞-categories with a symmetric monoidal structure requires an infinite amount of additional structure. The entirety of Appendix A.3 is devoted to the following theorem, which provides this upgraded structure:

Theorem A.2. The Mellin transform FMuk_T can be upgraded to an equivalence of symmetric monoidal categories with a W-action.

We will obtain Theorem A.2 as a consequence of a general uniqueness principle for symmetric monoidal structures on derived ∞-categories. To formulate it we first need to introduce some notation.

Notation A.3. Denote by Groth the category of Grothendieck abelian categories and colimit preserving functors, and by $\mathrm{Groth}_{\mathrm{proj}}$ the subcategory of Groth on those Grothendieck abelian categories with enough projectives and functors which preserve projective objects. We denote by $\mathrm{DGroth}_{\mathrm{proj}}$ the category defined informally as follows:

- Objects of $\mathrm{DGroth}_{\mathrm{proj}}$ are derived categories of Grothendieck abelian categories with enough projectives.

- A morphism $f : \mathcal{C} \to \mathcal{D}$ in $\mathrm{DGroth}_{\mathrm{proj}}$ is a colimit preserving functor that sends projective objects of $\mathcal{C}^\heartsuit$ to projective objects of $\mathcal{D}^\heartsuit$.

In addition to having morphisms, $\mathrm{Groth}_{\mathrm{proj}}$ has operations of arity n for any nonnegative integer n. Namely, for each finite family of source objects $\mathcal{C}_1, \ldots, \mathcal{C}_n$ and target object $\mathcal{C}$ an operation $\{\mathcal{C}_1, \ldots, \mathcal{C}_n\} \to \mathcal{C}$ is a functor

$$f : \mathcal{C}_1 \times \mathcal{C}_2 \times \ldots \times \mathcal{C}_n \to \mathcal{C}$$

with the following properties:

- f is colimit preserving on each variable.
- For each sequence of projective objects X_i in $\mathcal{C}_i$ the object $f(X_1, \ldots, X_n)$ is projective.

We may summarize the situation by saying that $\mathrm{Groth}_{\mathrm{proj}}$ has the structure of an operad.

Similarly, $\mathrm{DGroth}_{\mathrm{proj}}$ has the structure of an operad, where an operation $\{\mathcal{C}_1, \ldots, \mathcal{C}_n\} \to \mathcal{C}$ is a functor

$$f : \mathcal{C}_1 \times \mathcal{C}_2 \times \ldots \times \mathcal{C}_n \to \mathcal{C}$$

with the following properties:

- f is colimit preserving on each variable.
- For each sequence of projective objects X_i in $\mathcal{C}_i^\heartsuit$ the object $f(X_1, \ldots, X_n)$ belongs to $\mathcal{C}^\heartsuit$ and is projective.

We note that given an operation f in $\mathrm{DGroth}_{\mathrm{proj}}$ as above there is a corresponding operation $f^\heartsuit$ in $\mathrm{Groth}_{\mathrm{proj}}$ given by the following composition:

$$\mathcal{C}_1^\heartsuit \times \ldots \times \mathcal{C}_n^\heartsuit \hookrightarrow \mathcal{C}_1^{\leq 0} \times \ldots \times \mathcal{C}_n^{\leq 0} \xrightarrow{f} \mathcal{C}^{\leq 0} \xrightarrow{H^0} \mathcal{C}^\heartsuit$$

This forms part of a morphism of operads $\mathrm{DGroth}_{\mathrm{proj}} \to \mathrm{Groth}_{\mathrm{proj}}$. We are now ready to state the basic assertion that allows us to lift structures from the abelian setting to the derived setting:

Theorem A.4.

(1) $\mathrm{Groth}_{\mathrm{proj}}$ and $\mathrm{DGroth}_{\mathrm{proj}}$ are symmetric monoidal categories.
(2) The assignment $\mathcal{C} \mapsto \mathcal{C}^\heartsuit$ provides a symmetric monoidal equivalence $\mathrm{DGroth}_{\mathrm{proj}} = \mathrm{Groth}_{\mathrm{proj}}$.

Before giving the proof of Theorem A.4, we indicate how it can be used to deduce Theorem A.2. In what follows it will be convenient to shift the t-structure on $\mathrm{IndCoh}(\mathfrak{t}^*/X^\bullet(T))$ so that FMuk_T becomes t-exact.

Lemma A.5. For for any pair of projective objects $\mathcal{F}, \mathcal{G} \in \mathcal{D}(T)^\heartsuit$ the convolution $\mathcal{F} \star \mathcal{G}$ is a projective object of $\mathcal{D}(T)^\heartsuit$. Similarly, for any pair of projective objects $M, N \in \mathrm{IndCoh}(\mathfrak{t}^*/X^\bullet(T))^\heartsuit$, the tensor product $M \overset{!}{\otimes} N$ is a projective object of $\mathrm{IndCoh}(\mathfrak{t}^*/X^\bullet(T))^\heartsuit$.

Proof. Because the following diagram commutes

$$\begin{array}{ccc} T \times T & \longrightarrow & T \\ \downarrow & & \downarrow \\ T_{dR} \times T_{dR} & \longrightarrow & T_{dR} \end{array} \tag{18}$$

where the horizontal maps induced by multiplication and the vertical maps are the canonical maps, we see that in the category $\mathcal{D}(T)$, if $\mathcal{G}$ is the is the compact generator $(T \to T_{dR})_*^{\mathrm{IndCoh}}(\omega_T)$, then $\mathcal{G} \star \mathcal{G}$ is an infinite direct sum of copies of $\mathcal{G}$. The projection formula ([GR17c, Section 2.1.8]) similarly implies that if $\mathcal{G}' := q_*^{\mathrm{IndCoh}}(\omega_{\mathfrak{t}^*})$ then we have an isomorphism of $\mathcal{G}' \overset{!}{\otimes} \mathcal{G}'$ with

$$q_*^{\mathrm{IndCoh}}(\omega_{\mathfrak{t}^*}) \overset{!}{\otimes} q_*^{\mathrm{IndCoh}}(\omega_{\mathfrak{t}^*}) \simeq q_*^{\mathrm{IndCoh}}(\omega_{\mathfrak{t}^*} \overset{!}{\otimes} q^! q_*^{\mathrm{IndCoh}}(\omega_{\mathfrak{t}^*})) \simeq q_*^{\mathrm{IndCoh}} q^! q_*^{\mathrm{IndCoh}}(\omega_{\mathfrak{t}^*})$$

and so $\mathcal{G}' \overset{!}{\otimes} \mathcal{G}'$ is also an infinite direct sum of copies of $\mathcal{G}'$, by base change along the diagram (12). Now the claims follow since both $\mathcal{G}$ and $\mathcal{G}'$ are projective generators of the associated abelian categories (which we can see by and [GR] and the fact that $q^!$ is conservative by base change respectively) so any projective object is a direct summand of a direct sum of $\mathcal{G}$ or $\mathcal{G}'$ respectively. □

Proof of Theorem A.2. It follows from Lemma A.5 that $\mathcal{D}(T)$ is a nonunital commutative Vect-algebra in $\mathrm{DGroth}_{\mathrm{proj}}$. Note that it has an action of W induced from the action of W on T. Similarly, $\mathrm{IndCoh}(\mathfrak{t}^*/X^\bullet(T))$ is also a nonunital commutative Vect-algebra in $\mathrm{DGroth}_{\mathrm{proj}}$ with an action of W. By Theorem A.4 the usual W-equivariant symmetric monoidal structure on $\mathrm{FMuk}_T^{\heartsuit}$ may be upgraded to a W-equivariant nonunital symmetric monoidal structure on FMuk_T. This equivalence admits a unique unital symmetric monoidal extension by virtue of [Lur17, Theorem 5.4.3.5]. □

We now turn to the proof of Theorem A.4.

Notation A.6. For each object $\mathcal{C}$ of $\mathrm{Groth}_{\mathrm{proj}}$ we denote by $\mathcal{C}_{\mathrm{proj}}$ the full subcategory of $\mathcal{C}$ on the projective objects.

Lemma A.7. Let $\mathcal{C}$ be an object of $\mathrm{DGroth}_{\mathrm{proj}}$. Then:

(1) For every category with small colimits $\mathcal{D}$, restriction to $\mathcal{C}_{\mathrm{proj}}^{\heartsuit}$ provides an equivalence between the category $\mathrm{Funct}^L(\mathcal{C}^{\leq 0}, \mathcal{D})$ of colimit preserving functors $\mathcal{C}^{\leq 0} \to \mathcal{D}$ and the category $\mathrm{Funct}^{\oplus}(\mathcal{C}_{\mathrm{proj}}^{\heartsuit}, \mathcal{D})$ of small coproduct preserving functors $\mathcal{C}_{\mathrm{proj}}^{\heartsuit} \to \mathcal{D}$.
(2) For every stable category with small colimits $\mathcal{D}$, restriction to $\mathcal{C}_{\mathrm{proj}}^{\heartsuit}$ provides an equivalence $\mathrm{Funct}^L(\mathcal{C}, \mathcal{D}) = \mathrm{Funct}^{\oplus}(\mathcal{C}_{\mathrm{proj}}^{\heartsuit}, \mathcal{D})$.

Proof. Part (2) follows from part (1) since $\mathcal{C}$ is the stabilization of $\mathcal{C}^{\leq 0}$. To prove part (1) we apply the results from [Lur11] section 4.2. The category $\mathcal{C}^{\heartsuit}_{\text{proj}}$ is a socle in the sense of definition 4.2.9. The lemma follows from a combination of corollary 4.2.14 and proposition 4.2.15. □

Lemma A.8. Let $\mathcal{C}$ be an object of $\text{Groth}_{\text{proj}}$. Then for every $(1,1)$-category with small colimits $\mathcal{D}$, restriction to $\mathcal{C}_{\text{proj}}$ provides an equivalence $\text{Funct}^L(\mathcal{C},\mathcal{D}) = \text{Funct}^{\oplus}(\mathcal{C}_{\text{proj}},\mathcal{D})$.

Proof. Follows by applying part (1) of Lemma A.7 to the derived category of $\mathcal{C}$. □

Proof of Theorem A.4. We first show that the assignment $\mathcal{C} \mapsto \mathcal{C}^{\heartsuit}$ is an equivalence of operads. Since it is surjective, it is enough to show that for every sequence $\mathcal{C}_1,\ldots,\mathcal{C}_n,\mathcal{C}$ of objects of $\text{DGroth}_{\text{proj}}$ passage to hearts induces an equivalence

$$\text{Hom}_{\text{DGroth}_{\text{proj}}}(\{\mathcal{C}_1,\ldots,\mathcal{C}_n\},\mathcal{C}) = \text{Hom}_{\text{Groth}_{\text{proj}}}(\{\mathcal{C}_1^{\heartsuit},\ldots,\mathcal{C}_n^{\heartsuit}\},\mathcal{C}^{\heartsuit}).$$

The case $n=0$ is clear: in both cases a 0-ary operation consists of a projective object of $\mathcal{C}^{\heartsuit}$. Assume now that $n>0$. Then applying Lemma A.8 to the case when $\mathcal{D}$ is the category of functors $\mathcal{C}_2^{\heartsuit}\times\ldots\times\mathcal{C}_n^{\heartsuit}\to\mathcal{C}^{\heartsuit}$ which preserve colimits in each variable we see that restriction along the inclusion $(\mathcal{C}_1^{\heartsuit})_{\text{proj}}\to\mathcal{C}_1^{\heartsuit}$ provides an equivalence between $\text{Hom}_{\text{Groth}_{\text{proj}}}(\{\mathcal{C}_1^{\heartsuit},\ldots,\mathcal{C}_n^{\heartsuit}\},\mathcal{C}^{\heartsuit})$ and the space of functors

$$f:(\mathcal{C}_1^{\heartsuit})_{\text{proj}}\times\mathcal{C}_2^{\heartsuit}\times\mathcal{C}_3^{\heartsuit}\times\ldots\times\mathcal{C}_n^{\heartsuit}\to\mathcal{C}^{\heartsuit}$$

with the following properties:

- f preserves coproducts in the first variable.
- f preserves colimits in the variables $2,\ldots,n$.
- For every sequence of projective objects X_i in $\mathcal{C}_i^{\heartsuit}$ the object $f(X_1,\ldots,X_n)$ is projective.

Applying this reasoning inductively we conclude that

$$\text{Hom}_{\text{Groth}_{\text{proj}}}(\{\mathcal{C}_1^{\heartsuit},\ldots,\mathcal{C}_n^{\heartsuit}\},\mathcal{C}^{\heartsuit})$$

is equivalent to the space of functors

$$(\mathcal{C}_1^{\heartsuit})_{\text{proj}}\times(\mathcal{C}_2^{\heartsuit})_{\text{proj}}\times\ldots\times(\mathcal{C}_n^{\heartsuit})_{\text{proj}}\to\mathcal{C}^{\heartsuit}_{\text{proj}}$$

which preserve coproducts in each variable.

Similarly, an inductive application of part (2) of Lemma A.7 shows that $\text{Hom}_{\text{DGroth}_{\text{proj}}}(\{\mathcal{C}_1,\ldots,\mathcal{C}_n\},\mathcal{C})$ is also equivalent to the above space. This concludes the proof that the map $\text{DGroth}_{\text{proj}}\to\text{Groth}_{\text{proj}}$ is an equivalence of operads. It remains to show that these are symmetric monoidal categories.

Equip Groth with the structure of operad where an operation $\{\mathcal{C}_1,\ldots,\mathcal{C}_n\}\to\mathcal{C}$ is a functor

$$f:\mathcal{C}_1\times\ldots\times\mathcal{C}_n\to\mathcal{C}$$

which preserves colimits in each variable. This is in fact a symmetric monoidal category, and the inclusion Groth $\to \mathrm{Pr}^L$ preserves tensor products [Lur18b, Corollary C.5.4.19]. We may regard $\mathrm{Groth}_{\mathrm{proj}}$ as a suboperad of Groth. The unit of Groth is the category of abelian groups, which belongs to $\mathrm{Groth}_{\mathrm{proj}}$. To finish the proof it will suffice to show that if $\mathcal{C}$ and $\mathcal{D}$ are objects of $\mathrm{Groth}_{\mathrm{proj}}$ then the tensor product $\mathcal{C} \otimes \mathcal{D}$ (computed in Groth) is still a tensor product in $\mathrm{Groth}_{\mathrm{proj}}$. It is enough for this to prove that if X and Y are projective objects of $\mathcal{C}$ and $\mathcal{D}$ then $X \otimes Y$ is a projective object of $\mathcal{C} \otimes \mathcal{D}$.

The Yoneda embedding provides an equivalence between $\mathcal{C} \otimes \mathcal{D}$ and the category of limit preserving functors from $(\mathcal{C} \otimes \mathcal{D})^{\mathrm{op}}$ into the category of sets. This is equivalent to the category of functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{D}^{\mathrm{op}} \to \mathrm{Set}$ which preserve limits in each variable. Applying once again Lemma A.8 we obtain an equivalence between $\mathcal{C} \otimes \mathcal{D}$ and the category $\mathrm{Funct}^{\prod}(\{\mathcal{C}^{\mathrm{op}}_{\mathrm{proj}}, \mathcal{D}^{\mathrm{op}}_{\mathrm{proj}}\}, \mathrm{Set})$ of functors $\mathcal{C}^{\mathrm{op}}_{\mathrm{proj}} \times \mathcal{D}^{\mathrm{op}}_{\mathrm{proj}} \to \mathrm{Set}$ which preserve products in each variable. The functor $\mathcal{C} \otimes \mathcal{D} \to \mathrm{Set}$ corepresented by $X \otimes Y$ corresponds under this dictionary to the functor

$$\mathrm{ev}_{(X,Y)} : \mathrm{Funct}^{\prod}(\{\mathcal{C}^{\mathrm{op}}_{\mathrm{proj}}, \mathcal{D}^{\mathrm{op}}_{\mathrm{proj}}\}, \mathrm{Set}) \to \mathrm{Set}$$

of evaluation at (X, Y). Our goal is to show that $\mathrm{ev}_{(X,Y)}$ preserves geometric realizations. To do so it suffices to prove that $\mathrm{Funct}^{\prod}(\{\mathcal{C}^{\mathrm{op}}_{\mathrm{proj}}, \mathcal{D}^{\mathrm{op}}_{\mathrm{proj}}\}, \mathrm{Set})$ is closed under geometric realizations in the category $\mathrm{Funct}(\{\mathcal{C}^{\mathrm{op}}_{\mathrm{proj}}, \mathcal{D}^{\mathrm{op}}_{\mathrm{proj}}\}, \mathrm{Set})$ of functors $\mathcal{C}^{\mathrm{op}}_{\mathrm{proj}} \times \mathcal{D}^{\mathrm{op}}_{\mathrm{proj}} \to \mathrm{Set}$.

Let Ab be the category of abelian groups. We have a commutative square of categories

$$\begin{array}{ccc} \mathrm{Funct}^{\prod}(\{\mathcal{C}^{\mathrm{op}}_{\mathrm{proj}}, \mathcal{D}^{\mathrm{op}}_{\mathrm{proj}}\}, \mathrm{Ab}) & \longrightarrow & \mathrm{Funct}^{\prod}(\{\mathcal{C}^{\mathrm{op}}_{\mathrm{proj}}, \mathcal{D}^{\mathrm{op}}_{\mathrm{proj}}\}, \mathrm{Set}) \\ \downarrow & & \downarrow \\ \mathrm{Funct}(\{\mathcal{C}^{\mathrm{op}}_{\mathrm{proj}}, \mathcal{D}^{\mathrm{op}}_{\mathrm{proj}}\}, \mathrm{Ab}) & \longrightarrow & \mathrm{Funct}(\{\mathcal{C}^{\mathrm{op}}_{\mathrm{proj}}, \mathcal{D}^{\mathrm{op}}_{\mathrm{proj}}\}, \mathrm{Set}) \end{array}$$

where the horizontal arrows are induced by the forgetful functor $\mathrm{Ab} \to \mathrm{Set}$ and the vertical arrows are the inclusions. The top horizontal arrow is an equivalence since $\mathcal{C}^{\mathrm{op}}_{\mathrm{proj}}$ and $\mathcal{D}^{\mathrm{op}}_{\mathrm{proj}}$ are additive. The bottom horizontal arrow preserves geometric realizations since these are preserved by the forgetful functor $\mathrm{Ab} \to \mathrm{Set}$. We may therefore reduce to showing that left vertical arrow preserves geometric realizations. This follows from the fact that products in Ab are exact. □

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