APPROXIMATION THEOREMS FOR KEISLER MEASURES

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Abstract

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This dissertation is concerned with Keisler measures and their approximations. We investigate tame families of Keisler measures in varying contexts. We first restrict ourselves to the local NIP setting. There, we partially generalize a theorem of Hrushvoski, Pillay, and Simon and show that in this context, a measure is definable and finitely satisfiable (dfs) if and only if it is finitely approximated. We then consider generically stable measures outside of the NIP setting. We show that generically stable types correspond to $\{0, 1\}$-valued frequency interpretation measures, and we give examples of finitely approximated measures which are not frequency interpretation measures and local dfs measures which are not locally finitely approximated (joint with Gabriel Conant). We then introduce and describe "sequential approximations". We show that measures which are finitely satisifable in a countable model of an NIP theory admit this kind of approximation. We also prove that generically stable types admit a similar (and stronger) approximation. In the final chapter, we restrict ourselves to the group setting and introduce a convolution operation on Keisler measures. We classify all idempotent measures over stable groups and also show that a particular convolution algebra over an NIP group is isomorphic to a natural Ellis semigroup (joint with Artem Chernikov).

DEDICATION

To my parents, Kathy and Mark, and my sister, Hannah.

CONTENTS

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CHAPTER 1

INTRODUCTION

Progress in understanding NIP structures and especially NIP groups has rapidly advanced with the study of a particular family of finitely additive probability measures, so-called *Keisler measures*, over these structures. Initially, Keisler measures were introduced and first studied by H.J. Keisler [36], hence the name. Keisler demonstrated that some of the ideas and tools from stability theory could be extended to NIP structures by replacing types, which can be viewed as $\{0,1\}$-valued measures, with arbitrary finitely additive probability measures. Almost 20 years after Keisler's original paper, his work was revisited and greatly expanded upon by the work of Hrushovski, Peterzil, Pillay, and Simon [31–33]. The general theory of Keisler measures over NIP structures was developed in their work and led to the implementation of these objects across the field. Keisler measures played an essential role in the proof of the Pillay conjectures for o-minimal groups [31], provided a framework for the stable, distal, and NIP graph regularity lemmas [10, 11, 41, 42] as well as their group theoretic counterparts [15–17], and shaped our understanding of topological dynamics in NIP structures [7, 9, 49].

In contrast, little was (and still is) known about Keisler measures outside the class of NIP structures. This absence of any general theorems has left an interesting gap in the literature. The work in this dissertation provides and implements novel techniques for working with Keisler measures both inside and outside of the NIP context. In particular, we will demonstrates that concepts from functional analysis, combinatorics, and classical model theory are required to develop a general theory of

Keisler measures.

This thesis is broken into 6 chapters (including this one). The second chapter of this dissertation functions as a foundation for the rest of this text. There, we define Keisler measures and describe their basic topological and geometric properties. We then move on to describe the *zoo of Keisler measures*. We discuss the many different kinds of measures which we will come across throughout this text and show how these measures interact. Finally, we recall a collection of theorems about measures in the NIP context. Many of the results in this chapter fall into one of three categories: folklore results, propositions generalized from the type case to the measure context, and results originally proved by Hrushovski, Pillay, and Simon in [33]. Throughout the entire dissertation, we will liberally reference this chapter.

Chapter 3 is a modified version of my article entitled *Local Keisler measures and NIP formulas* [25]. The main theorem of this chapter is that φ-measures which are *locally definable and finitely satisfiable* over a small model are *locally finitely approximated*. Here, we move to the vantage point of classical functional analysis. The proof of our main theorem relies on a celebrated theorem of Bourgain, Fremlin, and Talagrand [5]. Many of the basic observations in our original paper have been moved to Chapter 2 since the proofs in the local case are similar to the global case.

Chapter 4 is a modified version of my article with Gabriel Conant entitled *Remarks on generic stability in independent theories* [14]. In NIP theories, generically stable Keisler measures can be characterized in several ways. We analyze these various forms of "generic stability" in arbitrary theories. Among other things, we show that the standard definition of generic stability for types coincides with the notion of a frequency interpretation measure. We also give combinatorial examples of types in NSOP theories that are finitely approximated but not generically stable, as well as φ-types in simple theories that are definable and finitely satisfiable in a small model, but not finitely approximated. Our proofs demonstrate interesting connections to classical results from Ramsey theory for finite graphs and hypergraphs.

Chapter 5 introduces a new class of measures, as well as a new class of types, which we call sequentially approximated measures and strongly sequentially approximated types. The first condition can be though of as a strengthening of finite satisfisfiability over a small model or a weakening of finite approximability. In general, if a measure is both sequentially approximated and definable, then it is finitely approximated. Moreover, we show that in NIP theories, any measure which is finitely satisfiable in a countable model is sequentially approximated. Strongly sequentially approximated types remain mysterious. We show that all generically stable types are strongly sequentially approximated over any model in which they invariant. However, in general, we find that the associated Keisler measure to a type can be sequentially approximated while the type is not strongly sequentially approximated.

The final chapter is a modified version of my preprint with Artem Chernikov entitled *Definable convolution and idempotent Keisler measures* which is currently in preparation [6]. We initiate a systematic study of the definable convolution operation on Keisler measures, generalizing the work of Newelski in the case of types. Adapting results of Glicksberg, we show that the supports of *idempotent dfs measures* are nice semigroups, and classify idempotent measures in stable groups as invariant measures on type-definable subgroups. We establish left-continuity of the convolution map in NIP theories and use it to show that the convolution semigroup is isomorphic to a particular Ellis semigroup in this context.

1.1 Notation

If r, s are real numbers, and $\epsilon > 0$, then we write $r \approx_{\epsilon} s$ to mean $|r - s| < \epsilon$. For the most part, our model theory notation is standard. We use $\mathcal{L}$ to denote a first order language, T to denote a first order theory in the language $\mathcal{L}$, and $\mathcal{U}$ to denote a sufficiently saturated model of T. Throughout this work, we will always have some fixed $\mathcal{L}$, T, and $\mathcal{U}$ in the background with some conditions on T. We use the letters x, y, z to denote finite tuples of variables (until the final chapter where we will require our tuples to have length 1). If a tuple of variables x is fixed, we use $\overline{x}$ to denote a *tuple of tuples*. We say that a subset A of $\mathcal{U}$ is a small if $\mathcal{U}$ is $|A|^+$-saturated. We write $M \prec \mathcal{U}$ to mean M is an elementary substructure of $\mathcal{U}$ and M is a small subset of $\mathcal{U}$. Given $A \subseteq \mathcal{U}$, we use the phrase "$\mathcal{L}(A)$-formula" to refer to formulas with parameters from A and "$\mathcal{L}$-formula" to refer to formulas without parameters. Moreover, if x is a tuple of variables then we let $\mathcal{L}_x(A)$ be the Boolean algebra of $\mathcal{L}(A)$-formulas (modulo logical equivalence[1]) with free-variables in x. Likewise, we let $\mathrm{Def}_x(A)$ denote the Boolean algebra of A-definable subsets of $\mathcal{U}^x$. We remark that $\mathrm{Def}_x(A)$ and $\mathcal{L}_x(A)$ are isomorphic in the obvious way and we will abuse notation and routinely identify definable sets with the formulas which define them. Similarly, if $(x_i)_{i\in\omega}$ is a sequence of distinct tuples of variables, we let $\mathcal{L}_{(x_i)_{i\in\omega}}(A)$ denote the Boolean algebra of formulas in this family of variables.

We say that a subset A of $\mathcal{U}^x$ is type-definable if there exists a small collection of formulas $\{\psi_i(x) : i \in I\}$ (i.e. $|I|$ is smaller than the saturation of $\mathcal{U}$) such that $A = \{a \in \mathcal{U}^x : \mathcal{U} \models \psi_i(a) \text{ for each } i \in I\}$. Moreover, if $B \subset \mathcal{U}$ and B contains all the parameters from each ψ_i, we say that A is type-definable over B. Additionally, if $\{\psi_i(x) : i \in I\}$ is a small collection of formulas, then we may write $r(x) = \bigwedge_{i\in I} \psi_i(x)$ where $r(\mathcal{U}) = \{a \in \mathcal{U}^x : \mathcal{U} \models \psi_i(a) \text{ for each } i \in I\}$.

We write $\varphi(x; y)$ for a partitioned formula with object variables x and parameter variables y. The formula $\varphi^*(y; x)$ will denote the exact same formula as $\varphi(x; y)$, but with the roles exchanged for parameter and variable tuples. There is a slight ambiguity in the literature between *instances of* φ and φ*-formulas*. We use these terms with the following convention: an **instance of** φ is a formula of the form $\varphi(x; b)$ with $b \in \mathcal{U}^y$. A φ**-formula** is a Boolean combination of instances of φ. We

[1]In particular, $\varphi(x)$ is identified with $\psi(x)$ if and only if $\mathcal{U} \models \forall x(\varphi(x) \leftrightarrow \psi(x))$.

denote the collection of φ-formulas as $\mathcal{L}_\varphi(\mathcal{U})$ (again, modulo logical equivalence). Similarly, $\mathcal{L}_\varphi(\mathcal{U})$ is a Boolean algebra isomorphic to the collection of subsets of $\mathcal{U}$ defined via a φ-formula. Finally, for a fixed partitioned $\mathcal{L}(\mathcal{U})$-formula $\varphi(x;y)$, we say that a subset A of $\mathcal{U}^x$ is φ-type-definable over M if M contains all the parameters from $\varphi(x;y)$ and there is a small collection of φ-formulas $\{\psi_i(x) : i \in I\}$ such that M contains all the parameters from each $\psi_i(x)$ and $A = \{a \in \mathcal{U}^x : \mathcal{U} \models \psi_i(a)$ for each $i \in I\}$.

1.1.1 Types and type spaces

If $A \subseteq \mathcal{U}$, then we let $S_x(A)$ be the collection of complete types in variable(s) x with parameters from A. If we want to emphasize the length of tuple instead of the tuple itself, we will write $S_n(A)$ for $S_x(A)$ when $|x| = n$. For a fixed partitioned $\mathcal{L}(A)$-formula $\varphi(x;y)$, we let $S_\varphi(A)$ be the collection of complete φ-types over A. We recall that both $S_\varphi(A)$ and $S_x(A)$ are a totally disconnected, compact Hausdorff spaces (also known as Stone spaces). For a formula $\varphi(x) \in \mathcal{L}_x(A)$, we write $\chi_{\varphi(x)}$ as the characteristic function from $S_x(A) \to [0,1]$. Namely,

$$\chi_{\varphi(x)}(p) = \begin{cases} 1 & \varphi(x) \in p, \\ 0 & \text{otherwise.} \end{cases}$$

If a, b are tuples (possibly infinite), then we write $\text{tp}(a/A)$ to denote the type of a over the parameters A and $a \equiv_A b$ to mean that $\text{tp}(a/A) = \text{tp}(b/A)$.

We now recall a few basic definitions and facts about global types. The facts presented here are well known and proofs can be found in most standard model theory texts (for instance [59]).

Definition 1.1. Let $p \in S_x(\mathcal{U})$.

1. p is called **invariant** if there exists a model $M \prec \mathcal{U}$ such that for any partitioned

$\mathcal{L}$-formula $\varphi(x;y)$, if $b \equiv_M b'$ then $\varphi(x;b) \in p$ if and only if $\varphi(x;b') \in p$. In this case, we say that p is **M-invariant**.

2. p is called **definable** if there exists a model $M \prec \mathcal{U}$ such that for any partitioned $\mathcal{L}$-formula $\varphi(x;y)$, there exists a formula $\psi(y)$ in $\mathcal{L}_y(M)$ such that for any $b \in \mathcal{U}^y$, we have that $\varphi(x;b) \in p$ if and only if $\mathcal{U} \models \psi(b)$. In this case, we say that p is **definable over** M.

3. Assume that $M \prec \mathcal{U}$. Then p is called **finitely satisfiable in** M if for every $\mathcal{L}_x(\mathcal{U})$-formula $\psi(x)$ in p, there exists some $b \in M$ such that $\mathcal{U} \models \psi(b)$. We denote the collection of global types which are finitely satisfiable in the elementary submodel M as $S_x(\mathcal{U}, M)$.

Definition 1.2. Let $p \in S_x(\mathcal{U})$ and $q \in S_y(\mathcal{U})$ be two global types such that p is invariant. Then, p is M-invariant for some M. We define the **Morley product** of p and q (denoted $p \otimes q$) as follows: for any formula $\mathcal{L}(\mathcal{U})$-formula $\varphi(x;y)$, choose $N \prec \mathcal{U}$ such that $M \prec N$ and N contains all the parameters from $\varphi(x;y)$. Then, $\varphi(x;y) \in p \otimes q$ if and only if $\varphi(x;b) \in p$ for some/any $b \models q|_N$. This product is well-defined (since p is M-invariant) and moreover does not depend on the choice of M or N.

The following fact is a standard exercise.

Fact 1.3. *If both p, q are M-invariant then $p \otimes q$ is M-invariant. Moreover, the Morley product on invariant types is associative.*

We now move on to discuss EM-types and Morley sequences. In this thesis, we will think of EM-types as types in countable many variable.

Definition 1.4. Fix an ordinal β and a β-indexed sequence $(a_\alpha)_{\alpha<\beta}$ of points in $\mathcal{U}^x$. Then the **Ehrenfeucht-Mostowski type** or **EM-type** of the sequence $(a_\alpha)_{\alpha<\beta}$ (over a parameter set $B \subseteq \mathcal{U}$), denoted $\mathrm{EM}((a_\alpha)_{\alpha<\beta}/B)$, is the following partial type:

$$\{\varphi(x_0, ..., x_n) \in \mathcal{L}_{(x_i)_{i\in\omega}}(B) : \mathcal{U} \models \varphi(a_{\alpha_0}, ..., a_{\alpha_n}) \text{ for any } \alpha_0 < ... < \alpha_n\}.$$

We remark that this partial type corresponds to a subset of $S_{(x_i)_{i<\omega}}(B)$.

Observation 1.5. It is clear from the definition above that for any ordinal β, any sequence of points $(a_\alpha)_{\alpha<\beta}$ in $\mathcal{U}^x$, and any $B \subseteq \mathcal{U}$, the type $\mathrm{EM}((a_\alpha)_{\alpha<\beta}/B)$ is complete if and only if the sequence $(a_\alpha)_{\alpha<\beta}$ is indiscernible over B.

Definition 1.6. Let $p \in S_x(\mathcal{U})$ and assume that p is invariant. Then, for any natural number n, we define p^n inductively

1. $p^0(x_0) = p(x_0)$.
2. For any $p^{n+1}(x_0, ..., x_{n+1}) = p(x_{n+1}) \otimes p^n(x_0, ..., x_n)$.
3. $p^\omega = \bigcup_{i\in\omega} p^n$.

A **Morley sequence** (of order type ω) in p over M is a sequence of points $(a_i)_{i\in\omega}$ such that each a_i is in $\mathcal{U}^x$ and for any $n \in \omega$, $(a_0, ..., a_n) \models p^n|_M$.

In general, we say that a sequence $(a_\alpha)_{\alpha<\beta}$ ordered by an ordinal β is a Morley sequence in p over M if the sequence is indiscernible over M and the EM-type of this sequence over M is given by $\{p^n|_M : n \in \omega\}$.

The following definition is due to Pillay and Tanović.

Definition 1.7 ([50])**.** Let $p \in S_x(\mathcal{U})$. Then p is called **generically stable** if there exists a model $M \prec \mathcal{U}$ such that p is M-invariant and for any ordinal β, any Morley sequence $(a_\alpha)_{\alpha<\beta}$ in p over M, and any $\mathcal{L}(\mathcal{U})$-formula $\psi(x)$, we have that

$$|\{\alpha : \mathcal{U} \models \psi(a_\alpha)\}| < \omega \text{ or } |\{\alpha : \mathcal{U} \models \neg\psi(a_\alpha)\}| < \omega.$$

In this case, we say that p is **generically stable over** M.

1.1.2 Stability and NIP

For the majority of this dissertation, we will work in and around the dividing line of the independence property. In Chapter 6, we will spend a few sections working with stable theories. We recall the definitions here.

Definition 1.8. Let $\varphi(x;y)$ be a partitioned $\mathcal{L}_x(\mathcal{U})$-formula. Then, $\varphi(x;y)$ is **unstable** if for every n, there exists a sequence $a_1, ..., a_n$ and $b_1, ..., b_n$ such that $\mathcal{U} \models \varphi(a_i; b_j)$ if and only if $i < j$. We say that a formula $\varphi(x;y)$ is **stable** if and only if it is not unstable. Furthermore we say that a theory T is **stable** is every $\mathcal{L}$-formula is stable.

Definition 1.9. Let $\varphi(x;y)$ be a partitioned $\mathcal{L}_x(\mathcal{U})$-formula. Then, for a subset $A \subset \mathcal{U}^x$, we say that $\varphi(x;y)$ **shatters** A if for every $K \subseteq A$, there exists some $b_K \in \mathcal{U}^y$ such that $\{a \in \mathcal{U}^x : \mathcal{U} \models \varphi(a; b_K)\} \cap A = K$. We say that $\varphi(x;y)$ has the **independence property** if $\varphi(x;y)$ shatters arbitrarily large finite subsets of $\mathcal{U}$. We say that $\varphi(x;y)$ is **dependent** or **NIP** (not the independence property) if and only if $\varphi(x;y)$ does not have the independence property. Furthermore we say that a theory T is **NIP** is every $\mathcal{L}$-formula is NIP.

It is not difficult to check that the definitions for a theory being stable or NIP are well-defined (i.e. do not depend on the choice of model). Therefore, every partitioned $\mathcal{L}$-formula is either stable/NIP or unstable/not NIP with respect to the theory T. We will deal with NIP theories and NIP formulas in greater detail in the following chapters. We recall a few very basic facts about stable and NIP formulas (see [59, Lemma 2.5, Lemma 2.9] for the NIP proofs).

Fact 1.10. *Let $\varphi(x;y)$ be a partitioned $\mathcal{L}(\mathcal{U})$-formula.*

1. *If $\varphi(x;y)$ is stable (NIP), then $\varphi^*(x;y)$ is stable (respectively, NIP).*
2. *Any Boolean combination of stable (NIP) formulas is stable (respectively, NIP).*

Miscellaneous Notation: We end this chapter by providing some miscellaneous notation which is used throughout the text.

For a subset $A^x \subset \mathcal{U}^x$, we denote the collection of n-tuples from A^x as $(A^x)^n$ and the collection of all finite tuples as $(A^x)^{<\omega} = \bigcup_{n\in\omega}(A^x)^n$.

Notation 1.11 (Convex Combination)**.** Let Y be a vector space (over $\mathbb{R}$) and $A \subseteq Y$. We let $\text{conv}(A)$ be the convex hull of A, and we let $\text{conv}_{\mathbb{Q}}(A)$ denote the collection of all rational convex combinations of elements from A, i.e.

$$\mathrm{conv}(A) = \left\{ \sum_{i=1}^{n} r_i a_i : a_i \in A \; ; \; n \in \mathbb{N} \; ; \; r_i \in \mathbb{R}^+ \; ; \; \sum_{i=1}^{n} r_i = 1 \right\},$$

and,

$$\mathrm{conv}_{\mathbb{Q}}(A) = \left\{ \sum_{i=1}^{n} r_i a_i : a_i \in A \; ; \; n \in \mathbb{N} \; ; \; r_i \in \mathbb{Q}^+ \; ; \; \sum_{i=1}^{n} r_i = 1 \right\}.$$

CHAPTER 2

KEISLER MEASURES

In this chapter, we define our main object of study: Keisler measures. As stated in the introduction, this chapter forms the necessary prerequisites for the rest of the thesis. We use this chapter as a repository for general propositions and helpful lemmas which come in handy in later chapters. After introducing Keisler measures, we describe some basic topological and geometric properties about the space of Keisler measures. We also present some basic facts concerning the relationship between a measure and its support. Following this, we enter the zoo of Keisler measures. There are many different kinds of Keisler measures and here we introduce them and explain how they relate to one another. In the last section, we recall some facts and theorems about Keisler measures in the context of NIP theories. Throughout this chapter, we always have a theory T and a sufficiently saturated model $\mathcal{U}$ of T in the background.

Definition 2.1. Let $A \subseteq \mathcal{U}$. Then a Keisler measure (in variables x) over A is a finitely additive probability measure on $\mathcal{L}_x(A)$. Namely, μ is a Keisler measure if and only if $\mu : \mathcal{L}_x(A) \to [0,1]$ and for any $\varphi(x), \psi(x)$ in $\mathcal{L}_x(A)$ we have

1. $\mu(x = x) = 1$.
2. $\mu(\neg\varphi(x)) = 1 - \mu(\varphi(x))$.
3. $\mu(\varphi(x) \vee \psi(x)) = \mu(\varphi(x)) + \mu(\psi(x)) - \mu(\varphi(x) \wedge \psi(x))$.

Moreover, we let $\mathfrak{M}_x(A)$ be the collection of Keisler measures on $\mathcal{L}_x(A)$.

Notation 2.2. If $\mu \in \mathfrak{M}_x(A)$, we sometimes write μ as μ_x or $\mu(x)$ to emphasize the variable the measure is in. If $|x| = |y|$, then we may also write $\mu(y)$ or μ_y which corresponds to the measure in $\mathfrak{M}_y(A)$ derived by simply changing the variable(s) from x to y. Also, like type spaces, if we want to emphasize the length of the tuple instead of the tuple itself, we write $\mathfrak{M}_x(A)$ simply as $\mathfrak{M}_n(A)$ when $|x| = n$. If $A \subseteq B$ and $\mu \in \mathfrak{M}_x(B)$, then we let $\mu|_A$ be the natural restriction of μ to $\mathcal{L}_x(A)$. We remark that $\mu|_A \in \mathfrak{M}_x(A)$.

Fact 2.3. *There is a one-to-one correspondence between finitely additive measures on a Boolean algebra and regular Borel probability measures on its associated Stone space. In our main setting, this implies that there is a unique correspondence between Keisler measures on $\mathcal{L}_x(A)$ and regular Borel probability measures on $S_x(A)$. To be pedantic, any Keisler measure naturally extends to a unique regular Borel probability measure on $S_x(A)$ while every regular Borel probability measure on $S_x(A)$ restricts to a unique Keisler measure on $\mathcal{L}_x(A)$. Throughout this thesis, we will identify the two without remark (see the discussion prior to Lemma 7.3 in [59]).*

In keeping with tradition, we sometimes abuse notation and identify a formula $\psi(x)$ in $\mathcal{L}_x(A)$ with the collection of types in $S_x(A)$ which contain $\psi(x)$, but this will be obvious from the context of our statement. We now describe the special kind of regularity that Keisler measures enjoy.

Observation 2.4. Suppose $\mu \in \mathfrak{M}_x(A)$. Then μ is regular as a Borel measure on $S_x(A)$ in the following sense:

1. If C is a closed subset of $S_x(A)$, then for every $\epsilon > 0$ there exists some $\mathcal{L}_x(A)$-formula $\psi(x)$ such that $C \subseteq \psi(x)$ and $|\mu(C) - \mu(\psi(x))| < \epsilon$.
2. If O is an open subset of $S_x(A)$, then for every $\epsilon > 0$ there exists some $\mathcal{L}_x(A)$-formula $\rho(x)$ such that $\rho(x) \subseteq O$ and $|\mu(O) - \mu(\rho(x))| < \epsilon$.
3. If B is a Borel subset of $S_x(A)$, then for every $\epsilon > 0$ there exists some open set O and some closed set C such that $C \subseteq B \subseteq O$, $|\mu(B) - \mu(C)| < \epsilon$ and $|\mu(B) - \mu(O)| < \epsilon$.

2.1 Basic topological and geometric structure

The space $\mathfrak{M}_x(A)$ comes equipped with two topologies (the *norm topology* and the *compact Hausdorff topology*). We will define both, but primarily use the compact Hausdorff topology in later sections and chapters. **After this subsection, all mentions of $\mathfrak{M}_x(A)$ as a topological space are in reference to the compact Hausdorff topology unless explicitly stated otherwise**. We begin by constructing the norm topology on $\mathfrak{M}_x(A)$. Let $\mathcal{M}_x(A)$ be the collection of all bounded[1] regular Borel (signed) measures on $S_x(A)$. Then, $\mathcal{M}_x(A)$ forms a real Banach space with the total variation norm. This norm induces the following metric on the space of Keisler measures,

$$d(\mu, \nu) = \sup_{\varphi(x) \in \mathcal{L}_x(A)} |\mu(\varphi(x)) - \nu(\varphi(x))|.$$

By Fact 2.3, the collection of Keisler measures is identified with a subset of $\mathcal{M}_x(A)$. In particular, $\mathfrak{M}_x(A)$ forms a norm-closed convex subset of this Banach space. The topology induced on $\mathfrak{M}_x(A)$ from the norm on $\mathcal{M}_x(A)$ is called the **norm topology**. Moreover we can characterize $\mathfrak{M}_x(A)$ as a subset of $\mathcal{M}_x(A)$ as follows: for any $\lambda \in \mathcal{M}_x(A)$, λ is a Keisler measure if and only if $||\lambda|| = 1$ and for any $\varphi(x) \in \mathcal{L}_x(A)$, we have that $\lambda(\varphi(x)) \geq 0$. More importantly, if $D \subseteq \mathfrak{M}_x(A)$, then it makes sense to consider $\mathrm{conv}(D)$ and $\mathrm{conv}_{\mathbb{Q}}(D)$.

We now describe a weaker topology on $\mathfrak{M}_x(A)$. Under this topology, the space $\mathfrak{M}_x(A)$ is a compact Hausdorff topology. There are several equivalent ways to describe the **compact Hausdorff topology** on $\mathfrak{M}_x(A)$. First, this topology is the topology induced from the product space $[0,1]^{\mathcal{L}_x(A)}$ where we think of each measure as a map from $\mathcal{L}_x(A)$ to $[0,1]$. Second, this topology on $\mathfrak{M}_x(A)$ is the coarsest topology such that for any continuous function $f : S_x(A) \to \mathbb{R}$, the map $\int f : \mathfrak{M}_x(A) \to \mathbb{R}$ is

[1] A measure μ is bounded if $\mu(S_x(A)) < \infty$.

continuous. In simple terms, a basic open subset U of $\mathfrak{M}_x(A)$ is of the form

$$U = \bigcap_{i=1}^{n} \Big\{ \mu \in \mathfrak{M}_x(A) : r_i < \mu(\varphi_i(x)) < s_i \Big\},$$

where each $\varphi_i(x) \in \mathcal{L}_x(A)$ and each r_i, s_i are in $\mathbb{R}$ for $i \leq n$. Again, in future sections we will almost always consider $\mathfrak{M}_x(A)$ as a topological space with this compact Hausdorff topology. We mention that the compact Hausdorff topology arrises naturally from functional analysis. The next fact follows directly from [54, Corollary 4.7.6].

Fact 2.5. *Let $C(S_x(A))$ be the collection of continuous functions from $S_x(A) \to \mathbb{R}$. We observe that since $S_x(A)$ is a compact Hausdorff space, the space of continuous*[2] *linear functionals from $C(S_x(A))$ to $\mathbb{R}$ is canonically isomorphic to $\mathcal{M}_x(A)$ by the Riesz representation theorem (sometimes called the Riesz-Markov-Kakutani representation theorem). Moreover, the compact Hausdorff topology defined above for $\mathfrak{M}_x(A)$ is exactly the restriction of the standard weak* topology on $\mathcal{M}_x(A)$ to $\mathfrak{M}_x(A)$.*

Every type in $S_x(A)$ can be viewed as a $\{0,1\}$-valued Keisler measure on $\mathcal{L}_x(A)$ in the obvious way. For a fixed type $p \in S_x(A)$, we write δ_p for the **Dirac measure concentrating on** p. This measure is defined as follows: for any $\varphi(x) \in \mathcal{L}_x(A)$,

$$\delta_p(\varphi(x)) = \begin{cases} 1 & \varphi(x) \in p, \\ 0 & \neg\varphi(x) \in p. \end{cases}$$

The map $\delta : S_x(A) \to \mathfrak{M}_x(A)$ which sends a type to its corresponding Keisler measure is injective and continuous and hence $S_x(A)$ is naturally embedded in $\mathfrak{M}_x(A)$. For any a in A^x, we write $\delta_{\mathrm{tp}(a/A)}$ simply as δ_a. It is obvious that if a is in A^x, then δ_a extends uniquely to a global measure and so we routinely associate δ_a with its global counterpart (i.e. $\delta_{\mathrm{tp}(a/\mathcal{U})}$). If $\overline{a} \in (A^x)^{<\omega}$ where $\overline{a} = (a_1, ..., a_n)$, then we write the associated average measure as $\mathrm{Av}(\overline{a})$ where for any $\varphi(x) \in \mathcal{L}_x(A)$,

[2]Continuous with respect to the supremum norm on $C(S_x(A))$.

$$\mathrm{Av}(\overline{a})(\varphi(x)) = \frac{|\{i : \mathcal{U} \models \varphi(a_i)\}|}{n}.$$

Again, as in the case of Dirac measures concentrating on realized types, if $\overline{a}$ is in $(A^x)^{<\omega}$ then $\mathrm{Av}(\overline{a})$ extends uniquely to a global measure (i.e. a measure on $\mathcal{L}_x(\mathcal{U})$) and we again routinely associate $\mathrm{Av}(\overline{a})$ with this measure. The following facts are standard and left to the reader.

Fact 2.6. *Fix $A \subseteq \mathcal{U}$. If we consider $\{\delta_a : a \in A^x\}$ either as a subset of $\mathfrak{M}_x(A)$ or as a subset of $\mathfrak{M}_x(\mathcal{U})$, then the following are true.*

1. *The norm topology on $\mathfrak{M}_x(A)$ refines the compact Hausdorff topology on $\mathfrak{M}_x(A)$.*
2. *The norm-closure of $\mathrm{conv}(\{\delta_a : a \in A^x\})$ is the following:*
$$\Big\{\sum_{i\in\omega} r_i\delta_{a_i} : a_i \in A^x;\ r_i \in \mathbb{R}_{\geq 0};\ \sum_{i\in\omega} r_i = 1\Big\}.$$
3. *The set $\mathrm{conv}_{\mathbb{Q}}(\{\delta_a : a \in A^x\})$ is a norm-dense subset of $\mathrm{conv}(\{\delta_a : a \in A^x\})$.*
4. *We have that $\mathrm{conv}_{\mathbb{Q}}(\{\delta_a : a \in A^x\}) = \{\mathrm{Av}(\overline{a}) : \overline{a} \in (A^x)^{<\omega}\}$.*

Notation 2.7. We write $\mathrm{conv}(A^x)$ for $\mathrm{conv}(\{\delta_a : a \in A^x\})$.

2.1.1 Supports of Keisler measures

We now move on to discussing an important collection of types connected to a Keisler measure: the support. The support of a measure can be thought of as the portion of the type space where the measure concentrates. More formally,

Definition 2.8. If μ is in $\mathfrak{M}_x(A)$, then we denote the support of μ as $\sup(\mu)$ where,

$$\sup(\mu) = \{p \in S_x(A) : \mu(\varphi(x)) > 0 \text{ for any } \varphi(x) \in p\}.$$

From time to time, one can reduce problems about Keisler measures to problems about types by showing that a particular property holds for all types in the support of a measure. We now state and prove some basic properties about supports. The propositions in this subsection are all more or less folklore. We provide the proofs for clarity and completeness.

Proposition 2.9. *Let μ be in $\mathfrak{M}_x(A)$. Then for any formula $\varphi(x)$ in $\mathcal{L}_x(A)$ such that $\mu(\varphi(x)) > 0$, there exists some $q \in \sup(\mu)$ such that $\varphi(x) \in q$. As consequence, we observe that* $\sup(\mu) \neq \emptyset$.

Proof. Assume that $\mu(\varphi(x)) > 0$. Notice that the collection $\Phi = \{\varphi(x)\} \cup \{\psi(x) \in \mathcal{L}_x(A) : \mu(\neg\psi(x)) = 0\}$ is finitely consistent. Therefore there exists a type q containing each formula from Φ. By construction, q is in the support of μ. □

Proposition 2.10. *Let $\mu \in \mathfrak{M}_x(A)$. Then* $\sup(\mu)$ *is a compact subset of $S_x(A)$ and* $\mu(\sup(\mu)) = 1$

Proof. Assume that $p \notin \sup(\mu)$. Then, there exists a formula $\varphi_p(x)$ such that $\varphi_p(x) \in p$ and $\mu(\varphi_p(x)) = 0$. Therefore

$$S_x(A) \backslash \sup(\mu) = \bigcup_{p \notin \sup(\mu)} \varphi_p(x).$$

So $\sup(\mu)$ is closed. Since $S_x(A)$ is compact, it follows that $\sup(\mu)$ is compact.

Now assume that $\mu(S_x(A) \backslash \sup(\mu)) > 0$. By regularity of μ, there exists a clopen subset $\psi(x)$ such that $\psi(x) \subseteq S_x(A) \backslash \sup(\mu)$ and $\mu(\psi(x))$ is positive. By Proposition 2.9, $\psi(x) \cap \sup(\mu)$ is non-empty and so we have a contradiction. We conclude that $\mu(\sup(\mu)) = 1$. □

Proposition 2.11. *Let $\mu \in \mathfrak{M}_x(A)$. Let $B \subseteq A$ and $r : S_x(A) \to S_x(B)$ be the natural restriction map. Then, for any q in* $\sup(\mu|_B)$*, there exists some $\hat{q} \in$* $\sup(\mu)$ *such that $r(\hat{q}) = q$.*

Proof. The map $r : S_x(A) \to S_x(B)$ is a continuous surjection between compact Hausdorff spaces. Since the continuous image of compact sets are compact, we observe that $r(\sup(\mu))$ is compact (and therefore closed). Notice that $r(\sup(\mu)) \subseteq \sup(\mu|_B)$. We only need to check that $r(\sup(\mu))$ is a dense subset of $\sup(\mu|_B)$. Assume that $\varphi(x) \in \mathcal{L}_x(B)$ and $\varphi(x) \cap \sup(\mu|_B) \neq \emptyset$. We need to show that $\varphi(x) \cap r(\sup(\mu)) \neq \emptyset$. Since $\varphi(x) \cap \sup(\mu|_B) \neq \emptyset$, we have that $\mu|_B(\varphi(x)) > 0$. So, $\mu(\varphi(x)) > 0$. By Proposition 2.9, there exists $q \in \sup(\mu)$ such that $\varphi(x) \in q$. Then, we have that $\varphi(x) \in r(q)$ and so $r(\sup(\mu)) \cap \varphi(x) \neq \emptyset$. We conclude that $r(\sup(\mu))$ is a closed dense subset of $\sup(\mu|_B)$ and so $r(\sup(\mu)) = \sup(\mu|_B)$. This completes the proof. $\square$

Remark 2.12. Let $\mu \in \mathfrak{M}_x(A)$ and $B \subseteq A$. If r is the restriction map from $S_x(A)$ to $S_x(B)$, then the $\mu|_B$ is the pushforward of μ along r, i.e. $r_*(\mu) = \mu|_B$. This is clear by the definition of the pushforward of a measure.

2.2 Zoo of Keisler measures

We now move to the global context and begin to describe the many different kinds of Keisler measures which appear in this dissertation. We are interested in *tameness properties* of measures and especially when our measures *are controlled by* a small submodel of $\mathcal{U}$. Instead of defining all the properties at once, we separate these properties into manageable subsections. As stated previously, many of the results in this section fall into one of three categories: folklore results, propositions generalized from the type case to the measure context, and results originally proved in [33] (some of the results are exposited in the paper [14]). We begin by defining the properties of invariance, definability, and finite satisfiability. We then move on to discuss Borel-definable measures and products of measures. We end this section with a by defining finitely approximated measures, frequency interpretation measures, and smooth measures.

2.2.1 Invariant, definable, and finitely satisfiable measures

The first three properties we describe are direct generalizations of tameness properties for types.

Definition 2.13. Fix $\mu \in \mathfrak{M}_x(\mathcal{U})$.

1. μ is **invariant** if there is $M \prec \mathcal{U}$ such that for any partitioned $\mathcal{L}$-formula $\varphi(x;y)$ and any $b, b' \in \mathcal{U}^y$, if $b \equiv_M b'$ then $\mu(\varphi(x;b)) = \mu(\varphi(x;b'))$. In this case, we also say μ is **M-invariant** or **invariant over** M.

2. μ is **definable** if there is $M \prec \mathcal{U}$ such that for any partitioned $\mathcal{L}$-formula $\varphi(x;y)$ and any $\epsilon > 0$, there exists formulas $\psi_1(y), ..., \psi_n(y)$ such each $\psi_i(y) \in \mathcal{L}_y(M)$, the collection $\{\psi_i(y) : i \leq n\}$ forms a partition of $\mathcal{U}^y$, and if $\models \psi_i(c) \wedge \psi_i(c')$, then $|\mu(\varphi(x;c)) - \mu(\varphi(x;c'))| < \epsilon$. In this case, we also say μ is **definable over** M.

3. μ is **finitely satisfiable in** $M \prec \mathcal{U}$ if for any $\mathcal{L}(\mathcal{U})$-formula $\varphi(x)$, if $\mu(\varphi(x)) > 0$ then $\mathcal{U} \models \varphi(a)$ for some $a \in M^x$. Similar to the case for types, we let $\mathfrak{M}_x(\mathcal{U}, M)$ denote the measures in $\mathfrak{M}_x(\mathcal{U})$ which are finitely satisfiable in M.

4. μ is **dfs** if there is $M \prec \mathcal{U}$ such that μ is both definable over M and finitely satisfiable in M. Similarly, if this is the case, we say that μ is **dfs over** M.

We now show that both definability and finite satisfiability imply invariance.

Proposition 2.14. *If μ is definable over M or finitely satisfiable in M, then μ is M-invariant.*

Proof. Let μ be definable over M and assume that b, b' are in $\mathcal{U}^y$ with $b \equiv_M b'$. Then, for any $\epsilon > 0$, we can find a partition $\mathcal{P}_\epsilon$ of $\mathcal{U}^y$ as in the definition of definability. Since $b \equiv_M b'$, we know that they must be in the same partition since all the formulas in the formation of the partition $\mathcal{P}_\epsilon$ are $\mathcal{L}(M)$-formulas. Therefore, $|\mu(\varphi(x;b)) - \mu(\varphi(x;b'))| < \epsilon$ for every ϵ which implies μ is M-invariant.

Suppose that μ is in $\mathfrak{M}_x(\mathcal{U}, M)$. Assume that μ is not M-invariant. Then, there exists a $\mathcal{L}$-formula $\varphi(x;y)$, an $\epsilon > 0$, and $b, b' \in \mathcal{U}^y$ such that $b \equiv_M b'$ and $\mu(\varphi(x;b)) - \mu(\varphi(x;b')) > \epsilon$. Then, $\mu(\varphi(x;b) \wedge \neg\varphi(x;b')) > 0$. By finite satisfiability, there exists some c in M^x such that $\mathcal{U} \models \varphi(c;b) \wedge \neg\varphi(c;b')$. Therefore, $b \not\equiv_M b'$ and we have a contradiction. □

We move to defining *fiber functions*. These functions are extremely useful and are used heavily throughout the rest of this text. They will play a central role in the discussion of product measures. We defined them here because they are also used in a nice topological characterization of definable measures. We use this latter characterization in the majority of the upcoming proofs.

Definition 2.15. Given $M \prec \mathcal{U}$, a partitioned $\mathcal{L}(M)$-formula $\varphi(x;y)$, and an M-invariant measure $\mu \in \mathfrak{M}_x(\mathcal{U})$, define the map $F^{\varphi}_{\mu,M} : S_y(M) \to [0,1]$ such that $F^{\varphi}_{\mu,M}(q) = \mu(\varphi(x;b))$ where $b \models q$ (this is well-defined by M-invariance). We will write $F^{\varphi}_{\mu,M}$ simply as F^{φ}_{μ} when there is no possibility of confusion.

The next lemma is used to show that definability for a measure is equivalent to a continuity condition. We prove this lemma in the generality of Stone spaces since it is used again in Chapter 3.

Lemma 2.16. *Let S be a totally disconnected compact Hausdorff space. Then, a map $f : S \to [0,1]$ is continuous if and only if for every $\epsilon > 0$, there exists a collection of clopen sets $\mathcal{P} = \{C_1, ..., C_m\}$ such that $\mathcal{P}$ forms a partition of S and for each $i \leq m$, if $b, b' \in C_i$, then $|f(b) - f(b')| < \epsilon$.*

Moreover, if f is continuous then for every $\epsilon > 0$, there exists a partition of clopen sets $\{C_1, ..., C_m\}$ of S such that if we choose b_i in each C_i and let $r_i = f(b_i)$, then

$$\sup_{q \in S} |f(q) - \sum_{i=1}^{m} r_i \chi_{C_i}(q)| < \epsilon$$

Proof. First, we prove the forward direction. Assume that $f : S \to [0,1]$ is continuous. Let $B = \{B_{\epsilon_i} : i \leq n\}$ be a finite collection of open intervals of length ϵ which cover $[0,1]$. Then, $f^{-1}(B_{\epsilon_i}) = U_i$. Then, $U_i = \bigcup_{j \in J_i} C_{i_j}$ where each C_{i_j} is clopen.

Now,

$$\bigcup_{i=1}^{n} f^{-1}(B_{\epsilon_i}) = \bigcup_{i=1}^{n}\bigcup_{j\in J_i} C_{i_j}$$

is an open cover of S. So, $\bigcup_{k=1}^{m} C_k$ for some k's in $\{i_j : i \leq n, j \in J_i\}$ is a finite subcover. If $b, b' \in C_k$, then $f(b), f(b') \in B_{\epsilon_i}$ and so $|f(b) - f(b')| < \epsilon$. Choosing the atoms of the Boolean algebra generated by $\{C_k : k \leq m\}$ gives us a partition.

Now, the other direction. We need to show that f is continuous. Let B_ϵ be an open interval of length ϵ. We want to show that $f^{-1}(B_\epsilon)$ is an open set. If $f^{-1}(B_\epsilon) = \emptyset$, then we are done. Assume that $p \in f^{-1}(B_\epsilon)$. Notice that $f(p) \in B_\epsilon$. Choose δ such that $(f(p)-\delta, f(p)+\delta) \subset B_\epsilon$. Let $\mathcal{P} = \{C_1, ..., C_m\}$ form a partition for δ and assume that $p \in C_p$. Then, $f(C_p) \subseteq (f(p) - \delta, f(p) + \delta)$ and so $C_p \subset f^{-1}(B_\epsilon)$. Repeating this process for each $p \in f^{-1}(B_\epsilon)$, we conclude that $f^{-1}(B_\epsilon) = \bigcup_{p\in B_\epsilon} C_p$. Therefore, $f^{-1}(B_\epsilon)$ is open.

For the moreover part, fix ϵ and let $\{C_1, ..., C_m\}$ be the partition of S found in the forward direction of the proof. For each $i \leq m$, choose $b_i \in C_i$ and set $r_i = f(b_i)$. For any $p \in S$, p is in exactly one element of the partition, say C_j. Then, we note that b_j and p are in the same partition. Therefore, we compute,

$$|f(p) - \sum_{i=1}^{m} r_i \chi_{C_i}(p)| = |f(p) - r_j \chi_{C_j}(p)| = |f(p) - r_j| = |f(p) - f(b_j)| < \epsilon$$

Since p was arbitrary, the inequality holds. □

Proposition 2.17. *Suppose that $\mu \in \mathfrak{M}_x(\mathcal{U})$. The measure μ is definable over M if and only if μ is M-invariant and for any partitioned $\mathcal{L}(M)$-formula $\varphi(x; y)$, the map $F_\mu^\varphi : S_y(M) \to [0, 1]$ is continuous.*

Proof. Assume μ is definable over M. By Proposition 2.14, μ is M-invariant. Fix a partitioned $\mathcal{L}(M)$-formula $\varphi(x; y)$ such that $\varphi(x; y) = \theta(x; y, a)$ for some $\mathcal{L}$-formula $\theta(x; y, z)$ and $a \in M^z$. Then $F_\mu^\theta : S_{yz}(M) \to [0, 1]$ is continuous by definability and

Lemma 2.16. Moreover, the map $i_a : S_y(M) \to S_{yz}(M)$ where $i_a(p)$ is the complete type extending $p \cup \{z = a\}$ is also continuous. Then, $F^{\varphi}_{\mu} = F^{\theta}_{\mu} \circ i_a$ and so F^{φ}_{μ} is continuous. The other direction follows directly from Lemma 2.16. □

From time to time, we might want to *change* the model we are working over. We will see in later chapters (especially Chapters 3 & 5) that it is advantageous to work over a countable model. The next proposition allows us to *change* the model we are working over while ensuring that our measure maintains the same properties over the new model (provided that our measure is already invariant over the *new model*).

Proposition 2.18. *Suppose that $\mu \in \mathfrak{M}_x(\mathcal{U})$. Assume that μ is M-invariant.*

1. *If μ is definable then it is definable over M.*
2. *If μ is finitely satisfiable in some small model N then it is finitely satisfiable in M.*

Proof. Assume that μ is definable. Fix a partitioned $\mathcal{L}$-formula $\varphi(x;y)$. Without loss of generality, we may assume that μ is definable over N where $M \subseteq N$. Then, the map $F^{\varphi}_{\mu,N} : S_y(N) \to [0,1]$ is continuous by definability. The map $F^{\varphi}_{\mu,M} : S_y(M) \to [0,1]$ is well-defined by M-invariance. Let $r : S_y(N) \to S_y(M)$ be the natural restriction map. Then r is a quotient map since it is a surjective continuous map between compact Hausdorff spaces. Moreover, we have that $F^{\varphi}_{\mu,N} = F^{\varphi}_{\mu,M} \circ r$. By the universal property of quotient maps, $F^{\varphi}_{\mu,M}$ is continuous and hence, μ is definable over M.

Now assume that μ is finitely satisfiable in some small model N and let $\mu(\varphi(x;b)) > 0$. Let N_1 realize a coheir of $\text{tp}(N/M)$ over Mb. By compactness, there exists a $b' \in \mathcal{U}^y$ such that $\text{tp}(N_1 b/M) = \text{tp}(Nb'/M)$. By invariance, we know that $\mu(\varphi(x;b)) = \mu(\varphi(x;b'))$. Since μ is finitely satisfiable in N, there exists some a in N^x such that $\mathcal{U} \models \varphi(a;b')$. Since $\text{tp}(N_1 b/M) = \text{tp}(Nb'/M)$, there exists a_1 in N_1^x such that $\models \varphi(a_1;b)$. By the coheir hypothesis, there exists a_0 in M^x such that $\models \varphi(a_0;b)$. □

We now demonstrate that measures which are finitely satisfiable in a small model also have a topological characterization.

Proposition 2.19. *Assume that $\mu \in \mathfrak{M}_x(\mathcal{U})$. Then, μ is finitely satisfiable in M if and only if μ is in the closure of* $\operatorname{conv}(M^x)$ *(viewed as a subset of $\mathfrak{M}_x(\mathcal{U})$).*

Proof. Assume μ is finitely satisfiable in M. Let U be a basic open subset of $\mathfrak{M}_x(\mathcal{U})$ containing μ. Then, there exists $\mathcal{L}(\mathcal{U})$-formulas $\varphi_1(x), ..., \varphi_n(x)$, and real numbers $r_i, ..., r_n, s_1, ..., s_n$ such that,

$$U = \bigcap_{i=1}^{n} \{\nu \in \mathfrak{M}_x(M) : r_i < \nu(\varphi_i(x)) < s_i\}.$$

The collection $\{\varphi_1(x), ..., \varphi_n(x)\}$ generates a finite Boolean algebra of $\mathcal{L}_x(\mathcal{U})$. Let $\theta_1(x), ..., \theta_m(x)$ be the atoms of this Boolean algebra and consider $\Theta = \{\theta_j(x) : \mu(\theta_j(x)) > 0\}$. Since μ is finitely satisfiable in M, we know that for each $\theta_j(x) \in \Theta$, there exists a_j in M^x such that $\mathcal{U} \models \theta_j(a_j)$. Consider the Keisler measure,

$$\nu = \sum_{\theta_j \in \Theta} \mu(\theta_j(x))\delta_{a_j}.$$

It is clear that $\nu \in U$ and so $\mu \in \operatorname{cl}(\operatorname{conv}(M^x))$.

Now, suppose that $\mu \in \operatorname{cl}(\operatorname{conv}(M^x))$. We want to show that μ is finitely satisfiable in M. Fix a formula $\psi(x) \in \mathcal{L}_x(\mathcal{U})$ and assume that $\mu(\psi(x)) > 0$. Consider the open set $U_\psi = \{\nu \in \mathfrak{M}_x(\mathcal{U}) : 0 < \nu(\psi(x)) < 2\}$. Since μ is in the closure of $\operatorname{conv}(M^x)$, there exists some $\mu_\psi = \sum_{i=1}^{n} r_i\delta_{a_i}$ where each a_i is in M^x and $\mu_\psi \in U_\psi$. Then, for some $i \leq n, \mathcal{U} \models \psi(a_i)$ which completes the proof. □

The last proposition of this section demonstrates that that dfs measures are φ^*-*definable.* Restricting to the case of types, the proposition shows that if p is dfs over a model M, then for any partitioned $\mathcal{L}_{xy}(M)$-formula $\varphi(x; y)$, there exists a φ^*-formula

$\psi(y)$ with parameters only from M such that $\models \psi(b)$ if and only if $\varphi(x;b) \in p$. As in Proposition 2.18, the proof uses the universal property of quotient maps.

Proposition 2.20. *Assume that μ is* dfs *over M and let $\varphi(x;y)$ be a partitioned $\mathcal{L}(M)$-formula. Then,*

1. *For any closed set $C \subseteq [0,1]$, the set $\{b \in \mathcal{U}^y : \mu(\varphi(x;b)) \in C\}$ is φ^*-type-definable over M.*
2. *Suppose $b \in \mathcal{U}^y$ and $\mu(\varphi(x;b)) > 0$. Then there is a φ^*-formula $\psi(y)$, with parameters from M, such that $\mathcal{U} \models \psi(b)$ and $\mu(\varphi(x;c)) > 0$ for any $c \in \psi(\mathcal{U})$.*

Proof. Let $r_\varphi \colon S_y(M) \to S_{\varphi^*}(M)$ be the natural restriction map. Recall that any continuous surjection between compact Hausdorff spaces is a quotient map, and so r_φ is a quotient map. We claim that $F := F_\mu^\varphi \circ r_\varphi^{-1}$ is a well-defined function from $S_{\varphi^*}(M)$ to $[0,1]$. In other words, we fix $c, c' \in \mathcal{U}^y$ such that $\mathrm{tp}_{\varphi^*}(c/M) = \mathrm{tp}_{\varphi^*}(c'/M)$ and show that $\mu(\varphi(x;c)) = \mu(\varphi(x;c'))$. Toward a contradiction, suppose $\mu(\varphi(x;c)) > \mu(\varphi(x;c'))$. Then $\mu(\varphi(x;c) \wedge \neg\varphi(x;c')) > 0$, and thus $\varphi(x;c) \wedge \neg\varphi(x;c')$ is realized in M, which contradicts $\mathrm{tp}_{\varphi^*}(c/M) = \mathrm{tp}_{\varphi^*}(c'/M)$.

Since μ is definable over M, we have that F_μ^φ is continuous. Now, by the universal property of quotient maps, F is continuous. This immediately implies the first statement by considering $F^{-1}(C)$. For the second statement, fix $b \in \mathcal{U}^y$ such that $\mu(\varphi(x;b)) > 0$. Then $F(\mathrm{tp}_{\varphi^*}(b/M)) > 0$. Fix $0 < \delta < F(\mathrm{tp}_{\varphi^*}(b/M))$ and consider $U = F^{-1}((\delta, 1])$. Then U is an open set in $S_{\varphi^*}(M)$ containing $\mathrm{tp}_{\varphi^*}(b/M)$, and so there is a φ^*-formula $\psi(y)$ over M such that $\mathrm{tp}_{\varphi^*}(b/M) \in \{p \in S_{\varphi^*}(\mathcal{U}) : \psi(y) \in p\} \subseteq U$. Now $\psi(y)$ is as desired. $\square$

2.2.2 Borel definability and products

In this subsection, we describe the basics of Borel-definability and products as well as relate these concepts to the properties from the previous subsection. In general, one can always construct a product type from an invariant type and an arbitrary type (see Definition 1.2). While it is true that for any pair of measures $\mu \in \mathfrak{M}_x(\mathcal{U})$ and $\nu \in \mathfrak{M}_y(\mathcal{U})$ we can construct the product measure $\mu \times \nu$ on $\mathcal{L}_x(\mathcal{U}) \times \mathcal{L}_y(\mathcal{U})$, this construction loses too much information. We would like to have a measure on $\mathcal{L}_{xy}(\mathcal{U})$ which naturally extends both of our measures. Now, we have good news and bad news. The good news is that for certain pairs of measures, we can assemble a product on the space $\mathcal{L}_{xy}(\mathcal{U})$. Unfortunately, construction of this product measures is more complicated than the type construction. This process relies on integrating fiber functions over a small submodel. Therefore, we will restrict to the collection of measures which have Borel fiber functions. This leads to the definition of Borel-definability.

Definition 2.21. Fix $\mu \in \mathfrak{M}_x(\mathcal{U})$. Then the measure μ is **Borel-definable** if there is $M \prec \mathcal{U}$ such that μ is M-invariant and for any partitioned $\mathcal{L}$-formula $\varphi(x;y)$, the map $F_\mu^\varphi : S_y(M) \to [0,1]$ is a Borel map. In this case, we say that μ is **Borel-definable over** M.

It is obvious from the definition that all definable measures are Borel, since all the fiber functions associated to a definable measure are continuous. We will see later that in the NIP context, all invariant measures are Borel-definable. We now recall some basic propositions which will help us define products.

Proposition 2.22. *Suppose that $\mu \in \mathfrak{M}_x(\mathcal{U})$. Assume that μ is Borel-definable over M. Then,*

1. *for every partitioned $\mathcal{L}(M)$-formula $\varphi(x;y)$, the map $F_{\mu,M}^\varphi : S_y(M) \to [0,1]$ is Borel.*
2. *Moreover, for any N such that $M \subseteq N$ we have that μ is Borel-definable over N.*

Proof. The proof of the first statement is similar to the proof of Proposition 2.17. The function F_μ^φ is the composition of a Borel function and a continuous function, and hence is Borel. In particular, if $\theta(x; y, b) = \varphi(x; y)$ and we consider the maps $F^{\theta}_{\mu} : S_{yz}(M) \to [0,1]$ and $i_b : S_y(M) \to S_{yz}(M)$ where $i_b(p)$ is the unique type extending $p \cup \{z = b\}$, then $F^{\varphi}_{\mu} = F^{\theta}_{\mu} \circ i_b$.

We now prove the second statement. For any $\mathcal{L}$-formula $\varphi(x; y)$, we have that $F^{\varphi}_{\mu,N}$ is equal to $F^{\varphi}_{\mu,M} \circ r$ where r is the restriction map. Therefore $F^{\varphi}_{\mu,N}$ is the composition of a continuous function and a Borel function, hence Borel. $\square$

Definition 2.23. Let $\mu \in \mathfrak{M}_x(\mathcal{U})$ and $\nu \in \mathfrak{M}_y(\mathcal{U})$ be Keisler measures, and suppose μ is Borel-definable over $M \prec \mathcal{U}$. We define the product $\mu \otimes \nu$ in $\mathfrak{M}_{xy}(\mathcal{U})$ such that, given an $\mathcal{L}(\mathcal{U})$-formula $\varphi(x; y)$,

$$\mu \otimes \nu(\varphi(x; y)) = \int_{S_y(N)} F^{\varphi}_{\mu} \, d\nu|_N,$$

where $N \prec \mathcal{U}$ contains M and any parameters in $\varphi(x; y)$ and $\nu|_N$ denotes the regular Borel probability measure on $S_y(N)$ associated to the restriction of ν to $\mathcal{L}_y(N)$ (we will write ν instead of $\nu|_N$ when there is no possibility for confusion).

In the context of the definition, the product $\mu \otimes \nu$ is well-defined and does not depend on the choice of N (see [59, Proposition 7.19] and also Proposition 6.4 in this thesis for a similar proof). We warn the reader that the product in general is not commutative, i.e. $\mu_x \otimes \nu_y \neq \nu_y \otimes \mu_x$. We now take the opportunity to clarify how the properties from our first section behave under products. Propositions 2.24 and 2.25 can be found in [33, Lemma 1.6] (the first without proof, the latter with).

Proposition 2.24. *Let $\mu \in \mathfrak{M}_x(\mathcal{U})$, $\nu \in \mathfrak{M}_y(\mathcal{U})$, and $\lambda \in \mathfrak{M}_z(\mathcal{U})$ be Keisler measures, and suppose μ and ν are definable over $M \prec \mathcal{U}$. Then $\mu \otimes \nu$ is definable over M, and $\mu \otimes (\nu \otimes \lambda) = (\mu \otimes \nu) \otimes \lambda$.*

Proof. We first show $\mu \otimes \nu$ is definable over M. Fix an $\mathcal{L}$-formula $\varphi(x, y; z)$. We need to show that the map $F^{\varphi}_{\mu \otimes \nu} : S_z(M) \to [0,1]$ is continuous. To demonstrate this, we will show that this map is a uniform limit of continuous functions, and hence continuous.

Fix $\epsilon > 0$. Since μ is definable, the map $F_\mu^\varphi : S_{yz}(M) \to [0,1]$ is continuous. Since $S_{yz}(M)$ is a Stone space, there are $\mathcal{L}(M)$-formulas $\psi_1(y,z), \ldots, \psi_n(y,z)$, which partition $S_{yz}(M)$, and real numbers $r_1, \ldots, r_n$ such that for any $p \in S_{yz}(M)$, $F_\mu^\varphi(p) \approx_\epsilon \sum_{i=1}^n r_i \chi_{\psi_i(y,z)}(p)$ by Lemma 2.16. Fix $p \in S_z(M)$, $c \models p|_M$, and $N \prec \mathcal{U}$ containing Mc. Let φ^c denote $\varphi(x,y;c)$ and ψ_i^c denote $\psi_i(y,c)$. Then

$$F_{\mu\otimes\nu}^\varphi(p) = \int_{S_y(N)} F_\mu^{\varphi^c}\, d\nu \approx_\epsilon \int_{S_y(N)} \sum_{i=1}^n r_i \chi_{\psi_i^c(y)}\, d\nu = \sum_{i=1}^n r_i \nu(\psi_i^c(y)) = \sum_{i=1}^n r_i F_\nu^{\psi_i}(p).$$

Since ν is definable over M, we have that each $F_\nu^{\psi_i}$ is continuous, and so $\sum_{i=1}^n r_i F_\nu^{\psi_i}$ is continuous. Therefore $F_{\mu\otimes\nu}^\varphi$ is the uniform limit of continuous functions.

Now, to verify associativity, let $\varphi(x,y,z)$ be any $\mathcal{L}(\mathcal{U})$-formula. We define $k_1 = (\mu\otimes(\nu\otimes\lambda))(\varphi(x,y,z))$ and $k_2 = ((\mu\otimes\nu)\otimes\lambda)(\varphi(x,y,z))$, and show $k_1 = k_2$. Let $N \prec \mathcal{U}$ contain M and any parameters in $\varphi(x,y,z)$. Fix $\epsilon > 0$, and let $\psi_1(y,z), \ldots, \psi_n(y,z)$ and $r_1, \ldots, r_n$ approximate $F_\mu^\varphi \colon S_{yz}(N) \to [0,1]$ as above. Then

$$\begin{aligned} k_1 = \int_{S_{yz}(N)} F_\mu^\varphi\, d(\nu\otimes\lambda) &\approx_\epsilon \int_{S_{yz}(N)} \sum_{i=1}^n r_i \chi_{\psi_i(y,z)}\, d(\nu\otimes\lambda) \\ &= \sum_{i=1}^n r_i (\nu\otimes\lambda)(\psi_i(y,z)) = \int_{S_z(N)} \sum_{i=1}^n r_i F_\nu^{\psi_i}\, d\lambda. \end{aligned}$$

Recall that $k_2 = \int_{S_z(N)} F_{\mu\otimes\nu}^\varphi\, d\lambda$. As above, we have $F_{\mu\otimes\nu}^\varphi(p) \approx_\epsilon \sum_{i=1}^n r_i F_\nu^{\psi_i}(p)$ for any $p \in S_z(N)$. Therefore

$$|k_2 - k_1| < \int_{S_z(N)} \left| F_{\mu\otimes\nu}^\varphi - \textstyle\sum_{i=1}^n r_i F_\nu^{\psi_i} \right| d\lambda + \epsilon < 2\epsilon. \qquad \square$$

Proposition 2.25. *Suppose $\mu \in \mathfrak{M}_x(\mathcal{U})$, $\nu \in \mathfrak{M}_y(\mathcal{U})$, and $M \prec \mathcal{U}$. If μ is Borel-definable over M and both μ and ν are finitely satisfiable in M, then $\mu\otimes\nu$ is finitely* *satisfiable in M.*

Proof. Fix a formula $\varphi(x;y) \in \mathcal{L}_{xy}(\mathcal{U})$. Assume that $\mu \otimes \nu(\varphi(x;y)) > 0$. Then,

$$\int_{S_y(N)} F_{\mu}^{\varphi} d(\nu|_N) > 0.$$

Then, there exists some $q \in \sup(\nu|_N)$ such that $F_{\mu}^{\varphi}(q) > 0$. Choose $b \in \mathcal{U}^y$ such that $b \models q$. Then $\mu(\varphi(x;b)) > 0$. By finite satisfiability of μ, there exists some a in M^x such that $\mathcal{U} \models \varphi(a,b)$. Then, $\varphi(a,y) \in q$, and since $q \in \sup(\nu|_N)$, we have that $\nu(\varphi(a,y)) > 0$. Since ν is finitely satisfiable in M, there exists $c \in M^y$ such that $\mathcal{U} \models \varphi(a,c)$. □

Corollary 2.26. *Assume $\mu \in \mathfrak{M}_x(\mathcal{U})$ and $\nu \in \mathfrak{M}_y(\mathcal{U})$. If μ and ν are* dfs *over M, then $\mu \otimes \nu$ is* dfs *over M.*

Proof. Follows directly from Proposition 2.24 and Proposition 2.25. □

Definition 2.27. Suppose that $\mu \in \mathfrak{M}_x(\mathcal{U})$ and μ is definable. Then, we define the following measures:

1. $\mu^0(x_0) = \mu(x_0)$.
2. $\mu^n = \mu^n(x_0, ..., x_n) = \mu(x_n) \otimes \mu^{n-1}(x_0, ..., x_{n-1})$.
3. $\mu^\omega = \bigcup_{i\in\omega} \mu^n$ (where μ^ω is a finitely additive measure on $\mathcal{L}_{(x_i)_{i\in\omega}}(\mathcal{U})$).

We note that μ^n and μ^ω are well-defined and definable by Proposition 2.24. Moreover, we let $\mathfrak{M}_\omega(\mathcal{U})$ be the collection of finitely additive measures on $\mathcal{L}_{(x_i)_{i\in\omega}}(\mathcal{U})$.

2.2.3 Finitely approximated, FIM, and smooth measures

In this section, we discuss the properties and relationships between finitely approximated measures, frequency interpretation measures (also known as FIM measures), and smooth measures. First, we remark that the notions of frequency interpretation measures and smooth measure first appear in [33] while finitely approximated measures are described implicitly. Finitely approximated measures were first explicitly studied by Chernikov and Starchenko in the context of the NIP regularity theorem [10]. Finitely approximated and FIM measures (locally) admit families of tame uniform approximations. To be more precise, after restricting to a partitioned formula, both FIM measures and finitely approximated measures resemble *frequency measures* i.e. measures of the form $\text{Av}(\overline{a})$. The difference between a FIM measure and a finitely approximated measure is the answer to the following question: For any particular partitioned formula, how difficult is it to find a uniform approximation of the form $\text{Av}(\overline{a})$? For FIM measures, one can easily find these approximations and there are many. Intuitively, one can find an approximation by simply taking a large random tuple (random with respect to the product measure). For an arbitrary finitely approximated measure finding an approximation might be more difficult.

On the other hand, smooth measures can be thought of as a generalization of realized types. Akin to realized types, these measures have unique extensions (by definition). We will later see examples of proofs where the classical role of a realized type is given to a smooth measure (see Propositions 2.43 and 5.27). Additionally, smooth measures also admit very nice approximations. While FIM and finitely approximated measures are characterized by admitting approximations of the form $\text{Av}(\overline{a})$, smooth measures can be characterized by admitting a small family of formulas which can be used to approximate the measure of all other formulas in $\mathcal{L}_x(\mathcal{U})$. This will be seen more clearly in Fact 2.29.

Definition 2.28. Let $\mu \in \mathfrak{M}_x(\mathcal{U})$.

1. μ is **finitely approximated** if there is $M \prec \mathcal{U}$ such that for any partitioned $\mathcal{L}$-formula $\varphi(x;y)$ and any $\epsilon > 0$, there exists some $\bar{a} \in (M^x)^{<\omega}$ such that for any $b \in \mathcal{U}^y$, $|\mu(\varphi(x;b)) - \text{Av}(\overline{a})(\varphi(x;b))| < \epsilon$. In this case, we call $\bar{a}$ a (φ, ϵ)**-approximation for** μ, and we say μ is **finitely approximated in** M.
2. μ is a **frequency interpretation measure** (or **FIM**) if there is $M \prec \mathcal{U}$ such that for any partitioned $\mathcal{L}$-formula $\varphi(x;y)$, there is a sequence $(\theta_n(x_1,\dots,x_n))_{n=1}^{\infty}$ of $\mathcal{L}(M)$-formulas satisfying the following properties:

(a) For any $\epsilon > 0$, there is some $n_{\epsilon,\varphi} \geq 1$ such that if $n \geq n_{\epsilon,\varphi}$, $\bar{a} \models \theta_n(\bar{x})$, and $b \in \mathcal{U}^y$, then $|\mu(\varphi(x;b)) - \mathrm{Av}(\bar{a})(\varphi(x;b))| < \epsilon$.

(b) $\lim_{n\to\infty} \mu^n(\theta_n(x_1, \ldots, x_n)) = 1$.

In this case, we say that μ is **FIM over** M.

3. μ is **smooth** if there exists $M \prec \mathcal{U}$ such that for any N where $M \subseteq N$, there exists a unique measure $\mu' \in \mathfrak{M}_x(N)$ such that $\mu'|_M = \mu|_M$. In this case, we also say that μ is **smooth over** M.

4. μ is **trivial** if it is in the closure (in the norm topology) of the convex hull of the Dirac measures of points in $\mathcal{U}^x$, i.e., there are sequences $(a_n)_{n=0}^{\infty}$ from $\mathcal{U}^x$ and $(r_n)_{n=0}^{\infty}$ from $[0,1]$ such that $\sum_{n=0}^{\infty} r_n = 1$ and $\mu = \sum_{n=0}^{\infty} r_n \delta_{a_n}$. If each a_n is in a submodel M, we say that μ is **trivial over** M.

In the definition of FIM, condition (a) implies that μ is finitely approximated in any $M \prec \mathcal{U}$ which contains parameters for the family of formulas $(\theta(x_1, ..., x_n))_{n=1}^{\omega}$. We will see in Proposition 2.30 that finitely approximated measures are definable and so the iterated product μ^n in condition (b) is well-defined. Moreover, from the definition above, it is unclear which type of approximation smooth measures have. The following fact makes this picture clear (see [33, Lemma 2.3] details).

Fact 2.29. *Let $\mu \in \mathfrak{M}_x(\mathcal{U})$. Then, μ is smooth over M if and only if for every partitioned formula $\varphi(x;y)$ in $\mathcal{L}$ and every $n \in \mathbb{N}$, there exists formulas $\psi_1^n(y), ..., \psi_m^n(y)$, $\theta_1^{n-}(x), \theta_1^{n+}(x), ..., \theta_m^{n-}(x), \theta_m^{n+}(x)$ in $\mathcal{L}(M)$ such that,*

1. *$\{\psi_i^n(y)\}_{i=1}^m$ partition $\mathcal{U}^y$.*
2. *If $\models \psi_j^n(b)$, then $\theta_j^{n-}(x) \subseteq \varphi(x;b) \subseteq \theta_j^{n+}(x)$.*
3. *For each $j \leq m$, $\mu(\theta_j^{n+}(x)) - \mu(\theta_j^{n-}(x)) < \frac{1}{n}$.*

We now describe the relationships between these new classes of measures.

Proposition 2.30. *Assume that $\mu \in \mathfrak{M}_x(\mathcal{U})$.*

1. *If μ is finitely approximated over M, then μ is* dfs *over M.*
2. *If μ is FIM over M, then μ is finitely approximated over M.*

3. *If μ is smooth over M, then μ is FIM over M.*

4. *If μ is trivial over M, then μ is smooth over M.*

Proof. We prove the first statement. Assume that μ is finitely approximated over M. We first show that μ is finitely satisfiable in M. Suppose that $\psi(x,y)$ is a $\mathcal{L}_{xy}$-formula and $\mu(\varphi(x;b)) > \epsilon > 0$ for some $b \in \mathcal{U}^y$. By finite approximability, there exists a tuple $\overline{a} = a_1, ..., a_n$ of elements in M^x which is a $(\varphi, \frac{\epsilon}{2})$-approximation for μ. Then, $|\mu(\varphi(x;b)) - \text{Av}(\overline{a})(\varphi(x;b))| < \frac{\epsilon}{2}$ and so $\text{Av}(\overline{a})(\varphi(x;b)) > 0$. Therefore, there must be some index $i \leq n$ such that $\models \varphi(a_i;b)$.

Notice that μ is M-invariant since μ is finitely satisfiable in M. By Proposition 2.17, it suffices to show that for every partitioned $\mathcal{L}$-formula $\varphi(x;y)$, the map $F_\mu^\varphi : S_y(M) \to [0,1]$ is continuous. If $\overline{a} = (a_1, ..., a_n)$ is a (φ, ϵ)-approximation for μ, then

$$\sup_{p \in S_y(M)} |F_\mu^\varphi(p) - F_{\text{Av}(\overline{a})}^\varphi(p)| = \sup_{p \in S_y(M)} |F_\mu^\varphi(p) - \frac{1}{n}\sum_{i=1}^{n} \chi_{\varphi(a_i,y)}(p)| < \epsilon.$$

It is clear that the map $\chi_{\varphi(a_i,y)} : S_y(M) \to [0,1]$ is continuous. Hence, the map $\frac{1}{n}\sum_{i=1}^{n} \chi_{\varphi(a_i,y)}$ is continuous. Therefore, F_μ^φ is the uniform limit of continuous functions and hence continuous.

The second statement is trivial from the definition. For the third statement, see Corollary 2.6 in [33]. The final statement is easy to show and left to the reader as an exercise. ☐

Let's now see how these families of measures interact with our notion of product. We begin by proving some lemmas about finitely approximated measures toward showing that the product of finitely approximated measures remains finitely approximated and these measures commute with one another.

Lemma 2.31. *Fix $M \prec \mathcal{U}$ and an $\mathcal{L}(M)$-formula $\varphi(x;y)$.*

1. *If $\overline{a} \in (M^x)^m$ and $\overline{b} \in (M^y)^n$ then* $\text{Av}(\overline{a}) \otimes \text{Av}(\overline{b}) = \text{Av}(\overline{b}) \otimes \text{Av}(\overline{a})$.

2. *Suppose $\nu \in \mathfrak{M}_y(\mathcal{U})$ is Borel-definable over M, and $a_1, ..., a_n \in M^x$. Then*

$$\int_{S_y(M)} F^{\varphi}_{\mathrm{Av}(\overline{a})} \, d\nu = \int_{S_x(M)} F^{\varphi^*}_{\nu} \, d\,\mathrm{Av}(\overline{a}),$$

i.e., $\mathrm{Av}(\overline{a}) \otimes \nu(\varphi(x;y)) = \nu \otimes \mathrm{Av}(\overline{a}))(\varphi(x;y))$.

3. *Fix $\mu \in \mathfrak{M}_x(\mathcal{U})$ and $\nu \in \mathfrak{M}_y(\mathcal{U})$. Assume that μ and ν are finitely approximated over M. Let $\bar{a}$ be a (φ, ϵ)-approximation for μ and $\bar{b}$ be a (φ^*, ϵ)-approximation for ν, then*

$$|\mu \otimes \nu(\varphi(x;y)) - \mathrm{Av}(\overline{a}) \otimes \mathrm{Av}(\overline{b})(\varphi(x;y))| < 2\epsilon.$$

Proof. The first statement is obvious from the definition. For the second statement, we compute;

$$\int_{S_y(M)} F^{\varphi}_{\mathrm{Av}(\overline{a})} \, d\nu = \int_{S_y(M)} \frac{1}{n} \sum_{i=1}^{n} F^{\varphi}_{a_i} \, d\nu = \frac{1}{n} \sum_{i=1}^{n} \int_{S_y(M)} F^{\varphi}_{a_i} \, d\nu$$
$$= \frac{1}{n} \sum_{i=1}^{n} \nu(\varphi(a_i;y)) = \frac{1}{n} \sum_{i=1}^{n} F^{\varphi^*}_{\nu}(\mathrm{tp}(a_i/M)) = \int_{S_x(M)} F^{\varphi^*}_{\nu} \, d\,\mathrm{Av}(\overline{a}).$$

For the final statement, we use the first and the second to compute the following;

$$(\mu \otimes \nu)(\varphi(x;y)) = \int_{S_y(M)} F^{\varphi}_{\mu} \, d\nu \approx_{\epsilon} \int_{S_y(M)} F^{\varphi}_{\mathrm{Av}(\overline{a})} \, d\nu = \int_{S_x(M)} F^{\varphi^*}_{\nu} \, d\,\mathrm{Av}(\overline{a})$$
$$\approx_{\epsilon} \int_{S_x(M)} F^{\varphi^*}_{\mathrm{Av}(\overline{b})} \, d\,\mathrm{Av}(\overline{a}) = \mathrm{Av}(\overline{a}) \otimes \mathrm{Av}(\overline{b})(\varphi(x;y)). \quad \square$$

Corollary 2.32. *Suppose $\mu \in \mathfrak{M}_x(\mathcal{U})$ and $\nu \in \mathfrak{M}_y(\mathcal{U})$. Assume that μ and ν are finitely approximated in $M \prec \mathcal{U}$. Then $\mu \otimes \nu = \nu \otimes \mu$ and $\mu \otimes \nu$ is finitely approximated in M.*

Proof. We first show that these measures commute. Fix an $\mathcal{L}(\mathcal{U})$-formula $\varphi(x;y)$. Choose a model N expanding M and containing all the parameters from $\varphi(x;y)$. Now apply parts 1 and 3 of Lemma 2.31:

$$\mu \otimes \nu(\varphi(x;y)) \approx_{2\epsilon} \mathrm{Av}(\overline{a}) \otimes \mathrm{Av}(\overline{b})(\varphi(x;y)) = \mathrm{Av}(\overline{b}) \otimes \mathrm{Av}(\overline{a}) \approx_{2\epsilon} \nu \otimes \mu(\varphi(x;y)).$$

To demonstrate that $\mu \otimes \nu$ is finitely approximated in M, we fix $\epsilon > 0$ and let $\phi(x, y; z)$ be an $\mathcal{L}$-formula. Let $\theta_1(x; y, z) = \phi(x, y, z)$ and $\theta_2(y; x, z) = \phi(x, y, z)$. Then a straightforward calculation shows that if $\bar{a} \in (M^x)^m$ is a $(\theta_1, \frac{\epsilon}{2})$-approximation for μ and $\bar{b} \in (M^y)^n$ is a $(\theta_2, \frac{\epsilon}{2})$-approximation for ν, then $((a_i, b_j))_{i \in [m], j \in [n]} \in (M^{xy})^{mn}$ is a (ϕ, ϵ)-approximation for $\mu \otimes \nu$. □

Warning 2.33. Since FIM measures are finitely approximated, we know that FIM measures commute. However, the product of FIM measures does not have to be FIM (see Corollary 4.18).

We recall a special property of smooth measures. In particular, smooth measures commute with all Borel-definable measures. This fact was originally proved by Hrushovski, Pillay, and Simon (see [33, Corollary 2.5]). The moreover portion can be found in [63, Corollary 3.17].

Fact 2.34. *Assume that $\mu \in \mathfrak{M}_x(\mathcal{U})$ and $\nu \in \mathfrak{M}_y(\mathcal{U})$. Moreover, assume that ν is Borel-definable over M and μ is smooth over M. Then, for any $\varphi(x; y) \in \mathcal{L}_{xy}(M)$, we have that,*

$$\int_{S_y(M)} F_\mu^\varphi d(\nu|_M) = \int_{S_x(M)} F_\nu^{\varphi^*} d(\mu|_M).$$

In particular, we have that $\mu \otimes \nu = \nu \otimes \mu$. Moreover, if μ and ν are smooth (over M) then $\mu \otimes \nu$ is smooth (over M).

We end this section by describing which kinds of properties descend to smaller models.

Proposition 2.35. *Assume that $\mu \in \mathfrak{M}_x(\mathcal{U})$. If μ is definable, FIM, smooth, finitely approximated, or* dfs *over/in M, then there exists a model M_0 such that $M_0 \prec M$, $|M_0| = |T| + \aleph_0$ and μ has that property over/in M_0.*

Proof. We notice that the properties of definability, FIM, and smoothness only require the existence of $(|T| + \aleph_0)$-many $\mathcal{L}_x(M)$-formulas (by Fact 2.29 and the definitions of definable and FIM). If we choose an elementary submodel M_0 of M containing the parameters from these formulas, then μ will have the desired property over M_0.

Finitely approximated measures only require the existence of $(|T| + \aleph_0)$-many elements of M. Choosing an elementary submodel M_0 of M with these elements demonstrates that μ is finitely approximated in M_0.

Finally by the first argument, if μ is dfs over M then there is some $M_0 \prec M$ of size $|T| + \aleph_0$ such that μ is definable over M_0. Then, μ is M_0-invariant and so by Proposition 2.18, μ is finitely satisfiable in M_0. $\square$

2.3 Keisler measures in NIP theories

In this section, we restrict our attention to the class of NIP theories. With the exception of Proposition 2.43, many of the results in this section are well known and were first proven by Hrushovski, Pillay, and Simon [33]. We will see that in NIP theories, many of our definitions collapse into a few distinct classes. Our first theorem collapses Borel-definable measures and invariant measures (see [59, Proposition 7.19, discussion after Definition 7.16]).

Fact 2.36 (T is NIP)**.** *Suppose that $\mu \in \mathfrak{M}_x(\mathcal{U})$ and let M be a small submodel of $\mathcal{U}$. Then the following are equivalent:*

1. *μ is M-invariant.*
2. *μ is Borel-definable over M.*
3. *Every type in $\sup(\mu)$ is invariant over M.*

Warning 2.37. The above does not hold in general and counterexamples are easy to come by in the Random Graph.

Fact 2.38 (T is NIP)**.** *Assume that μ, ν are M-invariant. Then, $\mu \otimes \nu$ is M-invariant and Borel-definable.*

Proof. By the Fact 2.36, we only need to show that $\mu \otimes \nu$ is M-invariant. Fix a partitioned $\mathcal{L}$-formula $\varphi(x, y; z)$. Let a and b be elements in $\mathcal{U}^z$ such that $\text{tp}(a/M) = \text{tp}(b/M)$. For sake of contradiction, assume that $\mu \otimes \nu(\varphi(x, y, a)) > \mu \otimes \nu(\varphi(x, y, b))$. Fix N such that $M \prec N$ and a, b are elements of N. Let $F_{\mu}^{\varphi_a} : S_y(N) \to [0, 1]$ via $F_{\mu}^{\varphi_a}(q) = \mu(\varphi(x, c, a))$ and define $F_{\mu}^{\varphi_b}$ similarly. Then,

$$\int_{S_y(N)} F_{\mu}^{\varphi_a} d(\nu|_N) > \int_{S_y(N)} F_{\mu}^{\varphi_b} d(\nu|_N) \implies \int_{S_y(N)} F_{\mu}^{\varphi_a} - F_{\mu}^{\varphi_b} d(\nu|_N) > 0.$$

Therefore, there exists some $q \in \sup(\nu|_N)$ such that $F_{\mu}^{\varphi_a}(q) > F_{\mu}^{\varphi_b}(q)$. Let $d \models q$. Then, we have that $\mu(\varphi(x, d, a)) > \mu(\varphi(x, d, b))$. Notice that if we show that $\text{tp}(da/M) = \text{tp}(db/M)$, then $\mu(\varphi(x, d, a)) = \mu(\varphi(x, d, b))$ since μ is M-invariant, and this will give us a contradiction with the previous sentence. Assume that $\psi(y, z) \in \text{tp}(da/M)$. Then, $\psi(y, a) \in q$. By Proposition 2.11, there exists some $\hat{q} \in \sup(\nu)$ such that $\hat{q}|_N = q$. Since $q \subset \hat{q}$, $\psi(y, a) \in \hat{q}$. By Fact 2.36, $\hat{q}$ is M-invariant. Since $\text{tp}(a/M) = \text{tp}(b/M)$, we have that $\psi(y, b) \in \hat{q}$ and so $\psi(y, b) \in q$ because $b \in N^z$. Since $d \models q$, we have that $\psi(y, z) \in \text{tp}(db/M)$ and so $\mu(\varphi(x, d, a)) = \mu(\varphi(x, d, b))$ which completes the proof.

We refer the reader to the discussion prior to [59, Exercise 20] for an alternative proof. □

We now come to what should be called *The fundamental theorem of generically stable measure in NIP theories.* In a very real sense, one could read this thesis as a large footnote to this theorem. The majority of the work throughout the last five years has gone to generalizing portions of this theorem into different contexts as well as finding counterexamples outside to NIP setting. We are grateful that Hrushovski, Pillay and Simon proved this theorem and we thank them for their work. This theorem states the following.

Theorem 2.39 (T is NIP)**.** *The following are also equivalent.*

1. μ is dfs *over M.*

2. μ is finitely approximated over M.

3. μ is FIM over M.

*Moreover, we say that μ is **generically stable over** M **(in the context of a NIP theory)** if any/all of* $1-3$ *hold.*

The significance of Theorem 2.39 should not be understated and the proof is non-trivial. Hrushovski, Pillay, and Simon present two proofs of this theorem in [33]. One proof goes through Ben Yaacov's work on continuous VC classes [4] while the other is closely related to classical VC theory and uses Chebyshev's inequality. In Chapter 3, we will prove a local version of this result (relating the local versions of dfs and finitely approximated). At the time of writing, we do not know a local version of FIM (see the introduction to Chapter 3).

The following results further impress the notion that smooth measures can play the role of realized types (at least in NIP theories). The first result argues that every measure can be extended to a smooth (think realized) measure [59, Proposition 7.9].

Fact 2.40 (T is NIP)**.** *Let $M \prec \mathcal{U}$ and $\mu \in \mathfrak{M}_x(M)$. Then, μ admits a smooth extension, i.e. there exists $\nu \in \mathfrak{M}_x(\mathcal{U})$ such that ν is smooth and $\nu|_M = \mu$.*

We now take the ideology that a measure can be realized by a smooth one seriously. We suggest that the curious reader check the analogous statement of Proposition 2.41 and 2.43 for types and see where realizations are used. They will notice that the proof is similar to ours except for the fact that the elements in $\mathcal{U}$ have been replaced by smooth measures. We will double down on this intuition in Chapter 5. Using the previous facts about smooth measures, we can now provide a *new proof* of the associativity of the product $\otimes$ on invariant measures.

Proposition 2.41 (T is NIP)**.** *The product $\otimes$ is associative on invariant measures.*

Proof. Assume that $\mu \in \mathfrak{M}_x(\mathcal{U})$, $\nu \in \mathfrak{M}_y(\mathcal{U})$, and $\lambda \in \mathfrak{M}_z(\mathcal{U})$. If μ and ν are invariant, then $(\mu \otimes \nu) \otimes \lambda = \mu \otimes (\nu \otimes \lambda)$. Without loss of generality, we may suppose that μ, ν are invariant over M and $\varphi(x, y, z) \in \mathcal{L}_{xyz}(M)$. It suffices to show

$$(\mu \otimes \nu) \otimes \lambda(\varphi(x, y, z)) = \mu \otimes (\nu \otimes \lambda)(\varphi(x, y, z)).$$

Fix ϵ. By Fact 2.40, we may choose small models N, N_1 such that $M \prec N \prec N_1 \prec \mathcal{U}$ and measures $\hat{\lambda} \in \mathfrak{M}_z(\mathcal{U})$, $\hat{\nu} \in \mathfrak{M}_y(\mathcal{U})$ such that $\hat{\lambda}|_M = \lambda|_M$, $\hat{\nu}|_N = \nu|_N$, $\hat{\lambda}$ is smooth over N, and $\hat{\nu}$ is smooth over N_1. Since $\hat{\lambda}$ is smooth over N, $\hat{\lambda}$ is finitely approximated in N_1 (Proposition 2.30) and so for the partitioned formula $\theta(z; x, y) = \varphi(x, y, z)$, there exists some n and $\bar{c} \in (N^z)^n$ such that $\bar{c}$ is a (θ, ϵ)-approximation for $\hat{\lambda}$. Likewise, for the partitioned formula $\theta_1(y; x, z)$, there exists some m and $\bar{b} \in (N_1^y)^m$ such that $\bar{b}$ is a (θ_1, ϵ)-approximation of $\hat{\nu}$. For each $i \leq n$, we let $F_\mu^{\varphi_{c_i}} : S_y(N) \to [0, 1]$ via $F_\mu^{\varphi_{c_i}}(p) = \mu(\varphi(x, d, c_i))$ where $d \models p$. We first approximate $(\mu \otimes \nu) \otimes \lambda(\varphi(x, y, z))$ by using Fact 2.34.

$$(\mu \otimes \nu) \otimes \lambda(\varphi(x, y, z)) = \int_{S_z(M)} F_{\mu\otimes\nu}^{\varphi} d(\lambda|_M) = \int_{S_z(M)} F_{\mu\otimes\nu}^{\varphi} d(\hat{\lambda}|_M)$$

$$= \int_{S_z(N)} F_{\mu\otimes\nu}^{\varphi} d(\hat{\lambda}|_N) = \int_{S_{xy}(N)} F_{\hat{\lambda}}^{\theta} d(\mu \otimes \nu|_N) \approx_\epsilon \int_{S_{xy}(N)} F_{\mathrm{Av}(\bar{c})}^{\theta} d(\mu \otimes \nu|_N)$$

$$= \frac{1}{n} \sum_{i=1}^{n} \mu \otimes \nu(\varphi(x, y, c_i)) = \frac{1}{n} \sum_{i=1}^{n} \int_{S_y(N)} F_\mu^{\varphi_{c_i}} d(\nu|_N)$$

$$= \frac{1}{n} \sum_{i=1}^{n} \int_{S_y(N)} F_\mu^{\varphi_{c_i}} d(\hat{\nu}|_N) = \frac{1}{n} \sum_{i=1}^{n} \int_{S_y(N_1)} F_\mu^{\varphi_{c_i}} d(\hat{\nu}|_{N_1})$$

$$= \frac{1}{n} \sum_{i=1}^{n} \int_{S_x(N_1)} F_{\hat{\nu}}^{\varphi^*_{c_i}} d(\mu|_{N_1}) \approx_\epsilon \frac{1}{n} \sum_{i=1}^{n} \int_{S_x(N_1)} F_{\mathrm{Av}(\bar{b})}^{\varphi^*_{c_i}} d(\mu|_{N_1})$$

$$= \frac{1}{m \cdot n} \sum_{j=1}^{m} \sum_{i=1}^{n} \mu(\varphi(x, b_j, c_i)).$$

We now approximate the other product. We first argue that $\hat{\nu} \otimes \hat{\lambda}$ extends $(\nu \otimes \lambda)|_M$. Fix $\psi(y;z) \in \mathcal{L}_{yz}(M)$. Again, by Fact 2.34,

$$\nu \otimes \lambda(\psi(y,z)) = \int_{S_z(M)} F_\nu^\psi d(\lambda|_M) = \int_{S_z(M)} F_\nu^\psi d(\hat{\lambda}|_M)$$

$$= \int_{S_z(N)} F_\nu^\psi d(\hat{\lambda}|_N) = \int_{S_y(N)} F_{\hat{\lambda}}^{\psi^*} d(\nu|_N) = \int_{S_y(N)} F_{\hat{\lambda}}^{\psi^*} d(\hat{\nu}|_N)$$

$$= \hat{\lambda} \otimes \hat{\nu}(\psi^*(z;y)) = \hat{\nu} \otimes \hat{\lambda}(\psi(y;z)).$$

By Fact 2.34, the product $\hat{\nu} \otimes \hat{\lambda}$ is a smooth measure (smooth over N_1) which extends $(\nu \otimes \lambda)|_M$. By 3 of Corollary 2.32 and Fact 2.34, we compute the following;

$$\mu \otimes (\nu \otimes \lambda)(\varphi(x,y,z)) = \int_{S_{yz}(M)} F_\mu^\varphi d(\nu \otimes \lambda|_M) = \int_{S_{yz}(M)} F_\mu^\varphi d(\hat{\nu} \otimes \hat{\lambda}|_M)$$

$$= \int_{S_{yz}(N_1)} F_\mu^\varphi d(\hat{\nu} \otimes \hat{\lambda}|_{N_1}) = \int_{S_x(N_1)} F_{\hat{\nu}\otimes\hat{\lambda}}^{\varphi^*} d(\mu|_{N_1}) \approx_{2\epsilon} \int_{S_x(N_1)} F_{\mathrm{Av}(\bar{b})\otimes\mathrm{Av}(\bar{c})}^{\varphi^*} d(\mu|_{N_1})$$

$$= \frac{1}{m \cdot n} \sum_{j=1}^{m} \sum_{i=1}^{n} \mu(\varphi(x, b_j, c_i)).$$

We conclude that $|\mu \otimes (\nu \otimes \lambda)(\varphi(x,y,z)) - (\mu \otimes \nu) \otimes \lambda(\varphi(x,y,z))| < 5\epsilon$ which finishes the proof. $\square$

Observation 2.42 (T is NIP)**.** Suppose $\mu \in \mathfrak{M}_x(\mathcal{U})$. If μ is invariant, then for any $n > 1$, the measures μ^n and μ^ω (from Definition 2.27) are well-defined by Proposition 2.41.

This final proposition was proved in our work on convolution algebras with Artem Chernikov.

Proposition 2.43 (T is NIP)**.** *Let $M \prec \mathcal{U}$ and suppose that $\mathfrak{M}_x^{inv}(\mathcal{U}, M)$ is the collection of global M-invariant measures (endowed with the induced compact Hausdorff topology). If $\nu \in \mathfrak{M}_y(\mathcal{U})$ and $\varphi(x;y)$ is any partitioned $\mathcal{L}_{xy}(\mathcal{U})$-formula, then the map*

$-\otimes\nu(\varphi(x;y)) : \mathfrak{M}_x^{inv}(\mathcal{U},M) \to [0,1]$ *is continuous.*

Proof. Choose N_0 such that $M \preceq N_0$, and N_0 contains all the parameters from φ. Then, choose N submodel that $N_0 \subset N$. There exists some $\hat{\nu} \in \mathfrak{M}_y(\mathcal{U})$ such that $\hat{\nu}|_{N_0} = \nu|_{N_0}$ and $\hat{\nu}$ is smooth over N by Fact 2.40. Let $\bar{b}$ be some (φ^*, ϵ)-approximation for $\hat{\nu}$ over N (i.e. $\bar{b}$ is some element in $(N^y)^{<\omega}$). Then for any $\mu \in \mathfrak{M}_x^{inv}(\mathcal{U},M)$, μ is invariant over both N_0 and N. By the observation that integrating over either space yields the same result, we have the following.

$$\mu \otimes \nu(\varphi(x;y)) = \int_{S_y(N_0)} F^{\varphi}_{\mu,N_0} d(\nu|_{N_0}) = \int_{S_y(N_0)} F^{\varphi}_{\mu,N_0} d(\hat{\nu}|_{N_0}) = \int_{S_y(N)} F^{\varphi}_{\mu,N} d(\hat{\nu}|_N).$$

By Fact 2.34,

$$\int_{S_y(N)} F^{\varphi}_{\mu,N} d(\hat{\nu}|_N) = \int_{S_x(N)} F^{\varphi^*}_{\hat{\nu},N} d(\mu|_N).$$

Now, we can use our (φ^*, ϵ)-approximation,

$$\int_{S_x(N)} F^{\varphi^*}_{\hat{\nu},N} d(\mu|_N) \approx_\epsilon \int_{S_x(N)} F^{\varphi^*}_{\mathrm{Av}(\bar{b})} d(\mu|_N) = \frac{1}{n}\sum_{i=1}^{n} \mu(\varphi(x;b_i)) = \frac{1}{n}\sum_{i=1}^{n} \int_{S_y(\mathcal{U})} \chi_{\varphi(x;b_i)} d\mu.$$

Clearly, each map $\int \chi_{\varphi(x;b_i)} : \mathfrak{M}_x(\mathcal{U}) \to [0,1]$ is continuous by the definition of the topology on this space. Therefore, each map $\int \chi_{\varphi(x;b_i)} : \mathfrak{M}_x^{inv}(\mathcal{U},M) \to [0,1]$ is continuous and the sum is continuous. Since $\mathrm{Av}(\bar{b})$ is independent of the choice of μ,

$$\sup_{\mu \in \mathfrak{M}_x^{inv}(\mathcal{U},M)} |\mu \otimes \nu(\varphi(x;y)) - \frac{1}{n}\sum_{i=1}^{n} \int_{S_y(\mathcal{U})} \chi_{\varphi(x;b_i)} d\mu| < \epsilon.$$

Therefore, $-\otimes\nu(\varphi(x;y))$ is a uniform limit of continuous functions and hence continuous. □

On a personal note, I spent a long time trying to prove Proposition 2.43. If one restricts to the case where $|M| = \aleph_0$, then one can use BFT (Lemma 3.29) and the dominated convergence theorem to prove the proposition. However, as the proof above shows, this approach is a red herring for the general case. It was not until I fully internalized the belief that smooth measures can be thought of as realized types (in NIP theories) that I was able to prove this result.

CHAPTER 3

LOCAL KEISLER MEASURES AND NIP FORMULAS

This chapter is a modified version of my article *Local Keisler measures and NIP formulas* [25]. As previously stated, the connection between finitely additive probability measures and NIP theories was first noticed by Keisler in his seminal paper [36]. Around 20 years later, the work of Hrushovski, Peterzil, Pillay, and Simon greatly expanded this connection in [31], [32], and [33]. In particular, they introduce the notion of *generically stable measures*. These measures exhibit properties similar to those found in the context of stable theories. One of the most striking results to come from this line of research is that dfs measures, finitely approximated measures, and FIM measures are the same in the context of NIP theories. We recall that Hrushovski, Pillay, and Simon [33] showed the following,

Theorem. *Assume that T is an NIP theory. Let $\mathcal{U}$ be a sufficiently saturated model of T, M be a small elementary substructure of $\mathcal{U}$, and $\mu \in \mathfrak{M}_x(\mathcal{U})$. Then the following are equivalent.*

1. *μ is* dfs *over M.*
2. *μ is finitely approximated in M.*
3. *μ is FIM over M.*

Moreover, we say that μ is generically stable over M (in the context of a NIP theory) if any/all of the above statements hold.

The purpose of this chapter is to prove a local version of the theorem above. We show that properties 1 and 2 are equivalent in the local context. At the time of writing, there is no known local definition for FIM. More specifically, we prove the following:

Theorem (Main theorem)**.** *Let T be a first order theory. Let $\mathcal{U}$ be a sufficiently saturated model of T, M be a small elementary substructure of $\mathcal{U}$, and $\varphi(x;y)$ a partitioned, NIP $\mathcal{L}$-formula. Let μ be a finitely additive measure on the collection of φ-definable sets with parameters from $\mathcal{U}$, i.e. $\mathcal{L}_{\varphi}(\mathcal{U})$. Then μ is* dfs *over M (as in Definition 3.33) if and only if μ is finitely appoximated in M (as in Definition 3.34).*

The proof of the global theorem cannot be directly applied to prove the above theorem. In the global case, one considers iterated products of dfs measures and computes the measure of a specific existential formula relative to this product. A priori, there is not a robust enough notion of product in the local context in which one can measure existential formulas. Therefore, we need to use a different technique to prove our main theorem.

The proof of our main theorem involves translating the concepts of definable and finitely satisfiable into the framework of functional analysis. From this vantage point, we can apply an important result of Bourgain, Fremlin, and Talagrand [5], namely, Theorem 3.4 in this dissertation. The connection between NIP formulas and Theorem 3.4 was first noticed by Chernikov and Simon [9] as well as independently by Ibarlucíca [34]. Furthermore, work extending this connection for types has been done by Simon [62] as well as the *NIP in a model* case by Khanaki and Pillay [38]. We extend this connection to local measures via Ben Yaacov's work on continuous VC classes [4].

This chapter is organized as follows: In section 3.1, we provide all necessary functional analysis background necessary for this chapter. In section 3.2, we connect NIP formulas, continuous VC classes, and the important result of Bourgain, Fremlin, and Talagrand mentioned earlier in the introduction. In section 3.3, we begin by exporting some important definitions into the local context. We then translate these properties into the language of analysis. Using the theorems from functional analysis outlined in section 3.1 and the connection established in section 3.2, we will prove our main theorem.

3.1 Background in analysis

We recall some definitions and theorems from functional analysis. We refer the reader to [18] as a standard reference on the subject. Let X be a set, and let $\mathbb{R}^X$ denote the collection of all functions from X to $\mathbb{R}$. Then $\mathbb{R}^X$ is a topological space with the standard product topology. If $A \subseteq \mathbb{R}^X$, we let $\mathrm{cl}_p(A)$ be the topological closure of A in $\mathbb{R}^X$. Let $(f_i)_{i\in\omega}$ be a sequence in $\mathbb{R}^X$. We recall two different notions of convergence:

1. $(f_i)_{i\in\omega}$ *converges pointwise* to a function f, written $f_i \to f$, if for every $b \in X$ and for every $\epsilon > 0$ there is some natural number N such that for any $n > N$, $|f_n(b) - f(b)| < \epsilon$.
2. $(f_i)_{i\in\omega}$ *converges uniformly* to a function f, written $f_i \to_u f$, if for every $\epsilon > 0$ there is some natural number N so that for any $n > N$, $\sup_{b\in X} |f_n(b) - f(b)| < \epsilon$.

Now, assume that X is a topological space and let $C(X)$ denote the space of continuous from X to $\mathbb{R}$. If X is a compact Hausdorff space, then $C(X)$ is a Banach space with the uniform norm, $||\cdot||_\infty$, where $||f||_\infty = \sup_{x\in X} |f(x)|$. Again, let $(f_i)_{i\in\omega}$ be a sequence of points in $C(X) \subset \mathbb{R}^X$. We say that $(f_i)_{i\in\omega}$ *converges weakly* to a function f, written $f_i \to_w f$, if for all continuous linear functionals $G : C(X) \to \mathbb{R}$, we have that $\lim_{i\in\omega} G(f_i) = G(f)$.

We note if X is a compact Hausdorff space, such as $S_y(M)$, then for any sequence of functions $(f_i)_{i\in\omega}$ in $C(X)$, one may determine whether this sequence converges pointwise, weakly, or uniformly. We now recall some theorems from functional analysis which connect these notions of convergence. Our first theorem is a trivial consequence of Mazur's lemma, a proof of which can be found in any basic text on functional analysis (for instance, [19]). This theorem connects the notions of uniform convergence and weak convergence. We will refer to the following theorem simply as Mazur's lemma.

Theorem 3.1 (Mazur's Lemma)**.** *Let $(Y, ||\cdot||)$ be a Banach space, $y \in Y$, $(a_i)_{i\in\omega}$ be a sequence of elements in Y, and $A = \{a_i : i \in \omega\}$. If $a_i \to_w y$, then there is a sequence of $z_i \in \mathrm{conv}_{\mathbb{Q}}(A)$ such that $\lim_{i\to\infty} ||z_i - y|| = 0$.*

In particular, if X is a compact Hausdorff space, $(f_i)_{i\in\omega}$ is a sequence in $C(X)$, and $f_i \to_w g$, then there exists $h_i \in \mathrm{conv}_{\mathbb{Q}}(\{f_i : i \leq \mathbb{N}\})$ such that $h_i \to_u g$. Our next theorem connects the notions of pointwise convergence and weak convergence. This theorem is a routine application of the dominated convergence theorem and the Riesz representation theorem (see [56, Theorem 18.4.1] for details).

Theorem 3.2. *Let X be a compact Hausdorff space, $f \in C(X)$, and $(f_i)_{i\in\omega}$ be a sequence in $C(X)$. Then the following are equivalent:*

1. *$f_i \to_w f$.*
2. *$f_i \to f$ and $\sup_{i\in\omega} ||f_i||_\infty < \infty$.*

The next theorem is a translation of the celebrated result by Bourgain, Fremlin, and Talagrand [5] which we alluded to in the introduction. This particular translation is due to Khanaki and Pillay [38]. A much more general statement is proven in [5] than the one we provide. The connection between this theorem and NIP formulas is well known and has been expanded upon in [62], [37], and [38]. This theorem is slightly more technical than the last two, but essentially it allows one to find pointwise convergent sequences under a particular tameness condition. Before we can state the theorem, we define the tameness condition.

Definition 3.3. Let $A \subset \mathbb{R}^X$. Then we say that A is **sequentially independent** if there exists a sequence $(f_i)_{i\in\omega}$ of elements in A, an $r \in \mathbb{R}$, and an $\epsilon > 0$ such that for every $I \subseteq \omega$, there exists some b_I in X such that,

$$\{n \in \omega : f_n(b_I) \leq r\} = I \text{ and } \{n \in \omega : f_n(b_I) \geq r + \epsilon\} = \omega \backslash I.$$

If A is not sequentially independent, we say that A is **sequentially dependent**.

Theorem 3.4 ([5], translated by [38])**.** *Let X be a compact Hausdorff space, $A \subseteq C(X)$, and $|A| \leq \aleph_0$. Assume that $\sup_{f \in A} ||f||_\infty < \infty$. Then the following are equivalent:*

1. *A is sequentially dependent.*
2. *for each $f \in \mathbb{R}^X$, if $f \in \text{cl}_p(A)$, then there exists a sequence of elements $(f_i)_{i\in\omega}$ from A such that $f_i \to f$.*

We now briefly discuss a family of special functions. For a fixed theory T, a monster model $\mathcal{U}$ of T, and a model $M \prec \mathcal{U}$, we define the following functions.

Definition 3.5. If $\varphi(x; y)$ is a partitioned $\mathcal{L}$-formula and a is in M^x, we define the map $F_a^\varphi : S_y(M) \to \{0, 1\} \subset \mathbb{R}$ via

$$F_a^\varphi(q) = F_{\delta_a}^\varphi(q) = \chi_{\varphi(a,y)}(q) = \begin{cases} 1 & \varphi(a, y) \in q, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, we let $\mathbb{F}_M^\varphi = \{F_a^\varphi : a \in M^x\}$.

Remark 3.6. It is clear that for each a in M^x the map F_a^φ is continuous. In this chapter, we will always view F_a^φ as a map from $S_y(M)$ to $\mathbb{R}$. We let $\text{conv}(\mathbb{F}_M^\varphi)$ be the convex hull of $\mathbb{F}_M^\varphi$ in $C(S_y(M))$.

Definition 3.7. Let $f \in \text{conv}_{\mathbb{Q}}(\mathbb{F}_M^\varphi)$. A **representative sequence** for f is a sequence of points $a_1, ..., a_m \in M^x$ so that for every $b \in \mathcal{U}^y$, $\text{Av}(\overline{a})(\varphi(x; b)) = f(\text{tp}(b/M))$.

Remark 3.8. Every element f in $\text{conv}_{\mathbb{Q}}(\mathbb{F}_M^\varphi)$ has many representative sequences. If $f \in \text{conv}_{\mathbb{Q}}(\mathbb{F}_M^\varphi)$, then $f = \sum_{i=1}^n r_i F_{c_i}^\varphi$ where each c_i is in M^x, $r_i \in \mathbb{Q}^+$, and $\sum_{i=1}^n r_i = 1$. Let m be the smallest number so that for every $i \leq n$, $m \cdot r_i \in \mathbb{N}$. Let

$$\mathbf{a}_i = \underbrace{c_i, ..., c_i}_{m \cdot r_i \text{ times}}$$

Then, the concatenation of the $\mathbf{a}_i$'s is a representative sequence for f.

3.2 NIP, VC, and BFT

We now connect the notions of *Vapnik-Chervonenkis dimension* (or *VC dimension*), NIP formulas, and sequential dependence. In particular, we show that if $\varphi(x; y)$ is NIP, then $\text{conv}(\mathbb{F}_M^\varphi)$, as a subset of $\mathbb{R}^{S_y(M)}$, is sequentially dependent. This result, Theorem 3.21, follows implicitly by results in [4]. Here, we give a direct proof of this theorem. We begin by fixing some notation.

Notation 3.9. Let $\varphi(x; y)$ be a partitioned formula. Let $\mathcal{F}_\varphi$ be the family of definable subsets of $\mathcal{U}^x$ of the form $\{\varphi(x; b) : b \in \mathcal{U}^y\}$. Likewise, we let $\mathcal{F}_{\varphi^*} = \{\varphi(a, y) : a \in \mathcal{U}^x\}$.

We now continue by recalling some basic VC theory. The purpose here is to clearly state the VC theorem. The VC theorem acts an an intermediary connecting the NIP formulas with the result of Bourgain, Fremlin, and Talagrand.

Definition 3.10. Let X be a set and let $\mathcal{F}$ be a family of subsets of X. Then, the **VC-dimension of** $\mathcal{F}$ is the largest n such that there exists $A \subset X$, $|A| = n$, and,

$$\{K \subset A : \exists F \in \mathcal{F} \text{ so that } F \cap A = K\} = \mathcal{P}(A).$$

We denote the VC-dimension of $\mathcal{F}$ as $\dim_{VC}(\mathcal{F})$. If no such n exists, then $\dim_{VC}(\mathcal{F}) = \infty$.

The following observation was first made by Laskowski [40].

Fact 3.11. *A formula $\varphi(x; y)$ is NIP if and only if $\mathcal{F}_\varphi$ and $\mathcal{F}_{\varphi^*}$ have finite VC-dimension.*

Definition 3.12. Let X be a set, $\mathcal{F}$ be a family of subsets of X, and μ be a measure on $\mathcal{P}(X)$. Then, we say that a sequence of elements $a_1, ..., a_n$ in X is an **ϵ-approximation for μ over $\mathcal{F}$** (or **$(\mathcal{F}, \epsilon)$-approximation for μ**) if for every $F \in \mathcal{F}$, $|\mu(F) - \mathrm{Av}(\bar{a})(F)| < \epsilon$.

Now we may state the VC Theorem.

Theorem 3.13 ([65])**.** *Let X be a set and $\mathcal{F}$ a family of subsets of X. Assume that $\mathcal{F}$ has finite VC-dimension. Let μ be a probability measure concentrating on a finite subset of X, i.e. $\mu(A) = 1$ for some finite set A. Then, for every $\epsilon > 0$, there exists some constant $C_{\epsilon,d}$, depending only on ϵ and $d = \dim_{VC}(\mathcal{F})$, and a sequence of points $c_1, ..., c_m$ in X such that $m \leq C_{d,\epsilon}$ and $c_1, ..., c_m$ is an $(\mathcal{F}, \epsilon)$-approximation of μ.*

Remark 3.14. We notice that if $\mathcal{F}$ is a family of subsets of X with VC-dimension d, then for every $\epsilon > 0$ and for every probability measure μ concentrating on a finite set, there exists some $K_{d,\epsilon}$ depending only of d and ϵ and a sequence of points $a_1, ..., a_m$ in X where $m = K_{d,\epsilon}$ such that $a_1, ..., a_m$ is an ϵ-approximation of μ over $\mathcal{F}$. In other words, for any measure concentrating on a finite set, we can find an ϵ-approximation of **exactly** length $K_{d,\epsilon}$, e.g. we may take $K_{d,\epsilon} = C_{d,\epsilon}!$ since one can just concatenate sequences of length less than $C_{d,\epsilon}$ with themselves many times.

We now define dependence in the continuous context and explain how this relates to sequential dependence.

Definition 3.15 (Shattering)**.** Let X, Y be sets, $f : X \times Y \to [0, 1]$, $r \in (0, 1)$ and $\epsilon > 0$. Let $A \subset X$. We say that f (r, ϵ)-shatters A if for every $K \subseteq A$, there exists some b_K in Y so that

$$\{a \in X : f(a, b_K) \leq r\} \cap A = K,$$

and,

$$\{a \in X : f(a, b_K) \geq r + \epsilon\} \cap A = A \setminus K.$$

Moreover, we say that B witnesses the (r, ϵ)-shattering of A if for each $K \subseteq A$, B contains b_K as above.

The following definition of dependence is equivalent to the one given in [4].

Definition 3.16. Let X, Y be sets and let $f : X \times Y \to [0, 1]$. We say that f is (r, ϵ)**-independent** if for every $n \in \mathbb{N}$, there exists $A \subset X$ where $|A| > n$ and f (r, ϵ)-shatters A. We say that f is **independent** if there exists some $r \in (0, 1)$ and $\epsilon > 0$ so that f is (r, ϵ)-independent. Finally, we say that f is **dependent** if f is not independent.

Fact 3.17. *If $f : X \times Y \to [0, 1]$ is dependent, then $\mathbf{X} = \{f(a, y) : a \in X\} \subset \mathbb{R}^Y$ is sequentially dependent.*

Proof. This follows directly from the definitions. □

We prove the theorem advertised in the introduction of this section, namely Theorem 3.21. We begin by defining a function. Showing that this function is dependent in the correct context yields our theorem.

Definition 3.18. Let $f : X \times Y \to \{0, 1\}$. Then, we define the map $\Psi_f : \mathrm{conv}(X) \times \mathrm{conv}(Y) \to [0, 1]$ via

$$\Psi_f(\mu, \nu) = \int f(x, y) d\mu d\nu = \sum_{i=1}^{n} \sum_{j=1}^{m} r_i s_j f(a_i, b_j),$$

where $\mu = \sum_{i=1}^{n} r_i \delta_{a_i}$ and $\nu = \sum_{j=1}^{m} s_j \delta_{b_j}$.

Lemma 3.19. *Assume that* $\varphi(x;y)$ *is an NIP formula. Consider* $f : \mathcal{U}^x \times \mathcal{U}^y \to \{0,1\}$ *via,*

$$f(a,b) = \begin{cases} 1 & \mathcal{U} \models \varphi(a,b), \\ 0 & \textit{otherwise}. \end{cases}$$

Then the map $\Psi_f : \mathrm{conv}(\mathcal{U}^x) \times \mathrm{conv}(\mathcal{U}^y) \to [0,1]$ *is dependent.*

Proof. Assume that the map Ψ_f is independent. We will find a Boolean combination of φ which is not NIP. Since Ψ_f is independent, Ψ_f is (r_0, ϵ_0)-independent for some r_0 in $(0,1)$ and $\epsilon_0 > 0$. Let $r = r_0 + \frac{\epsilon_0}{3}$, $\epsilon = \frac{\epsilon_0}{3}$, and $\delta = \frac{\epsilon}{6} = \frac{\epsilon_0}{18}$. Then Ψ_f is (r,ϵ)-independent because $[r, r+\epsilon] \subset [r_0, r_0+\epsilon_0]$. Now, $d_1 = \dim_{VC}(\mathcal{F}_\varphi)$ and $d_2 = \dim_{VC}(\mathcal{F}_{\varphi^*})$ where $d_1, d_2 < \infty$ since $\varphi(x;y)$ is NIP. Let $n_\star = K_{\delta,d_1}$ and let $m_\star = K_{\delta,d_2}$ described in Remark 3.14. Choose the smallest $w \in \{1,2,...,n_\star m_\star\}$ such that $\frac{w}{n_\star m_\star} \geq r$. Let $n_\star \times m_\star = \{(i,j) : i \leq n_\star, j \leq m_\star\}$ and let $W = [n_\star \times m_\star]^w$, i.e. the subsets of $n_\star \times m_\star$ of size $|w|$. Now, consider the formula:

$$\theta(x_1, ..., x_{n_\star}, y_1, ..., y_{m_\star}) \equiv \neg \bigvee_{\alpha \in W} \bigwedge_{(i,j)\in\alpha} \varphi(x_i, y_j)$$

Notice that $\theta(\overline{x}, \overline{y})$ takes in two sequences of elements and decides whether $\varphi(x;y)$ holds on a certain proportion of pairs of elements. In other words, if $a_1, ..., a_{n_\star}$ is a sequence in $\mathcal{U}^x$ and $b_1, ..., b_{m_\star}$ is a sequence in $\mathcal{U}^y$, then $\theta(\overline{a}, \overline{b})$ determines whether $\Psi_f(\mathrm{Av}(\overline{a}), \mathrm{Av}(\overline{b}))$ is greater than r.

We now show that θ is not NIP. Assume that Ψ_f (r_0, ϵ_0)-shatters A. Then, for each $K \subseteq A$, there exists ν_K in $\mathrm{conv}(\mathcal{U}^y)$ so that,

$$\{\mu \in \mathrm{conv}(\mathcal{U}^x) : \Psi_f(\mu, \nu_K) \leq r\} \cap A = K,$$

and,

$$\{\mu \in \mathrm{conv}(\mathcal{U}^x) : \Psi_F(\mu, \nu_K) \geq r + \epsilon\} \cap A = A \backslash K.$$

Then, for each $\mu \in A$, let $a_1^\mu, ..., a_{n_\star}^\mu$ be a δ-approximation for that particular μ over $\mathcal{F}_\varphi$ of length exactly $n_\star$. Moreover, for each $K \subset A$, let $b_1^K, ..., b_{m_\star}^K$ be a δ-approximation for ν_K over $\mathcal{F}_{\varphi^*}$ of length exactly $m_\star$. Let $A_\star = \{(a_1^\mu, ..., a_{n_\star}^\mu) : \mu \in A\} \subset \mathcal{U}^{|x| \cdot n_\star}$ and $B_\star = \{(b_1^K, ..., b_{m_\star}^K) : K \subset A\} \subset \mathcal{U}^{|y| \cdot m_\star}$. Then

$$|\Psi_f(\mu, \nu_K) - \Psi_f(\mathrm{Av}(\overline{a^\mu}), \mathrm{Av}(\overline{b^K}))| < 2\delta = \frac{\epsilon}{3}.$$

Now, if $(a_1^\mu, ..., a_{n_\star}^\mu)$ is in $A_\star$ and $(b_1^K, ..., b_{m_\star}^K)$ is in $B_\star$, then by a standard computation we notice that,

$$\mathcal{U} \models \theta(\overline{a^\mu}, \overline{b^K}) \iff \Psi_f(Av(\overline{a^\mu}), Av(\overline{b^K})) \leq r \iff \Psi_f(\mu, \nu_k) \leq r_0,$$

as well as,

$$\mathcal{U} \models \neg\theta(\overline{a^\mu}, \overline{b^K}) \iff \Psi_f(Av(\overline{a^\mu}), Av(\overline{b^K})) \geq r + \epsilon \iff \Psi_f(\mu, \nu_k) \geq r_0 + \epsilon.$$

We conclude that θ is not NIP. However, by the Fact 1.10, φ is also not NIP. Therefore, we have a contradiction. □

Recall that $\mathbb{F}_M^\varphi = \{F_a^\varphi : a \in M\}$ as defined in Definition 3.5.

Corollary 3.20. *Assume that $\varphi(x; y)$ is an NIP formula. Then the map* Eval : $\mathrm{conv}(\mathbb{F}_M^\varphi) \times S_y(M) \to [0, 1]$ *where* $\mathrm{Eval}(f, p) = f(p)$ *is dependent.*

Proof. Assume not. Then Eval is (r, ϵ)-independent for some r and ϵ. Assume that $A \subseteq \mathrm{conv}(\mathbb{F}_M^\varphi)$ is (r, ϵ)-shattered. Then, for each subset K of A, there exists p_K in $S_y(M)$ where $\{f \in \mathrm{conv}(\mathbb{F}_M^\varphi) : f(p_K) \leq r\} \cap A = K$ and $\{f \in \mathrm{conv}(\mathbb{F}_M^\varphi) : f(p_K) \geq r + \epsilon\} \cap A = A \backslash K$. Since f is in $\mathrm{conv}(\mathbb{F}_M^\varphi)$, we note that $f = \sum_{i=1}^n r_i F_{a_i}^\varphi$ for some a_i in M^x and $r_i > 0$. Let $\mu_f = \sum_{i=1}^n r_i \delta_{a_i}$. For each $K \subset A$, we let $b_K \in \mathcal{U}^y$ be a realization of p_K. Let $A_\star = \{\mu_f : f \in A\} \subset \mathrm{conv}(\mathcal{U}^x)$. Then, for any $\mu_f \in A_\star$, we have,

$$\Psi_f(\mu_f, \delta_{b_K}) = \text{Eval}(f, \text{tp}(b_K/M)) = \text{Eval}(f, p_K).$$

Therefore if Eval is independent then so is the map Ψ_f. Since $\varphi(x;y)$ is NIP, this contradicts Lemma 3.19. ☐

Theorem 3.21. *Assume that $\varphi(x;y)$ is an NIP formula. Then $\text{conv}(\mathbb{F}_M^{\varphi}) \subseteq \mathbb{R}^{S_y(M)}$ is sequentially dependent.*

Proof. This follows directly from Corollary 3.20 and Fact 3.17. ☐

3.3 Local Keisler measures

We now fix T a countable[1] first order theory, $\mathcal{U}$ a sufficiently saturated model of T, M a small elementary submodel of $\mathcal{U}$, and $\varphi(x;y)$ a partitioned $\mathcal{L}$-formula. We do **not** require $\varphi(x;y)$ to be NIP unless explicitly stated otherwise. In the first subsection, we define the weak notion of *φ-dfs* (Definition 3.25). Assuming that $\varphi(x;y)$ is NIP, we show that if μ is a finitely additive probability measure on $\mathcal{L}_\varphi(\mathcal{U})$ and μ is φ-dfs stable over some M where M is countable, then for every $\epsilon > 0$, there exists a sequence of points $a_1, ..., a_n$ of elements in M^x such that this sequence is an ϵ-approximation for μ over $\mathcal{F}_\varphi$ (Theorem 3.30). In the second subsection, we then use Theorem 3.30 to prove our main result described in the introduction. Some of the proofs in this section are almost identical to the general Keisler measure case and so we reference Chapter 2 liberally.

3.3.1 Basic definitions and properties

Let us begin with the definition of a φ-measure and then describe two possible notions for what it means for a φ-measure to be *invariant*.

[1]For uncountable theories, we simply take the reduct to a countable language containing the formula we are working with.

Definition 3.22. For any partitioned $\mathcal{L}$-formula $\varphi(x;y)$, a φ-measure is a finitely additive probability measure from $\mathcal{L}_\varphi(\mathcal{U})$. We denote the collection of all φ-measures as $\mathfrak{M}_\varphi(\mathcal{U})$.

Definition 3.23. Let $\mu \in \mathfrak{M}_\varphi(\mathcal{U})$.

1. μ is **M-invariant** if for every φ-formula $\psi(x;\bar{b})$ and for any $\bar{c} \in \mathcal{U}^{\bar{y}}$ such that $\text{tp}(\bar{b}/M) = \text{tp}(\bar{c}/M)$, we have $\mu(\psi(x;\bar{b})) = \mu(\psi(x;\bar{c}))$.
2. μ is **φ-invariant over** M if for every $b, c \in \mathcal{U}^y$ such that $\text{tp}(b/M) = \text{tp}(c/M)$, we have that $\mu(\varphi(x;b)) = \mu(\varphi(x;c))$.

The collection of measures which are φ-invariant over M contains the collection of M-invariant measures. Notice that φ-invariance only mentions instances of φ and does not mention all φ-formulas (i.e. Boolean combinations of instances of φ). We refer the reader to section 3.4 of this chapter for a counterexample. Our next definition connects φ-invariant measures over M to functions from $S_y(M)$ to $\mathbb{R}$.

Definition 3.24. Assume that $\mu \in \mathfrak{M}_\varphi(\mathcal{U})$ and μ is φ-invariant over M. Then, we define the function $F_\mu^\varphi : S_y(M) \to [0,1]$ via $F_\mu^\varphi(p) = \mu(\varphi(x;b))$ where $b \models p$.

As in the global case, Definition 3.24 allows us to transfer problems involving finitely additive measures to questions involving functions. We will soon see that special kinds of measures correspond to special kinds of functions. Let us now describe these special measures.

Definition 3.25. Let $\mu \in \mathfrak{M}_\varphi(\mathcal{U})$.

1. We say that μ is **φ-definable over** M if for every $\epsilon > 0$ there exist $\mathcal{L}(M)$-formulas $\psi_1(y), ..., \psi_n(y)$, such that:
   (a) The collection $\{\psi_i(y) : i \leq m\}$ forms a partition of $\mathcal{U}^y$.
   (b) If $\mathcal{U} \models \psi_i(e) \wedge \psi_i(c)$, then $|\mu(\varphi(x;e)) - \mu(\varphi(x;c))| < \epsilon$.
2. We say that μ is **finitely satisfiable in** M if for every φ-formula $\psi(x)$ such that $\mu(\psi(x)) > 0$, there exists some a in M^x such that $\mathcal{U} \models \psi(a)$.

3. We say that μ is **φ-dfs over** M if μ is both φ-definable and finitely satisfiable in M.

We now connect the kinds of measures defined above with functions from $S_y(M)$ to $\mathbb{R}$.

Proposition 3.26. *Let $\mu \in \mathfrak{M}_{\varphi}(\mathcal{U})$. Then μ is φ-definable over M if and only if μ is φ-invariant and the map $F_{\mu}^{\varphi} : S_y(M) \to [0,1]$ is continuous.*

Proof. This follows directly from Lemma 2.16 and the assumption that M is small. □

Proposition 3.27. *If $\mu \in \mathfrak{M}_{\varphi}(\mathcal{U})$ and μ is finitely satisfiable in M, then μ is M-invariant, and in particular, μ is φ-invariant.*

Proof. The proof is similar to the proof of Proposition 2.14. □

Recall that if $A \subseteq \mathbb{R}^X$, then $\mathrm{cl}_p(A)$ is the topological closure of A in $\mathbb{R}^X$. We now connect this closure property with finite satisfiability.

Proposition 3.28. *Let $\mu \in \mathfrak{M}_{\varphi}(\mathcal{U})$ and assume that μ is finitely satisfiable in M. Let $X = S_y(M)$. Then $F_{\mu}^{\varphi} \in \mathrm{cl}_p(\mathrm{conv}_{\mathbb{Q}}(\mathbb{F}_M^{\varphi})) \subset \mathbb{R}^X$.*

Proof. By Proposition 3.27, the map F_{μ}^{φ} is well defined. Let U be some open subset of $\mathbb{R}^X$ containing F_{μ}^{φ}. Without loss of generality, there exists $q_1, ..., q_n \in S_y(M)$ and real numbers $r_1, ..., r_n, s_1, ..., s_n$ such that

$$U = \bigcap_{i=1}^{n} \{f \in \mathbb{R}^X : r_i < f(q_i) < s_i\}.$$

Choose some $b_i \models q_i$ for each q_i. Since U contains F_{μ}^{φ}, we have that $r_i < \mu(\varphi(x; b_i)) < s_i$. We notice that $\varphi(x; b_1), ..., \varphi(x; b_n)$ generate a finite Boolean algebra. Let $\theta_1(x), ..., \theta_m(x)$ be the atoms of this Boolean algebra. If $\mu(\theta_i(x)) > 0$ then by finite satisfiability we can find a_i in M^x such that $\mathcal{U} \models \theta_i(a_i)$. Define

$$f(y) = \sum_{\{i:\mu(\theta_i(x))>0\}} \mu(\theta_i(x)) F^{\varphi}_{a_i}(y).$$

Then, $f \in \mathrm{conv}(\mathbb{F}^{\varphi}_M) \cap U$. Since $\mathrm{conv}_{\mathbb{Q}}(\mathbb{F}^{\varphi}_M)$ is a dense subset of $\mathrm{conv}(\mathbb{F}^{\varphi}_M)$ with the induced topology from $\mathbb{R}^X$, there exists some $g \in \mathrm{conv}_{\mathbb{Q}}(\mathbb{F}^{\varphi}_M) \cap U$. $\square$

Lemma 3.29. *Suppose that $\varphi(x;y)$ is NIP. Let $\mu \in \mathfrak{M}_{\varphi}(\mathcal{U})$. Assume that μ is finitely satisfiable in M and $|M| = \aleph_0$. Then there exists some sequence $(f_i)_{i\in\omega}$ of elements in $\mathrm{conv}_{\mathbb{Q}}(\mathbb{F}^{\varphi}_M)$ such that $f_i \to F^{\varphi}_{\mu}$.*

Proof. By Proposition 3.28, F^{φ}_{μ} is in $\mathrm{cl}_p(\mathrm{conv}_{\mathbb{Q}}(F^{\varphi}_M))$. Since $\varphi(x;y)$ is NIP, Theorem 3.21 holds. Since $|\mathrm{conv}_{\mathbb{Q}}(\mathbb{F}^{\varphi}_M)| = \aleph_0$, the first statement of Theorem 3.4 is satisfied. Therefore the second statement of Theorem 3.4 holds, so there exists a sequence of elements $f_i \in \mathrm{conv}(\mathbb{F}^{\varphi}_M)$ such that $f_i \to F^{\varphi}_{\mu}$. $\square$

We now prove the main theorem of this subsection.

Theorem 3.30. *Assume that $\varphi(x;y)$ is NIP. Let $\mu \in \mathfrak{M}_{\varphi}(\mathcal{U})$. Assume that μ is φ-dfs over M and $|M| = \aleph_0$. Then for every $\epsilon > 0$, there exists a sequence $a_1, ..., a_n$ of elements in M^x such that for any $b \in \mathcal{U}^y$*

$$|\mu(\varphi(x;b)) - \mathrm{Av}(\overline{a})(\varphi(x;b))| < \epsilon.$$

Proof. Fix ϵ. By Lemma 3.29, there exists a sequence $(f_i)_{i\in\omega}$ of elements in $\mathrm{conv}_{\mathbb{Q}}(\mathbb{F}^{\varphi}_M)$ such that $f_i \to F^{\varphi}_{\mu}$. By Fact 3.26, we know that F^{φ}_{μ} is continuous. Since each f_i is also continuous, we may apply Theorem 3.2. Therefore, $f_i \to_w F^{\varphi}_{\mu}$. By Mazur's lemma, there exists a sequence $g_i \in \mathrm{conv}_{\mathbb{Q}}(\mathbb{F}^{\varphi}_M)$ such that $g_i \to_u F^{\varphi}_{\mu}$. Choose g_n such that

$$\sup_{q\in S_y(M)} |F^{\varphi}_{\mu}(q) - g_n(q)| < \epsilon.$$

Notice that $g_n = \sum_{i=1}^{n} r_i F_{c_i}^{\varphi}$ and let $a_1, ..., a_n$ be a representative sequence for g_n. Then for each $b \in \mathcal{U}^y$,

$$|\mu(\varphi(x;b)) - \mathrm{Av}(\overline{a})(\varphi(x;b))| = |F_{\mu}^{\varphi}(\mathrm{tp}(b/M)) - g_n(\mathrm{tp}(b/M))|.$$

Since M is small, we may conclude that

$$\sup_{b \in \mathcal{U}} |\mu(\varphi(x;b)) - \mathrm{Av}(\bar{a})(\varphi(x;b))| = \sup_{q \in S_y(M)} |F_{\mu}^{\varphi}(q) - g_n(q)| < \epsilon.$$

$\square$

Question 3.31. Suppose that $\varphi(x;y)$ is NIP, $\mu \in \mathfrak{M}_{\varphi}(\mathcal{U})$, and μ is φ-dfs over a model N (where N is not necessarily countable). Does the conclusion of Theorem 3.30 hold?

3.3.2 Main result

The purpose of this section is to prove our main theorem as described in the introduction. We again do **not** require $\varphi(x;y)$ to be NIP in this section unless explicitly stated. We will see that if we strengthen our definition of *φ-definability* and in turn our definition of *φ-dfs*, we can prove our main result. We begin this section by considering different families of φ-definable sets.

Definition 3.32. For a fixed partitioned $\mathcal{L}$-formula $\varphi(x;y)$, we define Δ_{φ} as the Boolean algebra of partitioned formulas generated by $\{\varphi(x;y_i) : i \in \omega, |y_i| = |y|\}$. If $\theta(x;\overline{y})$ is an element of Δ_{φ}, we let $\mathcal{L}_{\theta}(\mathcal{U})$ be the Boolean subalgebra of $\mathcal{L}_x(\mathcal{U})$ generated by $\{\theta(x;\overline{b}) : \overline{b} \in \mathcal{U}^{\overline{y}}\}$. Moreover, we let I_{θ} denote the obvious restriction map from $\mathfrak{M}_{\varphi}(\mathcal{U})$ to $\mathfrak{M}_{\theta}(\mathcal{U})$. For notational purposes, if $\mu \in \mathfrak{M}_{\varphi}(\mathcal{U})$ and $\theta(x;\overline{y})$ is in Δ_{φ}, then we write $I_{\theta}(\mu)$ simply as μ_{θ}.

For example, the formulas $\varphi(x;y_1) \wedge \varphi(x;y_2)$, $\varphi(x;y_1) \triangle \varphi(x;y_2)$, and $\bigvee_{i=1}^{14} \varphi(x;y_i)$

are all elements of Δ_φ. We now give the appropriate definitions for dfs and finite approximability in the local context as well as some relations between the properties we have already defined.

Definition 3.33. Let $\mu \in \mathfrak{M}_\varphi(\mathcal{U})$. Then μ is **definable over** M if for every $\theta(x;\overline{y})$ in Δ_φ and $\epsilon > 0$ there exist $\mathcal{L}(M)$-formulas $\rho_1(\overline{y})$,...,$\rho_m(\overline{y})$, such that

1. The collection $\{\rho_i(\overline{y}) : i \leq m\}$ forms a partition of $\mathcal{U}^{\overline{y}}$.
2. if $\mathcal{U} \models \rho_i(\overline{e}) \wedge \rho_i(\overline{c})$, then $|\mu(\theta(x;\overline{e})) - \mu(\theta(x;\overline{c}))| < \epsilon$.

We say that μ is **dfs over** M if μ is definable and finitely satisfiable in M.

Definition 3.34. Let $\mu \in \mathfrak{M}_\varphi(\mathcal{U})$. Then, μ is **finitely approximated in** M if for every $\theta(x;\overline{y})$ in Δ_φ and for every $\epsilon > 0$, there exists a sequence $a_1, ..., a_n$ in M^x such that for any $b \in \mathcal{U}^{\overline{y}}$

$$|\mu(\theta(x;\overline{b})) - \mathrm{Av}(\overline{a})(\theta(x;\overline{b}))| < \epsilon.$$

Proposition 3.35 (Basic Properties)**.** *Let $\mu \in \mathfrak{M}_\varphi(\mathcal{U})$.*

1. *μ is M-invariant if and only if for every $\theta(x;\overline{y})$ in Δ_φ, the measure $\mu_\theta \in \mathfrak{M}_\theta(\mathcal{U})$ is θ-invariant over M.*
2. *μ is definable over M if and only if for every $\theta(x;\overline{y})$ in Δ_φ, the measure $\mu_\theta \in \mathfrak{M}_\theta(\mathcal{U})$ is θ-definable over M.*
3. *μ is definable over M if and only if μ is M-invariant and for every $\theta(x;\overline{y})$ in Δ_φ the map $F^{\theta}_{\mu_\theta} : S_{\overline{y}}(M) \to [0,1]$ is continuous.*
4. *μ is* dfs *over M if and only if for each $\theta(x;\overline{y}) \in \Delta_\varphi$, the measure $\mu_\theta \in \mathfrak{M}_\theta(\mathcal{U})$ (as a θ-measure) is* dfs *over M.*
5. *If μ is finitely approximated over M, then μ is* dfs *over M.*

Proof. 2 and 4 follow directly from the definitions. 1 follows from the fact that M is small. 3 follows from Proposition 3.26 and 2. The proof of 5 is identical to the general Keisler measure case shown in 1 of Proposition 2.30. □

We now present some properties for local measures which were proven for global measures case in Chapter 2. These properties will allow us to reduce our main result to the countable case and so we may apply Theorem 3.30.

Proposition 3.36. *Let $\mu \in \mathfrak{M}_\varphi(\mathcal{U})$. Assume that μ is finitely satisfiable over N and μ is M-invariant. Then, μ is finitely satisfiable in M.*

Proof. The proof is similar to the proof of Proposition 2.18. □

Proposition 3.37. *Let $\mu \in \mathfrak{M}_\varphi(\mathcal{U})$. If μ is* dfs *over M, then there exists $M_0 \prec M$ such that $|M_0| = \aleph_0$ and μ is* dfs *over M_0.*

Proof. The proof is similar to the proof of Proposition 2.35. □

Now we may essentially reduce our main result to Theorem 3.30.

Lemma 3.38. *Assume that $\varphi(x; y)$ is NIP. Let $\mu \in \mathfrak{M}_\varphi(\mathcal{U})$. If μ is* dfs *over M, then for every $\epsilon > 0$ there exists a sequence $a_1, ..., a_n$ in M^x such that for any $b \in \mathcal{U}^y$,*

$$|\mu(\varphi(x; b)) - \mathrm{Av}(\overline{a})(\varphi(x; b))| < \epsilon.$$

Proof. By Theorem 3.37, μ is dfs over some M_0 where $M_0 \prec M$ and $|M_0| = \aleph_0$. Then, we may apply Theorem 3.30 to μ and M_0. Since $M_0 \subseteq M$, we are done. □

Theorem 3.39. *Assume that $\varphi(x; y)$ is NIP. Let $\mu \in \mathfrak{M}_\varphi(\mathcal{U})$. Then μ is* dfs *over M if and only if μ is finitely approximated over M.*

Proof. By 5 of Proposition 3.35, if μ is finitely approximated over M, then μ is dfs over M. We only need to show the other direction. Assume μ is dfs over M. By 4 of Proposition 3.35, for any $\theta(x; \overline{y})$ in Δ_φ, μ_θ is dfs over M. By construction, $\mu(\theta(x; \overline{b})) = \mu_\theta(\theta(x; \overline{b}))$ for every $\overline{b} \in \mathcal{U}^{\overline{y}}$. By Lemma 3.38, for every $\epsilon > 0$ there are $a_1, ..., a_n$ in M^x so that for every $\overline{b} \in \mathcal{U}^{\overline{y}}$,

$$|\mu_\theta(\theta(x; \overline{b})) - \mathrm{Av}(\overline{a})(\theta(x; \overline{b}))| < \epsilon.$$

Since $\mu(\theta(x;b)) = \mu_\theta(\theta(x;b))$, we conclude that μ is finitely approximated over M. $\square$

3.4 Example

In section 3.3, we claimed that φ-invariant over M is not equivalent to M-invariant. In this section, we provide a concrete example in the Random Graph. Actually, our counterexample shows something stronger, namely we construct a measure which is φ-definable (over $\emptyset$), but not M-invariant. We begin by recalling a general fact about finitely additive measures (see [54, Theorem 3.6.1] for details).

Fact 3.40. *Fix a set X and let $\mathbb{A}$ and $\mathbb{D}$ be two Boolean algebras on X. Let μ_1, μ_2 be finitely additive measures on $\mathbb{A}, \mathbb{D}$ respectively. Let $\mathbb{B}$ be a Boolean algebra on X containing both $\mathbb{A}, \mathbb{D}$. Then, there exists a finitely additive measure μ on $\mathbb{B}$ which extends μ_1 and μ_2 if and only if*

$$\mu_1(A) \geq \mu_2(D) \text{ for any } A \in \mathbb{A},\ D \in \mathbb{D}, \text{ such that } A \supseteq D,$$

and

$$\mu_1(E) \leq \mu_2(F) \text{ for any } E \in \mathbb{A},\ F \in \mathbb{D}, \text{ such that } E \subseteq F.$$

Corollary 3.41. *Assume that $\mathbb{B}$ is a Boolean algebra on a set X. Let $\mathbb{A}_1, \mathbb{A}_2$ be subalgebras of $\mathbb{B}$ such that $\mathbb{A}_1 \cap \mathbb{A}_2 = \{\emptyset, X\}$. Let μ_1, μ_2 be finitely additive measures on $\mathbb{A}_1$, $\mathbb{A}_2$ respectively. Assume that for each $B \in \mathbb{A}_1$ and $C \in \mathbb{A}_2$, such that $B, C \neq \emptyset$, we have that $B \cap C \neq \emptyset$. Then, there exists a finitely additive measure μ on $\mathbb{B}$ such that $\mu|_{\mathbb{A}_1} = \mu_1$ and $\mu|_{\mathbb{A}_2} = \mu_2$.*

Proof. We show that the condition for Theorem 3.40 holds. Let $A \in \mathbb{A}_1, B \in \mathbb{A}_2$. Then, $B \not\subset A$ since $B \cap A^c \neq \emptyset$. Likewise for the other direction. Therefore, there is no obstruction to amalgamation. $\square$

Theorem 3.42. *Let $(\mathcal{U}; R)$ be a monster model of the Random Graph with edge* *relation $R(x,y)$ and M a small elementary submodel. Then, there exists a measure $\mu \in \mathfrak{M}_R(\mathcal{U})$ such that μ is R-definable over M, but μ is not M-invariant.*

Proof. Let $a, b \in \mathcal{U}$ such that $\text{tp}_{R^*}(a/M) = tp_{R^*}(b/M)$ and $\mathcal{U} \models R(a,b)$. Now, let $\mathbb{A}_1$ be the Boolean algebra generated by $\{R(x,c) : c \in \mathcal{U}, c \neq a, b\}$ and let $\mathbb{A}_2$ be the Boolean algebra generated by $\{R(x,a), R(x,b)\}$. Let μ be the "Lebesgue measure" on $\mathcal{L}_x(\mathcal{U})$ restricted to $\mathcal{L}_R(\mathcal{U})$. In particular, if $c_1, ..., c_n$ is a sequence of points in $\mathcal{U}$ with $c_i \neq c_j$ for $i \neq j$, then for any $K \in \mathcal{P}(n)$,

$$\mu\Big(\bigwedge_{c_i \in K}^{k} R(x,c_i) \wedge \bigwedge_{c_j \notin K} \neg R(x,c_j)\Big) = \frac{1}{2^n}.$$

We let μ_1 on $\mathbb{A}_1$ be $\mu|_{\mathbb{A}_1}$. For any choice of $r > 4$, we define μ_2 as follows:

1. $\mu_2(x = x) = 1$ and $\mu_2(\emptyset) = 0$.
2. $\mu_2(R(x,a)) = \frac{1}{2}$.
3. $\mu_2(R(x,b)) = \frac{1}{2}$.
4. $\mu_2(R(x,a) \wedge R(x,b)) = \frac{1}{r}$.
5. $\mu_2(\neg R(x,a) \wedge \neg R(x,b)) = \frac{1}{r}$.
6. $\mu_2(R(x,a) \wedge \neg R(x,b)) = \frac{1}{2} - \frac{1}{r}$.
7. $\mu_2(\neg R(x,a) \wedge R(x,b)) = \frac{1}{2} - \frac{1}{r}$.

By Corollary 3.41, there exists ν on $\mathcal{L}_\varphi(\mathcal{U})$ such that ν extends both μ_1 and μ_2. Notice that for every element $d \in \mathcal{U}$, we have that $\nu(R(x,d)) = \frac{1}{2}$. Therefore, ν is $R-$invariant (even $R-$definable). However, ν is not $M-$invariant. Consider $c_1, c_2 \in \mathcal{U}$ such that $c_i \neq a, b$, $\text{tp}_{R^*}(c_i/M) = \text{tp}_{R^*}(a/M)$, and $\mathcal{U} \models R(c_1, c_2)$. Let $\theta(x, y_1, y_2) = R(x,y_1) \wedge R(x,y_2)$. Now, $\text{tp}_{\theta^*}(c_1c_2/M) = tp_{\theta^*}(ab/M)$ and we have that:

$$\nu(R(x,a) \wedge R(x,b)) = \frac{1}{r} \neq \frac{1}{4} = \nu(R(x,c_1) \wedge R(x,c_2)).$$

□

CHAPTER 4

REMARKS ON GENERIC STABILITY

This chapter is joint work with Gabriel Conant and is a modified version of our paper *Remarks on generic stability in independent theories* [14]. An extremely useful characterization of stability for a complete theory is that any global type is definable and finitely satisfiable in some small model. On the other hand, the class of stable theories is highly restrictive, and a great deal of current research in model theory has focused on finding stable-like phenomena in unstable environments. In NIP theories, although not every type is necessarily definable and finitely satisfiable, the class of types with these properties is still quite resilient, and such types are now referred to as *generically stable*.

Generically stable types in NIP theories were first identified by Shelah [58], and then thoroughly studied by Hrushovski and Pillay [32] and Usvyatsov [64]. This investigation was extended to Keisler measures in NIP theories in [31] and [32], culminating in the work of Hrushovski, Pillay, and Simon [33] where generically stable Keisler measures were defined. As mentioned previously, it is shown in [33] that a global Keisler measure μ is dfs if and only if μ is finitely approximated if and only if μ is a frequency interpretation measure (Theorem 2.39) and we call this class of measures *generically stable* (in the NIP context).

A standard hypothesis in the NIP setting is that definability and finite satisfiability (in a small model) are opposite extremes on the spectrum of invariant types and measures, and so the synthesis of both properties forms a stable refuge in an unstable world. So it is not unreasonable to explore a similar motif beyond NIP theories, and especially in other tame regions like simplicity or NTP_2.

In this chapter, we study the above forms of "generic stability" in the wilderness outside of NIP. Generically stable types in arbitrary theories were defined by Pillay and Tanović in [50] and, in Section 4.1, we reconcile this definition with the setting of measures. Specifically, we show that a global type is generically stable if and only if it is a frequency interpretation measure (Proposition 4.2), which establishes a concrete connection between generic stability for measures in NIP theories and for types in arbitrary theories.

In section 4.2, we analyze theories in which every dfs Keisler measure is trivial (we call such theories *dfs-trivial*). We show that dfs-triviality reduces to measures in one variable (Proposition 4.9), and that dfs-nontriviality is preserved in reducts (Theorem 4.12). Finally, we give examples of dfs-trivial theories, including the theory of the Random Graph, the theory T^r_s of the generic K^r_s-free r-hypergraph for $s > r \geq 3$, and the theory T^*_{feq} of a generic parameterized equivalence relation (Corollary 4.14).

We then turn to the classes of dfs measures, finitely approximated measures, and FIM measures. As these three classes coincide in NIP theories, we focus on separating them in general theories. For instance, the question of whether finitely approximated measures coincide with frequency interpretation measures in arbitrary theories was asked by Chernikov and Starchenko in [10, Remark 3.6], and the examples below give a negative answer.

In Section 4.3, we first recall an example, due to Adler, Casanovas, and Pillay [1], of a theory with a generically stable global type p such that $p \otimes p$ is not generically stable. This theory is a variant of T^*_{feq} in which equivalence classes have size two. We note that this gives a non-simple theory with a finitely approximated 2-type that is not generically stable (and thus not frequency interpretable). We then exhibit similar behavior with a 1-type in the theory of the generic K_s-free graph for $s \geq 3$. Specifically, we consider the global type of a disconnected vertex, which is clearly not generically stable, and use lower bounds on the Ramsey numbers of Erdős and Rogers [24] to show this type is finitely approximated.

At this point, it still remains open whether there is a theory with a definable and finitely satisfiable *global* Keisler measure that is not finitely approximated. However, if we shift our focus to the *local level* of φ-types and φ-measures, then interesting examples emerge. This viewpoint is also motivated by the main theorem of Chapter 3 where we show that if $\varphi(x;y)$ is an NIP *formula*, then any definable and finitely satisfiable Keisler measure on φ-definable sets is finitely approximated. In Section 4.3.3, we show that this fails for φ-types in simple theories. In particular, we consider the theory T^r_s for some $s > r \geq 3$, and define the φ-type $p_R = \{\varphi(\bar{x};b) : b \in \mathcal{U}\}$ where $\varphi(x_1, \ldots, x_{r-1}; y)$ is $\neg R(\bar{x}, y) \wedge \bigwedge_{i \neq j} x_i \neq x_j$. Using the Ramsey property for finite K^{r-1}_s-free $(r-1)$-hypergraphs (due to Nešetřil and Rödl [44]), we show that p_R is finitely satisfiable in any small model. We then show that p_R is not finitely approximated by adapting an averaging argument of Erdős and Kleitman [23] on maximal cuts in $(r-1)$-hypergraphs to the setting of weighted hypergraphs.

A recurring theme in our results is that generic stability in the wild is very uncommon, and more fragile than in NIP theories. Regarding the interaction between dfs measures and finitely approximated measures, our examples suggest a much weaker connection outside of NIP, at least at the local level. On the other hand, all of our examples of measures that are finitely approximated, but not frequency interpretable, live in theories with TP_2. So perhaps there is hope for an NIP-like connection for these notions in NTP_2 or simple theories.

4.1 Generically stable types

In NIP theories, a Keisler measure $\mu \in \mathfrak{M}_x(\mathcal{U})$ is called *generically stable* if it satisfies the equivalent properties in Theorem 2.39. Generically stable *types* in NIP theories were initially studied by Shelah [58], and then in more depth by Hrushovski and Pillay [32] and Usyvatsov [64]. In [50], Pillay and Tanović give a definition of generic stability for types in arbitrary theories (Definition 1.7). This notion is further studied by Adler, Casanovas, and Pillay in [1]. An equivalent formulation of generic stability for types in arbitrary theories is given by García, Onshuus, and Usvyatsov in [26].

For ease of presentation, we recall an (obvious) equivalent definition of generic stability here. Given an infinite ordinal α and a sequence $(a_i)_{i<\alpha}$ in $\mathcal{U}^x$, we let $\mathrm{Av}(a_i)_{i<\alpha}$ denote the *average type* of $(a_i)_{i<\alpha}$ over $\mathcal{U}$, i.e., the partial type of $\mathcal{L}(\mathcal{U})$-formulas $\varphi(x)$ such that $\{i<\alpha : \mathcal{U} \models \neg\varphi(a_i)\}$ is finite.

Definition 4.1 ([50])**.** A type $p \in S_x(\mathcal{U})$ is **generically stable** if there is $M \prec \mathcal{U}$ such that p is M-invariant and $\mathrm{Av}(a_i)_{i<\alpha}$ is a complete type for any Morley sequence $(a_i)_{i<\alpha}$ in p over M and any infinite ordinal α. In this case, we also say p is **generically stable over** M.

We make two remarks. First, the use of ordinals other than ω is necessary in Definition 4.1. For example, if T is NIP then any invariant global type satisfies the conclusion of the definition when $\alpha=\omega$, but if T unstable then there is some invariant global type that is not definable (or finitely satisfiable in any small model), and hence not generically stable (see, e.g., [48, Theorem 2.15]). Second, since Definition 4.1 involves Morley sequences, it does not immediately transfer to measures[1]. The next result clarifies both of these remarks.

Proposition 4.2. *Given $p \in S_x(\mathcal{U})$ and $M \prec \mathcal{U}$, the following are equivalent.*

1. *p is generically stable over M.*
2. *p is M-invariant and $p = \mathrm{Av}(a_i)_{i<\omega}$ for any Morley sequence $(a_i)_{i<\omega}$ in p over M.*
3. *δ_p is a frequency interpretation measure over M.*

[1]Randomizations might be a possible future avenue to explore.

Proof. $2 \Rightarrow 1$. Assume 2. To show 1, it suffices to consider Morley sequences indexed by $\omega+\omega$ (we leave this as an exercise for the reader). So fix an $\mathcal{L}(\mathcal{U})$-formula $\varphi(x)$ and a Morley sequence $(a_i)_{i<\omega+\omega}$ in p over M. If $\varphi(x) \in p$ then, by 2, $\{i < \omega : \mathcal{U} \models \neg\varphi(a_i)\}$ and $\{\omega \leq i < \omega+\omega : \mathcal{U} \models \neg\varphi(a_i)\}$ are finite, and so $\{i < \omega+\omega : \mathcal{U} \models \neg\varphi(a_i)\}$ is finite. If $\varphi(x) \notin p$ then $\neg\varphi(x) \in p$ and so, by the same reasoning, $\{i < \omega + \omega : \mathcal{U} \models \varphi(a_i)\}$ is finite.

$1 \Rightarrow 3$. Assume 1, and fix an $\mathcal{L}$-formula $\varphi(x; y)$. We construct a sequence $(\theta_n)_{n\geq 1}$ as in Definition 2.28. By Definition 4.1 and compactness, there is some n_φ such that for any Morley sequence $(a_i)_{i<\omega}$ in p over M, and any $b \in \mathcal{U}^y$, either $\varphi(x; b) \in p$ and $|\{i < \omega : \mathcal{U} \models \neg\varphi(a_i; b)\}| \leq n_\varphi$, or $\neg\varphi(x; b) \in p$ and $|\{i < \omega : \mathcal{U} \models \varphi(a_i; b)\}| \leq n_\varphi$ (see [32, Proposition 3.2] or [50, Proposition 1] for details). Note that this implies that p is definable over $(a_i)_{i<\omega}$, and thus definable over M by M-invariance (see Proposition 2.18 or [59, Lemma 2.18]). So we may choose an $\mathcal{L}(M)$-formula $\psi(y)$ such that, for all $b \in \mathcal{U}^y$, $\varphi(x; b) \in p$ if and only if $\mathcal{U} \models \psi(b)$.

Given $i \geq 1$, let $n_i = n_\varphi i$. We will define a sequence $(\theta_{n_i}(x_1, \ldots, x_{n_i}))_{i=1}^\infty$ of $\mathcal{L}(M)$-formulas such that, for all $i \geq 1$, $\theta_{n_i}(x_1, \ldots, x_{n_i}) \in p^{(n_i)}$ and

$$\text{if } \bar{a} \models \theta_{n_i}(x_1, \ldots, x_{n_i}) \text{ and } b \in \mathcal{U}^y \text{ then } |\delta_p(\varphi(x; b)) - \mathrm{Av}_{\bar{a}}(\varphi(x; b))| \leq \tfrac{1}{i} \qquad (\dagger)$$

(recall that δ_p is the Keisler measure where we view p as a $\{0,1\}$-valued measure in $\mathfrak{M}_x(\mathcal{U})$). First, we note that this suffices to prove 3. Indeed, given $(\theta_{n_i})_{i=1}^\infty$ as above and $n \geq n_\varphi$, let $\theta_n(x_1, \ldots, x_n)$ be $\theta_{n_i}(x_1, \ldots, x_{n_i}) \wedge \bigwedge_{j \leq n} x_j = x_j$ where $n_i \leq n < n_{i+1}$. Note that $\theta_n(x_1, \ldots, x_n) \in p^{(n)}$ for all n. Also, if $n \geq n_\varphi$, $\bar{a} \models \theta_n(x_1, \ldots, x_n)$, and $b \in \mathcal{U}$ then, using the triangle inequality and $(\dagger)$, one can show $|\delta_p(\varphi(x; b)) - \mathrm{Av}_{\bar{a}}(\varphi(x; b))| < \frac{3}{i}$ where i is such that $n_i \leq n < n_{i+1}$. So it suffices to construct $(\theta_{n_i})_{i=1}^\infty$ as above.

Fix $i \geq 1$, and define the $\mathcal{L}(M)$-formula

$$\varphi(x_1, \ldots, x_{n_i}; y) := \bigvee_{\substack{I \subseteq [n_i] \\ |I| > n_\varphi}} \left(\bigwedge_{j \in I} \left(\varphi(x_j; y) \wedge \neg\psi(y) \right) \vee \bigwedge_{j \in I} \left(\neg\varphi(x_j; y) \wedge \psi(y) \right) \right).$$

Then $p^{(n_i)}(x_1, \ldots, x_{n_i}) \wedge \varphi(x_1, \ldots, x_{n_i}; y)$ is inconsistent. Now, if we decide to set $\theta_{n_i}(x_1, \ldots, x_{n_i}) := \forall y \neg\varphi(x_1, \ldots, x_{n_i}; y)$, we have $\theta_{n_i}(x_1, \ldots, x_{n_i}) \in p^{(n_i)}$. It is straightforward to verify that $\theta_{n_i}(x_1, \ldots, x_{n_i})$ satisfies (†).

$3 \Rightarrow 2$. Assume 3. By definition, p is M-invariant. Fix a Morley sequence $(a_i)_{i<\omega}$ in p over M, and some $\varphi(x; b) \in p$. Let $I = \{i < \omega : \mathcal{U} \models \varphi(a_i)\}$. By 3, we may choose n sufficiently large and an $\mathcal{L}(M)$-formula $\theta(x_1, \ldots, x_n) \in p^{(n)}$ such that, for any $\bar{a}' \models \theta(\bar{x})$, $|\delta_p(\varphi(x)) - \mathrm{Av}_{\bar{a}'}(\varphi(x; b))| < 1$. Note, in particular, that $\theta(a_{i_1}, \ldots, a_{i_n})$ holds for any $i_1 < \ldots < i_n < \omega$. We now have $|\omega \backslash I| < n$ since, if not, then there are $i_1 < \ldots < i_n < \omega$ such that $\neg\varphi(a_{i_j}; b)$ holds for all $1 \leq j \leq n$, and so $\delta_p(\varphi(x; b)) - \mathrm{Av}((a_{i_1}, \ldots, a_{i_n})(\varphi(x; b)) = 1$, contradicting the choice of n and θ. □

The previous proposition can be taken as evidence that frequency interpretation measures provide a compatible generalization of the standard notion of generic stability for types to the class of all measures.

Remark 4.3. Suppose $p \in S_x(\mathcal{U})$ is generically stable over $M \prec \mathcal{U}$, and let $\varphi(x; y)$ be an $\mathcal{L}$-formula. Then we have $\mathcal{L}(M)$-formulas $(\theta_n)_{n=1}^{\infty}$ witnessing that p is a frequency interpretation measure over M (as in Definition 2.28). By Proposition 2.20, and the proof of Proposition 4.2, we see that θ_n is of the form $\forall y \neg\varphi(x_1, \ldots, x_n; y)$, where $\varphi(x_1, \ldots, x_n; y)$ is a Boolean combination of $\varphi(x_i, y)$ and a φ^*-formula $\psi(y)$ over M. In particular, if $\mathcal{L}_0 \subseteq \mathcal{L}$ contains $\varphi(x; y)$, then $p|_{\mathcal{L}_0}$ is still generically stable over M with respect to $T|_{\mathcal{L}_0}$.

Call a global M-invariant type $p \in S_x(\mathcal{U})$ *stable* over $M \prec \mathcal{U}$ if $p|_M$ is a *stable type*, i.e., there does not exist a formula $\varphi(x; y)$, an M-indiscernible sequence $(a_i)_{i<\omega}$

of realizations of $p|_M$, and a sequence $(b_i)_{i<\omega}$ from $\mathcal{U}^y$ such that $\mathcal{U} \models \varphi(a_i; b_j)$ if and only if $i \leq j$. It is not hard to show that $p \in S_x(\mathcal{U})$ is stable over $M \prec \mathcal{U}$ if and only if $p = \mathrm{Av}(a_i)_{i<\omega}$ for any indiscernible sequence $(a_i)_{i<\omega}$ of realizations of $p|_M$ (see, e.g., [1]). In particular, if $p \in S_x(\mathcal{U})$ is stable over $M \prec \mathcal{U}$, then it is generically stable over M. Using a similar proof (which we leave as an exercise), one obtains an analogous characterization of generic stability in terms of the order property.

Proposition 4.4. *Suppose $p \in S_x(\mathcal{U})$ is M-invariant for some $M \prec \mathcal{U}$. Then p is generically stable over M if and only if there does not exist a formula $\varphi(x; y)$, a Morley sequence $(a_i)_{i<\omega}$ in p over M, and a sequence $(b_i)_{i<\omega}$ from $\mathcal{U}^y$ such that $\mathcal{U} \models \varphi(a_i; b_j)$ if and only if $i \leq j$.*

4.2 dfs-trivial theories

Definition 4.5. Fix a variable sort x.

1. Let $\mathfrak{M}^{\mathrm{tr}}_x(\mathcal{U})$, $\mathfrak{M}^{\mathrm{dfs}}_x(\mathcal{U})$, $\mathfrak{M}^{\mathrm{fam}}_x(\mathcal{U})$, and $\mathfrak{M}^{\mathrm{fim}}_x(\mathcal{U})$ denote the spaces of trivial measures, dfs measures, finitely approximated measures, and frequency interpretation measures, respectively.
2. A set $\Omega \subseteq \mathfrak{M}_x(\mathcal{U})$ is **closed under localization** if, for any $\mu \in \Omega$ and any Borel subset $X \subseteq S_x(\mathcal{U})$ with $\mu(X) > 0$, Ω contains the Keisler measure
$$\varphi(x) \mapsto \mu(\varphi(x) \cap X)/\mu(X)$$
(we call this measure the **localization of μ at X**).

Note that, in the last definition above, we have identified $\mu \in \mathfrak{M}_x(\mathcal{U})$ with the associated Borel probability measure on $S_x(\mathcal{U})$. Note also that a type $p \in S_x(\mathcal{U})$ is trivial if and only if it is realized in $\mathcal{U}$.

Remark 4.6. $\mathfrak{M}^{\mathrm{tr}}_x(\mathcal{U}) \subseteq \mathfrak{M}^{\mathrm{fim}}_x(\mathcal{U}) \subseteq \mathfrak{M}^{\mathrm{fam}}_x(\mathcal{U}) \subseteq \mathfrak{M}^{\mathrm{dfs}}_x(\mathcal{U})$, and each of these sets is closed under localization.

Proposition 4.7. *Suppose $\Omega \subseteq \mathfrak{M}_x(\mathcal{U})$ is closed under localization. Then $\Omega \subseteq \mathfrak{M}^{\mathrm{tr}}_x(\mathcal{U})$ if and only if, for any $\mu \in \Omega$, there is $b \in \mathcal{U}^x$ such that $\mu(x = b) > 0$.*

Proof. The left-to-right-direction is clear. So assume that for any $\mu \in \Omega$, there is $b \in \mathcal{U}^x$ such that $\mu(x = b) > 0$. Fix $\mu \in \Omega$ and let $S = \{b \in \mathcal{U}^x : \mu(x = b) > 0\}$.

We first argue that S is countable. Given $b \in X$, let $n(b) \in \mathbb{N}_{\geq 2}$ be such that $\frac{1}{n(b)} < \mu(x = b) \leq \frac{1}{n(b)-1}$. If S is uncountable, then there is some infinite $S_0 \subseteq S$ and $n \geq 2$ such that $n(b) = n$ for all $b \in S_0$. So if $Y \subseteq S_0$ has size n, then $\mu(Y) = \sum_{b \in Y} \mu(x = b) > 1$, which is a contradiction.

Let $\nu = \sum_{b \in S} \mu(x = b)\delta_b$. We will show $\mu = \nu$. First, suppose $X \subseteq S_x(\mathcal{U})$ is Borel and $X \cap S = \emptyset$ (here we identify $\mathcal{U}$ with the set of realized types in $S_x(\mathcal{U})$). Then we claim $\mu(X) = 0$. If not, then let $\mu_0 \in \mathfrak{M}_x(\mathcal{U})$ be the localization of μ at X. Then $\mu_0 \in \Omega$, and so there is some $b \in \mathcal{U}^x$ such that $\mu_0(x = b) > 0$, which contradicts $X \cap S = \emptyset$. Now, given a Borel set $X \subseteq S_x(\mathcal{U})$, we have $\mu(X) = \mu(X \backslash S) + \mu(X \cap S) = \mu(X \cap S) = \nu(X)$ as desired. □

For the rest of this section, we assume T is one-sorted.

Definition 4.8. A complete theory T is ***dfs*-trivial** if every dfs Keisler measure is trivial, i.e., $\mathfrak{M}_n^{\mathrm{dfs}}(\mathcal{U}) = \mathfrak{M}_n^{\mathrm{tr}}(\mathcal{U})$ for all $n \geq 1$.

Proposition 4.9. *T is* dfs*-trivial if and only if* $\mathfrak{M}_1^{\mathrm{dfs}}(\mathcal{U}) = \mathfrak{M}_1^{\mathrm{tr}}(\mathcal{U})$.

Proof. Fix $n \geq 1$ and suppose that every measure in $\mathfrak{M}_n^{\mathrm{dfs}}(\mathcal{U})$ is trivial. Suppose $\mu \in \mathfrak{M}_{n+1}^{\mathrm{dfs}}(\mathcal{U})$, and let $\mu_0 \in \mathfrak{M}_n(\mathcal{U})$ be the projection of μ to the first n variables, i.e., $\mu_0(\varphi(x_1, \ldots, x_n)) = \mu(\varphi(x_1, \ldots, x_n) \wedge x_{n+1} = x_{n+1})$. Note that $\mu_0 \in \mathfrak{M}_n^{\mathrm{dfs}}(\mathcal{U})$, and thus is trivial by assumption. Fix a countable set $I \subset \mathcal{U}^n$ and a function $r \colon I \to (0, 1]$ such that $\mu_0 = \sum_{i \in I} r_i \delta_i$. Fix $i \in I$, and let $\nu_i \in \mathfrak{M}_1(\mathcal{U})$ be such that $\nu_i(\varphi(x)) = \frac{1}{r_i}\mu(\varphi(x_{n+1}) \wedge (x_1, \ldots, x_n) = i)$. Then $\nu_i \in \mathfrak{M}_1^{\mathrm{dfs}}(\mathcal{U})$ for all $i \in I$, and so we have $\nu_i = \sum_{j=0}^{\infty} s_j^i \delta_{a_j^i}$ for some sequences $(a_j^i)_{j=0}^{\infty}$ from $\mathcal{U}$ and $(s_j^i)_{j=0}^{\infty}$ from $[0, 1]$. Now we claim that

$$\mu = \sum_{i \in I} \sum_{j=0}^{\infty} r_i s_j^i \delta_{(i, a_j^i)},$$

and so μ is trivial. Let $\bar{x} = (x_1, \ldots, x_n, x_{n+1})$ and, for $i \in I$, define the formula $\sigma_i(\bar{x}) := ((x_1, \ldots, x_n) = i) \wedge (x_{n+1} = x_{n+1})$. Then $\mu(\sigma_i(\bar{x})) = r_i$ for any $i \in I$. Since $\sum_{i \in I} r_i = 1$, it follows that for any $\mathcal{L}(\mathcal{U})$-formula $\varphi(x_1, \ldots, x_n, x_{n+1})$, we have

$$\begin{aligned}\mu(\varphi(\bar{x})) = \sum_{i \in I} \mu(\varphi(\bar{x}) \wedge \sigma_i(\bar{x})) = \sum_{i \in I} \mu(\varphi(i, x_{n+1}) \wedge \sigma_i(\bar{x})) = \sum_{i \in I} r_i \nu_i(\varphi(i, x)) \\ = \sum_{i \in I} \sum_{j=0}^{\infty} r_i s_j^i \delta_{a_j^i}(\varphi(i, x)) = \sum_{i \in I} \sum_{j=0}^{\infty} r_i s_j^i \delta_{(i, a_j^i)}(\varphi(\bar{x})). \quad \square\end{aligned}$$

Question 4.10. Does the analogue of Proposition 4.9 hold for finitely approximated measures or for frequency interpretation measures?

Remark 4.11. If T is NIP then T is dfs-nontrivial. This is a standard construction, which we briefly recall (see also, e.g., [59, Example 7.2]). Assume T is NIP, and let $(a_i)_{i \in [0,1]}$ be a non-constant indiscernible sequence in $\mathcal{U}$. Define $\mu \in \mathfrak{M}_1(\mathcal{U})$ so that $\mu(\varphi(x))$ is the Lebesgue measure of $\{i \in [0,1] : \mathcal{U} \models \varphi(a_i)\}$. Since T is NIP, it follows that μ is a well-defined nontrivial definable Keisler measure, and it is clearly finitely satisfiable in any $M \prec \mathcal{U}$ containing $(a_i)_{i \in [0,1]}$.

It should be mentioned that not every NIP theory admits a nontrivial dfs *type*. For example, in distal theories (which are NIP), any such type must be algebraic (see [60, Proposition 2.27]). However, if T is stable then any non-algebraic global type corresponds to a nontrivial dfs Keisler measure.

The next goal is to show that dfs-nontriviality is preserved under reducts. First, we make precise our use of the word "reduct". Let T_0 be a complete $\mathcal{L}_0$-theory in some one-sorted language $\mathcal{L}_0$ of small cardinality (relative to $\mathcal{U}$). Without loss of generality, we assume $\mathcal{L}_0$ is relational. We say T_0 is a **reduct** of T if there is some finite $F \subset \mathcal{U}$ and, for each n-ary relation $R \in \mathcal{L}_0$, an $\mathcal{L}(\mathcal{U})$-formula $\theta_R(x_1, \ldots, x_n)$ (with $|x_i| = 1$) such that $(\mathcal{U} \backslash F; (\theta_R)_{R \in \mathcal{L}_0}) \models T_0$.

Theorem 4.12. *If T_0 is a reduct of T, and T_0 is* dfs*-trivial, then T is* dfs*-trivial.*

Proof. Fix a finite set $F \subset \mathcal{U}$ and $\mathcal{L}(\mathcal{U})$-formulas $(\theta_R)_{R\in\mathcal{L}_0}$ such that if $\mathcal{U}_0$ is the $\mathcal{L}_0$-structure $(\mathcal{U}\backslash F; (\theta_R)_{R\in\mathcal{L}_0})$, then $\mathcal{U}_0 \models T_0$. We may add constants to T and assume that each θ_R is over $\emptyset$, and F is definable by an $\mathcal{L}$-formula $\chi(x)$ over $\emptyset$.

To show that T is dfs-trivial, it suffices by Propositions 4.7 and 4.9 to fix some $\mu \in \mathfrak{M}_1^{\text{dfs}}(\mathcal{U})$ and show that $\mu(x = b) > 0$ for some $b \in \mathcal{U}$. Toward a contradiction, suppose $\mu(x = b) = 0$ for all $b \in \mathcal{U}$. Let $M \prec \mathcal{U}$ be such that μ is definable and finitely satisfiable in M. Then $M_0 := M\backslash F \prec_{\mathcal{L}_0} \mathcal{U}_0$, and $\mathcal{U}_0$ is $|M_0|^+$-saturated as an $\mathcal{L}_0$-structure.

Now let μ_0 be the restriction of μ to $\mathcal{L}_0$-formulas over $\mathcal{U}_0$. We show that $\mu_0 \in \mathfrak{M}_1^{\text{dfs}}(\mathcal{U}_0)$, which contradicts the assumption that T_0 is dfs-trivial. In particular, we show μ_0 is definable (with respect to $\mathcal{L}_0$) and finitely satisfiable in M_0. Fix an $\mathcal{L}_0$-formula $\varphi(x; y)$. Suppose $b \in \mathcal{U}_0^y$ is such that $\mu_0(\varphi(x; b)) > 0$. Note that $\mu(\chi(x)) = 0$ by assumption, and so $\mu(\varphi(x; b) \wedge \neg\chi(x)) > 0$. So $\varphi(x; b) \wedge \chi(x)$ is realized in M, i.e., $\varphi(x; b)$ is realized in M_0. Now fix a closed set $C \subseteq [0, 1]$ and let $X = \{b \in \mathcal{U}_0^y : \mu_0(\varphi(x; b)) \in C\}$. Then $X = \{b \in \mathcal{U}^y : \mu(\varphi(x; b)) \in C\} \cap (\mathcal{U}\backslash F)^y$, and so X is $\mathcal{L}_0$-type-definable over M by Proposition 2.20(a). Since F is $\emptyset$-definable, X is $\mathcal{L}_0$-type-definable over M_0. □

Recall that the *Random Graph* is the Fraïssé limit of the class of finite graphs, and the *Random Bipartite Graph* is the Fraïssé limit of the class of finite bipartite graphs. In order to obtain a Fraïssé class in the latter case, we work in the language $\mathcal{L} = \{E, P, Q\}$ where E is the edge relation and P, Q are predicates for the bipartition.

Theorem 4.13.

1. *The theory of the Random Graph is* dfs-*trivial.*
2. *The theory of the Random Bipartite Graph is* dfs-*trivial.*

Proof. We prove part (b). The argument for part (a) is similar (and easier), so we leave it as an exercise. The case of the Random Graph is also alluded to by Chernikov and Starchenko in [10, Example 3.8].

Let T be the theory of the Random Bipartite Graph. By Propositions 4.7 and 4.9, it suffices to fix $\mu \in \mathfrak{M}_1^{\mathrm{dfs}}(\mathcal{U})$ and show that $\mu(x = b) > 0$ for some $b \in \mathcal{U}$. Toward a contradiction, suppose $\mu(x = b) = 0$ for all $b \in \mathcal{U}$. Fix $M \prec \mathcal{U}$ such that μ is definable and finitely satisfiable in M.

Suppose first that there is some $b \in \mathcal{U}$ such that $\mu(E(x,b)) > 0$. Without loss of generality, assume $b \in Q(\mathcal{U})$. By Proposition 2.20(b), there is a E^*-formula $\psi(y)$ over M such that $\mathcal{U} \models \psi(b)$ and $\mu(E(x,c)) > 0$ for any $c \in \psi(\mathcal{U})$. Without loss of generality, we may assume $\psi(y)$ is of the form

$$\bigwedge_{m\in A} E(m,y) \wedge \bigwedge_{m\in B} \neg E(m,y)$$

for some finite disjoint $A, B \subseteq M$. Note that $A \subseteq P(M)$. By saturation, there is $c \in \mathcal{U}$ such that $E(m,c)$ holds for all $m \in A$ and $\neg E(m,c)$ holds for all $m \in M \backslash A$. It follows that $\mathcal{U} \models \psi(c)$, and so $\mu(E(x,c)) > 0$. Therefore $\mu(E(x,c) \wedge x \notin A) > 0$. By finite satisfiability, there is some $m \in M \backslash A$ such that $\mathcal{U} \models E(m,c)$. Then $E(m,c)$ holds and $m \in M \backslash A$, which contradicts the choice of c.

Now suppose that $\mu(\neg E(x,b)) = 1$ for all $b \in \mathcal{U}$. Note that $\mu(P(x) \vee Q(x)) = 1$ and so, without loss of generality, we may assume $\mu(P(x)) > 0$. Let $E_0(x,y)$ be the formula $\neg E(x,y) \wedge ((P(x) \wedge Q(y)) \vee (P(y) \wedge Q(x)))$. Then $E_0(x,y)$, $P(x)$, and $Q(x)$ define a Random Bipartite Graph on $\mathcal{U}$. Moreover, if $b \in Q(\mathcal{U})$ then $\mu(E_0(x,b)) > 0$. So we may apply the argument above to obtain a contradiction. $\square$

We now give several examples of theories which are dfs-trivial because they have one of the above theories as a reduct. Recall that, given $r \geq 2$, an r*-uniform hypergraph* (or r*-graph*) is a set of vertices together with an irreflexive, symmetric r-ary relation R. For any fixed $r \geq 2$, the class of finite r-graphs is a Fraïssé class, and we let T^r be the theory of the Fraïssé limit. Given $s > r$, let K_s^r be the complete

r-graph on s vertices. Then, for any fixed $s > r$, the class of finite K_s^r-free r-graphs is a Fraïssé class, and we let T_s^r be the theory of the Fraïssé limit.

Corollary 4.14. *The following theories are* dfs*-trivial:*

1. *T^r for any $r \geq 2$,*
2. *T_s^r for any $s > r \geq 3$,*
3. *the theory of any pseudofinite field,*
4. *the theory of the random tournament,*
5. *the theory T^*_{feq} of a generic parameterized equivalence relation, and*
6. *any completion of* ZF*.*

Proof. (1) We show that T^2 is a reduct of T^r. Let $F \subset \mathcal{U}$ be a set of size $r-2$. Let $E(x,y)$ be $R(x,y,\bar{c})$ where $\bar{c}$ enumerates F. Suppose $A, B \subset \mathcal{U} \backslash F$ are finite and disjoint. Define a one-point extension $H = ABF \cup \{e\}$ of the induced hypergraph on ABF by adding only the edges $R(a,e,\bar{c})$ for all $a \in A$. Then H is an r-graph, and so we may assume H embeds in $\mathcal{U}$ over ABF. Now $E(a,e)$ holds for all $a \in A$ and $\neg E(b,e)$ holds for all $b \in B$.

(2) Note that if $r \geq 3$ and $\mathcal{U} \models T_s^r$ then the graph H constructed in (1) is K_s^r-free. So the same argument works to show that T^2 is a reduct of T_s^r.

(3) Let T be the theory of a pseudofinite field K, and let p be a prime different from the characteristic of K. By a result of Duret [20], T^2 is a reduct of T via the formula $E(x,y) := \exists z(x + y = z^p) \wedge x \neq y$.

(4) Recall that a tournament is a directed graph in which every pair of vertices is joined by exactly one directed edge. Let T be the theory of the random tournament (i.e., the Fraïssé limit of finite tournaments), and let $\mathcal{U} \models T$. We show that the theory of the Random Bipartite Graph is a reduct of T. Let R be the directed edge relation, and fix some $a \in \mathcal{U}$. Let $P = \{b \in \mathcal{U} : R(a,b)\}$ and let $Q = \{b \in \mathcal{U} : R(b,a)\}$. Note that P and Q partition $\mathcal{U} \backslash \{a\}$. Define a bipartite graph relation $E \subseteq P \times Q$ where

$E(b,c)$ holds if and only if $R(b,c)$. Then $(P,Q;E)$ satisfies the axioms of the Random Bipartite Graph.

(5) We show that the theory of the Random Bipartite Graph is a reduct of T^*_{feq} (see Section 4.3.1 for the definition of this theory). Let $E_z(x,y)$ be the parameterized equivalence relation, where x,y are in the object sort O and z is in the parameter sort P. Fix some $a \in O(\mathcal{U})$, and let $P_0 = O(\mathcal{U})\backslash\{a\}$ and $Q_0 = P(\mathcal{U})$. Define a bipartite graph relation $E_0 \subseteq P_0 \times Q_0$ where $E_0(b,c)$ holds if and only if $E_c(a,b)$. Then $(P_0, Q_0; E_0)$ satisfies the axioms of the Random Bipartite Graph.

(6) Let T be a completion of ZF, and let $\mathcal{U} \models T$. We show that T^2 is a reduct of T via the formula $E(x,y) := (x \in y \vee y \in x)$.[2] Fix finite disjoint $A, B \subset \mathcal{U}$. Define $c = A \cup \{B\}$, which is an element of $\mathcal{U}$. Then $E(a,c)$ holds for all $a \in A$ and, by the axiom of foundation, we have $\neg E(b,c)$ for all $b \in B$. $\square$

Noticeably absent from the previous corollary is T^2_s for $s \geq 3$. We will see in Section 4.3.2 that these theories are *not* dfs-trivial.

4.3 Examples

4.3.1 Parameterized equivalence relations

The purpose of this section is to develop an example of Adler, Casanovas, and Pillay [1]. Let $\mathcal{L}$ be a language with two sorts O and P (for "objects" and "parameters") and a ternary relation $E_z(x,y)$ on $O \times O \times P$ (with x,y of sort O and z of sort P). Let T_{feq2} be the incomplete theory asserting that for any z in P, $E_z(x,y)$ is an equivalence relation on O in which each class has size 2. Then T_{feq2} has a model completion, which we denote T^*_{feq2}. This theory was defined in [1, Example 1.7], and can also be constructed as the *generic variation* of the theory T^*_{eq2} of an equivalence relation with infinitely many classes of size 2. Generic variations were defined by

[2] This was observed by James Hanson.

Baudisch in [3], although we have used an equivalent two-sorted version as in [8, Section 6.2].

Note that T^*_{eq2} has quantifier elimination and eliminates $\exists^\infty$ (in fact, this theory is complete and strongly minimal). It follows that T^*_{feq2} is complete, model complete, and eliminates $\exists^\infty$ (see [3, Corollary 2.10, Theorem 3.1]). However, T^*_{feq2} does not eliminate quantifiers unless one adds a binary function $f\colon P \times O \to O$ such that, for any $z \in P$, $f_z(-)\colon O \to O$ swaps the two elements in each E_z-class (more precisely, $f_z(x) = y$ if and only if $E_z(x, y) \wedge x \neq y$).

Theorem 4.15. *T^*_{feq2} is not simple.*

Proof. To show that T^*_{feq2} is not simple we will in fact witness TP_2.

Let $\{b_{i,j}, c_i : i, j < \omega\} \subseteq O(\mathcal{U})$ be a collection of pairwise distinct objects. Given $i < \omega$ and $j < k < \omega$, the formula $E_x(b_{i,j}, c_i) \wedge E_x(b_{i,k}, c_i)$ is inconsistent since all E_x-classes have size 2. On the other hand, for any function $\sigma\colon \omega \to \omega$, the type $\{E_x(b_{i,\sigma(i)}, c_i) : i < \omega\}$ is consistent since we can find a parameter $a \in P(\mathcal{U})$ such that $\{b_{i,\sigma(i)}, c_i\}$ is an E_a-class for all $i < \omega$. Altogether, we have shown that the formula $\varphi(x; y_1, y_2) := E_x(y_1, y_2)$ has TP_2. □

Despite the similarities between the definitions of T^*_{feq2} and T^*_{feq}, the behavior of generically stable types is different (recall that T^*_{feq} is dfs-trivial by Corollary 4.14).

Remark 4.16. If $\mathcal{U} \models T^*_{feq2}$ then any definable subset of $O(\mathcal{U})$ is finite or cofinite (this is easily checked using quantifier elimination in the language with f named).

Fact 4.17 (Adler, Casanovas, & Pillay [1])**.** *If $\mathcal{U} \models T^*_{feq2}$ then there is a generically stable type $p \in S_1(\mathcal{U})$ such that $p \otimes p$ is not generically stable.*

Proof. See [1, Example 1.7]. The type p is the unique type in $S_1(\mathcal{U})$ that contains $O(x)$ and $\neg E_c(x, b)$ for all $b \in O(\mathcal{U})$ and $c \in P(\mathcal{U})$. By Remark 4.16, it is clear that p is generically stable, and it is shown in [1] that $p \otimes p$ is not generically stable. □

Since the type p in the previous fact is not realized in $\mathcal{U}$, we conclude that T^*_{feq2} is not dfs-trivial. We also obtain a separation between generic stability and finite approximation for types (recall that these notions are equivalent in NIP theories).

Corollary 4.18. *If $\mathcal{U} \models T^*_{feq2}$ then there is a type $q \in S_2(\mathcal{U})$ that is finitely approximated but not generically stable.*

Proof. Let $p \in S_1(\mathcal{U})$ be the type from Fact 4.17. Then $p \in \mathfrak{M}^{\mathrm{fim}}_1(\mathcal{U})$ by Proposition 4.2. So $q := p \otimes p \in \mathfrak{M}^{\mathrm{fam}}_2(\mathcal{U})$, since $\mathfrak{M}^{\mathrm{fim}}_1(\mathcal{U}) \subseteq \mathfrak{M}^{\mathrm{fam}}_1(\mathcal{U})$ and finitely approximated measures are closed under Morley products (see Proposition 2.32). □

4.3.2 K_s-free graphs

Fix $s \geq 3$ and let K_s be the complete graph on s vertices. Given a finite graph G, let $\alpha_s(G)$ denote the size of the largest subset of G which induces a K_{s-1}-free subgraph. Let $R_s(n)$ be the smallest integer N such that any graph G of size N either contains K_s or satisfies $\alpha_s(G) \geq n$.

Theorem 4.19 (Erdős & Rogers 1962 [24])**.** *$R_s(n) \geq \Omega(n^{1+c_s})$ for some $c_s > 0$. Thus there are K_s-free graphs $(G_i)_{i=0}^{\infty}$ such that $|G_i| \to \infty$ and $\alpha_s(G_i) = o(|G_i|)$.*

Remark 4.20. For $s = 3$, Theorem 4.19 was first proved by Erdős [22] in 1957, and it was eventually shown that $R_3(n) = \Theta(\frac{n^2}{\log n})$ (see [2] and [39]).

We use $\mathcal{L} = \{E\}$ for the language of graphs, and let $\mathcal{U} \models T^2_s$ be sufficiently saturated. By quantifier elimination for T^2_s, there is a unique type in $S_1(\mathcal{U})$ containing $\neg E(x, b)$ for all $b \in \mathcal{U}$. We let p_E denote this type. Note that p_E is definable over $\emptyset$ and not realized in $\mathcal{U}$.

Theorem 4.21. *The type p_E is finitely approximated, but is not generically stable.*

Proof. Let $(a_i)_{i<\omega}$ be a Morley sequence in p_E over some small model. Then $a_i \neq a_j \wedge \neg E(a_i, a_j)$ holds for all $i \neq j$, and so there is $b \in \mathcal{U}$ such that $E(a_i, b)$ holds if and only if i is even. So p_E is not generically stable.

Now we show that p_E is finitely approximated in the unique countable model M of T^2_s. Let $\varphi(x;\bar{y})$ be a formula in the language of graphs, with $\bar{y} = (y_1, \ldots, y_m)$. Without loss of generality, we may assume that some instance of $\varphi(x;\bar{y})$ is in p_E (otherwise every instance of $\neg\varphi(x;\bar{y})$ is in p_E, and so we may apply the argument below to $\neg\varphi(x;\bar{y})$).

By quantifier elimination, we may fix a quantifier-free formula $\psi(\bar{y})$, an integer $N \geq 1$, and $A_t, B_t, C_t, D_t \subseteq [m]$, for $t \in [N]$, such that

$$\varphi(x;\bar{y}) \equiv \bigvee_{t=1}^{N} \left(\bigwedge_{i\in A_t} \neg E(x,y_i) \wedge \bigwedge_{i\in B_t} x \neq y_i \wedge \bigwedge_{i\in C_t} E(x,y_i) \wedge \bigwedge_{i\in D_t} x = y_i \right) \wedge \psi(\bar{y}).$$

Since p_E contains an instance of $\varphi(x;\bar{y})$, it follows that there is some $t_* \in [N]$ such that $C_{t_*} = \emptyset = D_{t_*}$, and so $\varphi(x;\bar{b}) \in p_E$ for any $\bar{b} \in \mathcal{U}^m$ such that $\mathcal{U} \models \psi(\bar{b})$. Fix $\epsilon > 0$. We want to find $n \geq 1$ and $\bar{a} = (a_1, \ldots, a_n) \in M^n$ such that, for any $\bar{b} \in \mathcal{U}^m$,

$$\left|\delta_p(\varphi(x;\bar{b})) - \mathrm{Av}(\bar{a})(\varphi(x;\bar{b}))\right| < \epsilon. \tag{†}$$

Let $|A_{t_*}| = k$ and $|B_{t_*}| = \ell$. By Theorem 4.19, we may choose $n > \frac{2\ell}{\epsilon}$ and $G = \{a_1, \ldots, a_n\} \subset M$ such that $\alpha_s(G) < \frac{\epsilon}{2k}n$. Fix $\bar{b} \in \mathcal{U}^m$. If $\mathcal{U} \models \neg\psi(\bar{b})$ then $\varphi(x;\bar{b}) \notin p$ and $\mathcal{U} \models \neg\varphi(a_i;\bar{b})$ for all $i \in [n]$, so (†) holds trivially. So we can assume $\mathcal{U} \models \psi(\bar{b})$, which implies that $\varphi(x;\bar{b}) \in p$.

For $j \in A_{t_*}$, set $X_j = \{i \in [n] : E(a_i, b_j)\}$, and note that $\{a_i : i \in X_j\}$ induces a K_{s-1}-free subgraph of G (since $\mathcal{U}$ is K_s-free). In particular $|X_j| < \frac{\epsilon}{2k}n$ for all $j \in A_{t_*}$. Define the sets $Y = \{i \in [n] : a_i = b_j \text{ for some } j \in B_{t_*}\}$ and $Z = \{i \in [n] : \neg\varphi(a_i;\bar{b})\}$. Then we have $Z \subseteq Y \cup \bigcup_{j\in A_{t_*}} X_j$, which implies

$$|Z| \leq |Y| + \sum_{j\in A_{t_*}} |X_j| < \ell + \tfrac{\epsilon}{2}n < \epsilon n.$$

So (†) holds, as desired. □

Remark 4.22. From the proof of Theorem 4.21 we see that, given an $\mathcal{L}$-formula $\varphi(x;\bar{y})$, there is a sequence $(\theta_n(x_1,\ldots,x_n))_{n=1}^{\infty}$ of $\mathcal{L}$-formulas (over $\emptyset$) such that, for any $\epsilon>0$, if $n\geq n_{\epsilon,\varphi}$ then $|p_E(\varphi(x;\bar{b}))-\mathrm{Av}_{\bar{a}}(\varphi(x;\bar{b}))|<\epsilon$ for any $\bar{a}\models\theta_n(\bar{x})$ and $\bar{b}\in\mathcal{U}^{\bar{y}}$. In particular, let $\theta_n(x_1,\ldots,x_n)$ describe the isomorphism type of the graph G chosen with $\alpha_s(G)$ sufficiently small depending on ϵ and φ. Of course, since $\alpha_s(G)$ is small, G must contain (many) edges, and so $\theta_n(\bar{x})\notin p_E^{(n)}$.

Next, we show that if $\mathcal{U}\models T_s^2$ then $\mathfrak{M}_1^{\mathrm{dfs}}(\mathcal{U})$ coincides with $\mathfrak{M}_1^{\mathrm{fam}}(\mathcal{U})$, and is the convex hull of p_E and $\mathfrak{M}_1^{\mathrm{tr}}(\mathcal{U})$. So p_E is essentially the only non-trivial dfs measure in one variable. We also observe that every frequency interpretation measure in one variable is trivial.

Theorem 4.23. *Let $\mathcal{U}\models T_s^2$.*

1. $\mathfrak{M}_1^{\mathrm{fam}}(\mathcal{U})=\mathfrak{M}_1^{\mathrm{dfs}}(\mathcal{U})=\{rp_E+(1-r)\mu:\mu\in\mathfrak{M}_1^{\mathrm{tr}}(\mathcal{U}),\ r\in[0,1]\}$.

2. $\mathfrak{M}_1^{\mathrm{fim}}(\mathcal{U})=\mathfrak{M}_1^{\mathrm{tr}}(\mathcal{U})$.

Proof. Part (a). Let $\mu\in\mathfrak{M}_1(\mathcal{U})$ be definable and finitely satisfiable in $M\prec\mathcal{U}$.

Claim: If $\mu(x=b)=0$ for all $b\in\mathcal{U}$, then $\mu=p_E$.

Proof: Assume $\mu(x=b)=0$ for all $b\in\mathcal{U}$ and, toward a contradiction, suppose $\mu(E(x,b))>0$ for some $b\in\mathcal{U}$. There are two cases.

Suppose first that $b\notin M$. Let $\psi(y)$ be an $\mathcal{L}(M)$-formula such that $\psi(b)$ holds and, for any $c\in\mathcal{U}$, if $\psi(c)$ holds then $\mu(E(x,c))>0$. Let $A\subset M$ be the finite set of parameters in $\psi(y)$. Since $b\notin M$, we may find $c\in\mathcal{U}$ such that $c\equiv_A b$ and $\neg E(m,c)$ for all $m\in M\backslash A$. Then $\psi(c)$ holds and so $\mu(E(x,c)\wedge x\notin A)>0$. But $E(x,c)\wedge x\notin A$ is not realized in M.

Now suppose $b\in M$. Let $X=\{m\in M:E(m,b)\}$. Then X is K_{s-1}-free, so there is $c\in\mathcal{U}\backslash M$ such that $E(m,c)$ for all $m\in X$. By the above, $\mu(\neg E(x,c))=1$, and so $\mu(\neg E(x,c)\wedge E(x,b))>0$. But $\neg E(x,c)\wedge E(x,b)$ is not realized in M. $\dashv_{\text{claim}}$

Now, let $S = \{b \in \mathcal{U} : \mu(x = b) > 0\}$. As in the proof of Proposition 4.7, S is countable. By the claim, we may assume $S \neq \emptyset$, and so $\mu(S) > 0$. Let $\nu = \frac{1}{\mu(S)} \sum_{b \in S} \mu(x = b)\delta_b$, and note that $\nu \in \mathfrak{M}_1^{\mathrm{tr}}(\mathcal{U})$.

Let $X = \mathcal{U} \backslash S$. If $\mu(X) = 0$ then $\mu = \nu$, and we are finished. So assume $\mu(X) > 0$. Let μ_0 be the localization of μ at X. Then μ_0 is dfs and $\mu_0(x = b) = 0$ for all $b \in \mathcal{U}$. By the claim, $\mu_0 = p_E$. Note that $p_E(S) = 0$ since S is countable. Altogether, given $A \in \mathrm{Def}_1(\mathcal{U})$, we have

$$\begin{aligned}\mu(A) &= \mu(A \backslash S) + \mu(A \cap S) \\ &= \mu(X)\mu_0(A \backslash S) + \mu(S)\nu(A \cap S) = \mu(X)p_E(A) + (1 - \mu(X))\nu(A).\end{aligned}$$

So $\mu = \mu(X)p_E + (1 - \mu(X))\nu$, as desired.

Part (b). Suppose $\mu \in \mathfrak{M}_1^{\mathrm{fim}}(\mathcal{U})$. Then $\mu \in \mathfrak{M}_1^{\mathrm{dfs}}(\mathcal{U})$ and so, by Theorem 4.21 and the claim in part (a), we have $\mu(x = b) > 0$ for some $b \in \mathcal{U}$. So $\mathfrak{M}_1^{\mathrm{fim}}(\mathcal{U}) = \mathfrak{M}_1^{\mathrm{tr}}(\mathcal{U})$ by Proposition 4.7. □

4.3.3 K_s^r-free hypergraphs

We have now seen that if $s > r \geq 3$ then T_s^r is dfs-trivial, while T_s^2 is not dfs-trivial for any $s > 2$. The change in behavior from $r = 2$ to $r \geq 3$ is reminiscent of a similar disparity at the level of dividing lines. In particular, T^2 is simple, but T_s^2 is not simple for any $s \geq 3$ (in fact, T_s^2 has SOP_3 by Shelah [57]). On the other hand, T^r and T_s^r are both simple for any $s > r \geq 3$ (this was shown by Hrushovski [30]; see also [13, Section 7.1]).

Despite the fact that T_s^r is dfs-trivial for $s > r \geq 3$, we can find interesting behavior in these theories at the level of φ-types. First, let us recall some notions. Let T be a complete theory with monster model $\mathcal{U}$, and fix an $\mathcal{L}$-formula $\varphi(x; y)$. We let $S_\varphi(\mathcal{U})$ be the space of complete φ-types over $\mathcal{U}$. Given $p \in S_\varphi(\mathcal{U})$, we recall:

1. p is definable if the set $\{b \in \mathcal{U}^y : \varphi(x;b) \in p\}$ is definable (and thus, the same is true for any Boolean combination of $\varphi(x;y_i)$);
2. p is finitely satisfiable in $M \prec \mathcal{U}$ if any finite subset of p is realized in M;
3. p is finitely approximated if there is $M \prec \mathcal{U}$ such that, for any formula $\psi(x;z)$, which is a finite Boolean combination of $\varphi(x;y_i)$, and any $\epsilon > 0$, there are $a_1, \ldots, a_n \in M^x$ such that, for any $c \in \mathcal{U}^z$, $|\delta_p(\psi(x;c)) - \mathrm{Av}_{\bar{a}}(\psi(x;c))| < \epsilon$.

The definitions above for φ-types are consistent with the definitions for φ-measures when we view a type as a $\{0,1\}$-valued local Keisler measure on $\mathcal{L}_\varphi(\mathcal{U})$ (see Definition 3.33 and 3.34).

Recall that in Chapter 3, we proved a local version of the equivalence of 1 and 2 in Theorem 2.39. Specifically, if $\varphi(x;y)$ is NIP and μ is a local Keisler measure on φ-formulas, then μ is dfs if and only if μ is finitely approximated (see Theorem 3.39). We will show that the analogue of this fails for simple formulas. In fact, we will find a complete φ-type in a simple theory (specifically, T^r_s for $s > r \geq 3$) that is dfs, but is not finitely approximated. Before defining this type, we recall some results from graph theory. First, we state the following corollary of the *Ramsey property* for the class of finite K^r_s-free r-graphs.

Theorem 4.24 (Nešetřil & Rödl 1979 [43, 44])**.** *Given $s > r \geq 2$ and $n \geq 1$, there is a finite K^r_s-free r-graph G such that any edge-coloring of G with n colors admits a monochromatic copy of K^r_{s-1}.*

Next, we consider (vertex) colorings of weighted hypergraphs. In particular, given $r \geq 2$, a *weighted r-graph* is a pair $H = (V, w)$ where V is a finite vertex set and $w\colon [V]^r \to \mathbb{R}$ is a function (here $[V]^r$ is the set of r-element subsets of V). Suppose $H = (V, w)$ is a weighted r-graph. Set $w(V) = \sum_{\sigma \in [V]^r} w(\sigma)$. An *$r$-coloring* of H is a function $\chi\colon V \to [r]$. We say that an r-coloring χ *splits* $\sigma \in [V]^r$ if $\chi(u) \neq \chi(v)$ for all distinct $u, v \in \sigma$. The *weight* of an r-coloring χ, denoted $w(\chi)$, is the sum of $w(\sigma)$ over all $\sigma \in [V]^r$ such that χ splits σ. The next fact is due to Erdős and Kleitman [23] in the setting of unweighted hypergraphs.

Lemma 4.25. *Let $H = (V, w)$ be a finite weighted r-graph for some $r \geq 2$. Then there is an r-coloring χ of H such that $w(\chi) \geq \frac{r!}{r^r} w(V)$.*

Proof. Given an r-coloring χ of H and $\sigma \in [V]^r$, let $w_\chi(\sigma)$ be $w(\sigma)$ if χ splits σ, and 0 otherwise. So $w(\chi) = \sum_{\sigma \in [V]^r} w_\chi(\sigma)$. Let $n = |V|$. Then the number of r-colorings of H is r^n and, given $\sigma \in [V]^r$, the number of r-colorings of H that split σ is $r^{n-r} r!$. So we can compute the average weight of an r-coloring of H as follows:

$$\frac{1}{r^n} \sum_\chi w(\chi) = \frac{1}{r^n} \sum_\chi \sum_\sigma w_\chi(\sigma) = \frac{1}{r^n} \sum_\sigma \sum_\chi w_\chi(\sigma) = \frac{1}{r^n} \sum_\sigma \frac{r^n r!}{r^r} w(\sigma) = \frac{r!}{r^r} w(V).$$

Therefore some r-coloring of H has weight at least $\frac{r!}{r^r} w(V)$. $\square$

Now we fix $s > r \geq 3$. Let $M \models T^r_s$ be the Fraïssé limit of the class of finite K^r_s-free r-graphs, and let $\mathcal{U} \succ M$ be a sufficiently saturated elementary extension. Let $\varphi(x_1, \ldots, x_{r-1}; y)$ be the formula $\neg R(x_1, \ldots, x_{r-1}, y) \wedge \bigwedge_{i \neq j} x_i \neq x_j$. We define $p_R \in S_\varphi(\mathcal{U})$ to be the complete φ-type containing $\varphi(\bar{x}; b)$ for all $b \in \mathcal{U}$.

Theorem 4.26. *The φ-type p_R is* dfs*, but not finitely approximated.*

Proof. It is clear that p_R is definable. We show that p_R is finitely satisfiable in M. Fix $b_1, \ldots, b_n \in \mathcal{U}$. We want to find $a_1, \ldots, a_{r-1} \in M$ such that $\neg R(\bar{a}, b_i)$ holds for all $i \in [n]$, and $a_i \neq a_j$ for all distinct $i, j \in [r-1]$.

By Theorem 4.24, there is a finite K^{r-1}_s-free $(r-1)$-graph $G = (W, E)$ such that any edge-coloring of G with n colors admits a monochromatic copy of K^{r-1}_{s-1}. Define an r-graph $H = (W, R)$ such that, given $\sigma \in [W]^r$, $R(\sigma)$ holds if and only if $[\sigma]^{r-1} \subseteq E$. Then H is K^r_s-free since, if $A \in [W]^s$ is such that $[A]^r \subseteq R$ then $[A]^{r-1} \subseteq E$. So we may assume that H is an induced subgraph of $M \backslash \{b_1, \ldots, b_n\}$.

For $i \in [n]$, let $C_i = \{\tau \in [W]^{r-1} : R(\tau, b_i)\}$. Toward a contradiction, suppose $[W]^{r-1} = C_1 \cup \ldots \cup C_n$. Then we can define an edge coloring $c \colon E \to [n]$ such that $c(\tau) = \min\{i \in [n] : \tau \in C_i\}$. By choice of G, there is $A \in [W]^{s-1}$ and $\ell \in [n]$

such that $[A]^{r-1} \subseteq E$ and $c(\tau) = \ell$ for all $\tau \in [A]^{r-1}$. But then $[A \cup \{b_\ell\}]^r \subseteq R$, contradicting that $\mathcal{U}$ is K_s^r-free. So we may fix some $\sigma \in [W]^{r-1} \backslash (C_1 \cup \ldots \cup C_n)$. Let $\sigma = \{a_1, \ldots, a_{r-1}\}$. Then $a_1, \ldots, a_{r-1} \in M$, $\neg R(\bar{a}, b_i)$ for all $i \in [n]$, and $a_i \neq a_j$ for all distinct $i, j \in [r-1]$, as desired.

To show that p_R is not finitely approximated, we fix $\bar{a}^1, \ldots, \bar{a}^n \in \mathcal{U}^{r-1}$ and find some $b \in \mathcal{U}$ such that $|\{t \in [n] : \varphi(\bar{a}^t; b)\}| < (1 - \epsilon_r)n$, where $\epsilon_r = (r-1)^{1-r}(r-1)!$. After re-indexing if necessary, we may assume there is some $m \leq n$ such that, given $t \in [n]$, we have $|\{a_1^t, \ldots, a_{r-1}^t\}| = r - 1$ if and only if $t \leq m$.

Let $V = \{a_i^t : t \in [m],\ i \in [r-1]\}$. For $\sigma \in [V]^{r-1}$, set

$$I_\sigma = \left\{t \in [m] : \{a_1^t, \ldots, a_{r-1}^t\} = \sigma\right\}.$$

Define the weight function $w \colon [V]^{r-1} \to \{0, 1, \ldots, m\}$ such that $w(\sigma) = |I_\sigma|$. Note that $\{I_\sigma : \sigma \in [V]^{r-1}\}$ is a partition of $[m]$ (with some I_σ possibly empty), and so $w(V) = m$. By Lemma 4.25, there is an $(r-1)$-coloring χ of (V, w) such that $w(\chi) \geq \epsilon_r m$. Let $\Sigma = \{\sigma \in [V]^{r-1} : \chi \text{ splits } \sigma\}$.

We now define an r-graph (V', R') extending (V, R). Let $V' = V \cup \{v_*\}$, where v_* is a vertex not in V, and set $R' = R \cup \{\sigma \cup \{v_*\} : \sigma \in \Sigma\}$. Toward a contradiction, suppose (V', R') is not K_s^r-free. Then there is $A \in [V']^s$ such that $[A]^r \subseteq R'$. So $v_* \in A$ since (V, R) is K_s^r-free. Since $|A \cap V| = s - 1 > r - 1$, there are distinct $v_1, v_2 \in A \cap V$ such that $\chi(v_1) = \chi(v_2)$. Fix $\sigma \in [A \cap V]^{r-1}$ such that $v_1, v_2 \in \sigma$. Then χ does not split σ, and so $\sigma \cup \{v_*\} \notin R'$, which contradicts $[A]^r \subseteq R'$.

Finally, since (V', R') is K_s^r-free, it follows that there is some $b \in \mathcal{U}$ such that, given $\sigma \in [V]^{r-1}$, $R(\sigma, b)$ holds if and only if $\sigma \in \Sigma$. Let $I = \{t \in [m] : R(\bar{a}^t, b)\}$. Then $I = \bigcup_{\sigma \in \Sigma} I_\sigma$, and so $|I| = w(\chi) \geq \epsilon_r m$. So

$$|\{t \in [n] : \neg\varphi(\bar{a}^t; b)\}| = |I \cup \{m+1, \ldots, n\}| \geq \epsilon_r m + n - m \geq \epsilon_r n,$$

as desired. □

Note that p_R does not extend to a *global* dfs measure, since T_s^r is dfs-trivial and p_R cannot be extended to a global trivial measure.

Remark 4.27. The main reason to use hypergraphs in the above arguments was to work in a simple theory. However, a similar situation could be constructed in the theory T_s^2 for $s \geq 4$. Specifically, let $\varphi(x, y; z)$ be $\neg(E(x, z) \wedge E(y, z)) \wedge x \neq y$, and let $p \in S_\varphi(\mathcal{U})$ be the complete φ-type containing $\varphi(x, y; b)$ for all $b \in \mathcal{U}$. Then an argument similar to the $r = 3$ case of Theorem 4.26 shows that p is dfs, but not finitely approximated.

In light of all of the examples above, we make the following conjecture and ask some questions.

Conjecture 4.28. *There is a theory T, and a Keisler measure $\mu \in \mathfrak{M}_x(\mathcal{U})$ such that μ is* dfs*, but not finitely approximated.*

Question 4.29. Is there a simple (or even NTP_2) theory T and a *global* Keisler measure $\mu \in \mathfrak{M}_x(\mathcal{U})$ such that either μ is dfs but not finitely approximated, or μ is finitely approximated but not a frequency interpretation measure? Is there a type in a simple (or NTP_2) theory with either of these properties?

CHAPTER 5

SEQUENTIAL APPROXIMATIONS

One of the joys of working in a metric space is that the closure of a set coincides with its *sequential closure*. In particular, if X is a metric space, A is a subset of X, and a is in the closure of A, then there exists a sequence of elements in A such that this sequence converges to a. In [61], Simon showed that global types which are finitely satisfiable in a countable model of a countable NIP theory admit a similar property. Fix T a countable theory, $\mathcal{U}$ a monster model of T and M a small elementary submodel. Simon demonstrated the following ([61, Lemma 2.8]):

Theorem 5.1 (Simon)**.** *Assume that T is NIP. If $p \in S_x(\mathcal{U})$ and finitely satisfiable in M where $|M| = \aleph_0$, then there exists a sequence of points $(a_i)_{i\in\omega}$ such that each $a_i \in M^x$ and $\lim_{i\to\infty} \operatorname{tp}(a_i/\mathcal{U}) = p$.*

The purpose of this chapter is to *morally* generalize the proof of the above theorem in two different directions. By mimicking Simon's proof, we are able to prove the following,

1. Let T be any countable theory. If p is a type in $S_x(\mathcal{U})$ and is generically stable over M, then p admits a *strong sequential approximation* in M, i.e. there exists a sequence of points in M^x such that their corresponding types converge to p.
2. Assume that T is a countable NIP theory. Let μ be a Keisler measure in $\mathfrak{M}_x(\mathcal{U})$ and $|M| = \aleph_0$. If μ is finitely satisfiable in M, then μ admits a *sequential approximation* in M, i.e. there exists a sequence of points in $(M^x)^{<\omega}$ such that their corresponding average measures converge to μ.

The proofs of both of these theorems are slightly more involved than one would expect. For example, we already know many diverse and useful approximation theorems for measures in NIP theories. Additionally, we know from Proposition 4.2 that if p is generically stable over a model M and I is a Morley sequence in p over M, then p is the sequential limit of this sequence. However, stringing together different approximation techniques typically results in a *modes-of-convergence* problem.

As stated previously, the technique used to prove both these theorems mimics the argument used in [61, Lemma 2.8]. In the generically stable case, the set up is identical: let p be generically stable over a model M, and I be a Morley sequence in p over M. As in Simon's proof, we use both M and I to find an eventually indiscernible sequence of points in M^x which converge to $p|_{MI}$. The eventual EM-type of this sequence over M is precisely $p^{\omega}|_M$. Using generic stability (specifically, Proposition 4.2) and compactness, we conclude that this sequence must converge to p.

The proof of the Keisler measure case is slightly more exotic since there is no standard notion of a "Morley sequence in a Keisler measure". Therefore, we must replace I with another object. We will show that this can be resolved by replacing the Morley sequence by a countable model N_ω containing a family of smooth extensions of $\mu|_M$. This provides more evidence for the intuition that smooth measures can play the role of realized types, at least in the NIP context. After constructing a countable model with these "realizations", we find a sequence of points in $(M^x)^{<\omega}$ such that the corresponding average measures on these tuples converge to $\mu|_{N_\omega}$. After finding an ϵ-eventual indiscernible subsequence, we are able to readapt most of Simon's proof technique by making use of known approximation theorems, symmetry properties, and some basic integration techniques.

In addition to proving these two main theorems, we also describe some basic properties of strongly sequentially approximated types and sequentially approximated measures. At the end of the chapter, we examine some concrete examples outside the generically stable and NIP contexts. Most notably, we observe that there exists a type p such that its corresponding Keisler measure is sequentially approximated (even finitely approximated), but the type itself is not strongly sequentially approximated (in fact, the types described in both Corollary 4.18 and Theorem 4.21 exemplify this property).

This chapter is structured as follows: In section 5.1, we describe sequentially approximated measures and strongly sequentially approximated types. In section 5.2, we demonstrate that if p is generically stable over M, then p is strongly sequentially approximated in M. In section 5.3, we show that if T is a countable NIP theory, and μ is finitely satisfiable in a countable model M, then μ is sequentially approximated in M. In section 5.4, we exposit some concrete examples of types which are not (strongly) sequentially approximated but their associated measures are sequentially approximated.

5.1 Sequentially approximated types and measures

We begin this chapter by isolating the property of *sequential approximability.* We assume that T is countable, but make no other global assumptions on T unless specifically stated. As usual, $\mathcal{U}$ is a fixed sufficiently saturated model of T. We recall some basic facts about convergence. Recall that for any $A \subseteq \mathcal{U}$, both $S_x(A)$ and $\mathfrak{M}_x(A)$ are compact Hausdorff topological spaces with the topology induced from $[0,1]^{\mathcal{L}(\mathcal{A})}$.

Definition 5.2. Fix $A \subseteq \mathcal{U}$, $p \in S_x(A)$ and $\mu \in \mathfrak{M}_x(A)$.

1. We say that a sequence of types $(p_i)_{i\in\omega}$, where each p_i is in $S_x(A)$, **converges** to p if it converges in the Stone space topology on $S_x(A)$, which we write as "$\lim_{i\to\infty} p_i = p$ in $S_x(A)$". In particular, for every $\psi(x) \in \mathcal{L}_x(A)$, there exists some natural number N_φ such that for any $n > N_\varphi$, $\psi(x) \in p_n$ if and only if $\psi(x) \in p$. Moreover, we say that a sequence $(a_i)_{i\in\omega}$ of points in $\mathcal{U}^x$ **converges** to p if $\text{tp}(a_i/A)$ converges to p in $S_x(A)$.
2. We say that a sequence of measures $(\mu_i)_{i\in\omega}$, where each μ_i is in $\mathfrak{M}_x(A)$, **converges** to μ if it converges in the usual compact Hausdorff topology on $\mathfrak{M}_x(A)$,

which we write as "$\lim_{i\to\infty}\mu_i = \mu$ in $\mathfrak{M}_x(A)$". In particular, for every $\psi(x) \in \mathcal{L}_x(A)$ and $\epsilon > 0$, there exists some natural number $N_{\varphi,\epsilon}$ such that for any $n > N_{\varphi,\epsilon}$,

$$|\mu_n(\varphi(x)) - \mu(\varphi(x))| < \epsilon.$$

In addition, we say that a sequence of tuples $(\overline{a}_i)_{i\in\omega}$, where each tuple is an element of $(A^x)^{<\omega}$, **converges** to μ if $\mathrm{Av}(\overline{a}_i)$ converges to μ in $\mathfrak{M}_x(A)$.

The following proposition observes the relationships between finitely satisfiable types/measures and topological closure (in the compact Hausdorff topology).

Proposition 5.3. *If $p \in S_x(\mathcal{U})$, $\mu \in \mathfrak{M}_x(\mathcal{U})$, and p, μ are finitely satisfiable in a submodel M, then*

1. *The type p is in the closure of $\{tp(a/\mathcal{U}) : a \in M^x\}$ in $S_x(\mathcal{U})$.*
2. *The associated Keisler measure δ_p is in the closure $\{\delta_a : a \in M^x\}$ in $\mathfrak{M}_x(\mathcal{U})$.*
3. *The measure μ is in the closure of $\{\mathrm{Av}(\overline{a}) : \overline{a} \in (M^x)^{<\omega}\}$ in $\mathfrak{M}_x(\mathcal{U})$.*

Proof. The proof of 1 is standard and the proof of 2 follows directly from 1. Statement 3 follows from Proposition 2.19 and Fact 2.6. □

We now define strongly sequentially approximated types and sequentially approximated measures.

Definition 5.4. Let $p \in S_x(\mathcal{U})$ and $\mu \in \mathfrak{M}_x(\mathcal{U})$. We say that,

1. p is **strongly sequentially approximated** if there exists $M \prec \mathcal{U}$ and there exists a sequence of points $(a_i)_{i\in\omega}$ such that each a_i is in M^x and $\lim_{i\to\infty} \mathrm{tp}(a_i/\mathcal{U}) = p$ in $S_x(\mathcal{U})$. In this case, we say that μ is **strongly sequentially approximated in** M.
2. μ is **sequentially approximated** if there exists $M \prec \mathcal{U}$ and there M if there exists a sequence of tuples $(\overline{a}_i)_{i\in\omega}$ such that each $\overline{a}_i$ is in $(M^x)^{<\omega}$ and $\lim_{i\to\infty} \mathrm{Av}(\overline{a}_i) = \mu$ in $\mathfrak{M}_x(\mathcal{U})$. In this case, we say that p is **sequentially approximated in** M.

Warning 5.5. The definition above is only meaningful in the context of types and measures over large sets of parameters. Indeed, if M is a countable model and T is a countable theory, then for every $p \in S_x(M)$, there exists a sequence of points in M^x such that $\lim_{i\to\infty} \mathrm{tp}(a_i/M) = p$ in $S_x(M)$. The analogous statement also holds for measures.

Warning 5.6. By Proposition 5.3, a type p in $S_x(\mathcal{U})$ is strongly sequentially approximated over a model M, then the associated Keisler measure δ_p is sequentially approximated over M. The converse fails in general.

5.1.1 Basic properties

We now relate (strongly) sequentially approximated (types) measures to properties we already know. For intuition, sequential approximability should be thought of as a strong version of finite satisfiability over a small model or a weaker version of finite approximability.

Proposition 5.7. *Assume that $p \in S_x(\mathcal{U})$ and $\mu \in \mathfrak{M}_x(\mathcal{U})$.*

1. *If p and μ are (strongly) sequentially approximated in M, then p and μ are finitely satisfiable in M. Even more, p and μ are finitely satisfiable in a countable elementary submodel of M.*
2. *If μ is sequentially approximated in M, then μ is Borel-definable over M.*
3. *If μ is finitely approximated in M, then μ is sequentially approximated in M.*
4. *If T is NIP, then p is strongly sequentially approximated in M if and only if δ_p is sequentially approximated in M.*
5. *Assume that $k \subseteq \{1, 2, ..., n\}$ and let $\pi_k : S_n(\mathcal{U}) \to S_k(\mathcal{U})$ and $\rho_k : \mathfrak{M}_n(\mathcal{U}) \to \mathfrak{M}_k(\mathcal{U})$ be the obvious projection. If $p \in S_n(\mathcal{U})$ and p is strongly sequentially approximated, then $\pi_k(p)$ is strongly sequentially approximated. Similarly, if $\mu \in \mathfrak{M}_n(\mathcal{U})$ is sequentially approximated then so is $\rho_k(\mu)$.*

Proof. The first part of statement 1 is obvious. For the second part, we only need to choose a submodel containing a sequence which sequentially approximates our type/measure.

Proof of 2: By statement 1, μ is finitely satisfiable in M and hence M-invariant. So, for any partitioned formula $\varphi(x; y)$ in $\mathcal{L}$, the map $F_\mu^\varphi : S_y(M) \to [0, 1]$ is well-defined. Then, this statement follows from the observation that the map F_μ^φ is Baire-1 (since the sequence of continuous functions $\left(F_{\mathrm{Av}(\overline{a})}^\varphi\right)_{i\in\omega}$ converges pointwise to F_μ^φ). Therefore the map F_μ^φ is Borel and thus μ is Borel-definable over M.

Proof of 3: Let $(\varphi_n(x; y_n))_{n\in\omega}$ be an enumeration of the partitioned $\mathcal{L}$-formulas. For each n choose $\overline{a}_n$ in $(M^x)^{<\omega}$ such that for every $b \in \mathcal{U}^{y_j}$ where $j \leq n$, we have

$$|\operatorname{Av}(\overline{a}_n)(\varphi_j(x; b)) - \mu(\varphi_j(x; b))| < \frac{1}{n}.$$

It is clear that $\lim_{n\to\infty} \operatorname{Av}(\overline{a}_n) = \mu$ in $\mathfrak{M}_x(\mathcal{U})$.

Proof of 4: Again, the forward direction is trivial. We consider the converse. If δ_p is sequentially approximated if M then δ_p is finitely satisfiable in a countable submodel M_0 by statement 1. It is clear that p is finitely satisfiable in M_0. Hence, by Theorem 5.1, p is strongly sequentially approximated in M_0 and hence in M.

Proof of 5: Clear from the definition. Simply consider the sequence restricted to the appropriate coordinates. □

Proposition 5.8. *A measure μ is sequentially approximated and definable over M if and only if μ is finitely approximated over M.*

Proof. We first prove the forward direction. The proof is more or less identical to the proof of Theorem 3.30. Hence, we only sketch the argument. For any partitioned formula $\varphi(x; y)$ in $\mathcal{L}$, consider the map $F_\mu^\varphi : S_y(M) \to [0, 1]$. Let $(\overline{a}_i)_{i\in\omega}$ be a sequence of points in $(M^x)^{<\omega}$ such that $\lim_{i\to\infty} \operatorname{Av}(\overline{a}_i) = \mu$ in $\mathfrak{M}_x(\mathcal{U})$. Recall that each map $F_{\mathrm{Av}(\overline{a})}^\varphi : S_y(M) \to [0, 1]$ is continuous and the sequence $\left(F_{\mathrm{Av}(\overline{a}_i)}^\varphi\right)_{i\in\omega}$ converge pointwise to F_μ^φ. Since μ is definable, the map F_μ^φ is continuous. Then by a standard application of Mazur's lemma (Theorem 3.1), there exists a sequence of functions $(g_j)_{j\in\omega}$ such that each g_j is a rational convex combination of $\{F_{\mathrm{Av}(\overline{a}_i)}^\varphi : i \leq n_j\}$ for some natural number n_j and this sequence converges uniformly to F_μ^φ. Using this sequence, it is easy to show that μ is finitely approximated over M by choosing a representative sequence of some g_j sufficiently close to F_μ^φ.

For the converse, μ is definable over M by statement 1 in Proposition 2.30. Moreover, μ is sequentially approximated in M by statement 3 in Proposition 5.7. □

Finally, we show that sequentially approximated measures commute with definable measures. It is known in the context of NIP theories that definable measures commute with finitely satisfiable measures (see [59, Proposition 7.22]), however all known proofs make use of the fact that measures in NIP theories admit (local) uniform approximation by averaging on types. In general, it is not clear whether sequentially approximated measures admit such approximations.

Proposition 5.9. *Sequentially approximated and definable measures commute. In particular, if $\mu \in \mathfrak{M}_x(\mathcal{U})$, $\nu \in \mathfrak{M}_y(\mathcal{U})$, μ is sequentially approximated over M and ν is definable over M, then $\mu \otimes \nu = \nu \otimes \mu$.*

Proof. Fix a formula $\varphi(x; y)$ in $\mathcal{L}_{xy}(\mathcal{U})$. Assume that $M \subseteq N$ and N contains all the parameters from $\varphi(x; y)$. Since ν is definable over M (and therefore, definable over N), we know that the map $F_\nu^{\varphi^*} : S_x(N) \to [0, 1]$ is continuous. By Lemma 2.16, for every $\epsilon > 0$, there exists formulas $\psi_1(x), ..., \psi_n(x)$ in $\mathcal{L}_x(M)$ and real numbers $r_1, ..., r_n$ such that

$$\sup_{q \in S_y(N)} |F_\nu^{\varphi^*}(q) - \sum_{i=1}^{n} r_i \chi_{\psi_i(x)}(q)| < \epsilon.$$

Let $(\overline{a}_j)_{j \in \omega}$ be a sequence of points in $(M^x)^{<\omega}$ such that $\lim_{j \to \infty} \text{Av}(\overline{a}_j) = \mu$ in $\mathfrak{M}_x(\mathcal{U})$. Now, we compute

$$\nu \otimes \mu(\varphi(x; y)) = \int_{S_x(N)} F_\nu^{\varphi^*} d\mu \approx_\epsilon \int_{S_x(N)} \sum_{i=1}^{n} r_i \chi_{\psi_i(y)} d\mu = \sum_{i=1}^{n} r_i \mu(\psi_i(y))$$

$$= \sum_{i=1}^{n} r_i \lim_{j\to\infty} \operatorname{Av}(\psi_i(\overline{a}_j)) = \lim_{j\to\infty} \sum_{i=1}^{n} r_i \operatorname{Av}(\psi_i(\overline{a}_j)) = \lim_{j\to\infty} \int_{S_x(N)} \sum_{i=1}^{n} r_i \chi_{\psi_i(y)} d \operatorname{Av}(\overline{a}_j).$$

We also compute the other product. By the dominated convergence theorem, we have

$$\mu \otimes \nu(\varphi(x;y)) = \int_{S_y(N)} F_{\mu}^{\varphi} d\nu = \lim_{j\to\infty} \int_{S_y(N)} F_{\operatorname{Av}(\overline{a}_j)}^{\varphi} d\nu = \lim_{i\to\infty} \int_{S_x(N)} F_{\nu}^{\varphi} d \operatorname{Av}(\overline{a}_j).$$

Therefore, we have that

$$|\nu \otimes \mu(\varphi(x;y)) - \mu \otimes \nu(\varphi(x;y))| \leq \lim_{j\to\infty} \int_{S_x(N)} |F_{\nu}^{\varphi} - \sum_{i=1}^{n} r_i \chi_{\psi(y)}| d \operatorname{Av}(\overline{a}_j)$$

$$< \lim_{j\to\infty} \int_{S_x(N)} \epsilon d \operatorname{Av}(\overline{a}_j) = \lim_{j\to\infty} \epsilon = \epsilon.$$

□

5.1.2 Egorov's theorem

It is interesting to note that sequentially approximated measures are not too far away from finitely approximated measures. In particular, if we fix some measure on the parameter space, any sequentially approximated measure is *almost* finitely approximated. A direct application of Egorov's theorem gives this result.

Theorem 5.10 (Egorov's Theorem)**.** *Let (X, B, μ) be a finite measure space. Assume that f_i is a sequence of measurable functions from $X \to \mathbb{R}$ such that $(f_i)_{i\in\omega}$ converges to a function f pointwise. Then, for every $\epsilon > 0$ there exists a closed set $Y_\epsilon \subset X$ such that $f_i|_Y$ converges to $f|_Y$ uniformly on Y_ϵ and $\mu(X\backslash Y_\epsilon) < \epsilon$.*

A proof of Egorov's theorem can be found in [35, Theorem 3.2.4.1]. As literally a direct corollary, we have the following application.

Corollary 5.11. *Assume that p and μ are (strongly) sequentially approximated in M.*

Let ν be a measure on $S_y(M)$. Then, for every $\epsilon > 0$, there exists a set $X_\epsilon \subset S_y(M)$ such that,

1. *$\nu(X_\epsilon) > 1 - \epsilon$.*
2. *For every $\delta > 0$ and every partitioned formula $\varphi(x; y) \in \mathcal{L}$, there exists $\overline{a}_\delta$ in $(M^x)^{<\omega}$ such that for every $b \in \mathcal{U}^y$ such that $\text{tp}(b/M) \in X_\epsilon$, we have,*

$$|\mu(\varphi(x; b)) - \text{Av}(\overline{a}_\delta)(\varphi(x; b))| < \delta.$$

3. *For every $\delta > 0$ and every partitioned formula $\varphi(x; y) \in \mathcal{L}$, there exists a_δ in M^x such that for every $b \in \mathcal{U}^y$ such that $\text{tp}(b/M) \in X_\epsilon$, we have,*

$$\varphi(x; b) \in p \iff \models \varphi(a_\delta, b).$$

5.2 Generically stable types

In this section, we demonstrate that generically stable types admit a strong sequential approximation over models. We assume that T is countable unless explicitly stated otherwise. We begin with a discussion on eventually indiscernible sequences which were introduced in [61].

Definition 5.12. Let $(c_i)_{i\in\omega}$ be a sequence of points in $\mathcal{U}^x$ and $A \subset \mathcal{U}$. We say that $(c_i)_{i\in\omega}$ is an **eventually indiscernible sequence** over A if for any formula $\varphi(x_0, ..., x_n)$ in $\mathcal{L}(A)$, there exists some natural number N_φ such that for any $n_k > .... > n_0 > N_\varphi$ and $m_k > ... > m_0 > N$, we have that

$$\models \varphi(c_{n_0}, ..., c_{n_k}) \leftrightarrow \varphi(c_{m_0}, ..., c_{m_k}).$$

Fact 5.13. *Let $(b_i)_{i\in\omega}$ be a sequence of points in $\mathcal{U}^x$ and assume that $|A| = \aleph_0$. Then, there exists a subsequence $(c_i)_{i\in\omega}$ of $(b_i)_{i\in\omega}$ such that $(c_i)_{i\in\omega}$ is eventually indiscernible over A.*

Proof. The proof is a standard application of Ramsey's theorem and taking the diagonal. We prove a "continuous" version of this fact in the next section and the proof is analogous. See Proposition 5.21 for details. ☐

Now, for any eventually indiscernible sequence $(c_i)_{i\in\omega}$ over a set of parameters A, we can associate to this sequence a unique type in $S_\omega(A)$. We call this the *eventual Ehrenfeucht-Mostowski type* (or EEM-type) of $(c_i)_{i\in\omega}$ over A. We now give the formal definition.

Definition 5.14. Let $(b_i)_{i\in\omega}$ be a sequence of points in $\mathcal{U}^x$ and $A \subseteq \mathcal{U}$. Then the **eventual Ehrenfeucht-Mostowski type** of $(b_i)_{i\in\omega}$ over A, written $\mathrm{EEM}((b_i)_{i\in\omega})/A)$, is a (partial) type in $S_\omega(A)$ with the following definition: let $\varphi(x_{i_0}, ..., x_{i_k})$ be a formula in $\mathcal{L}(A)$ where the indices are ordered $i_0 < ... < i_k$. Then, $\varphi(x_{i_0}, ...x_{i_k}) \in \mathrm{EEM}((b_i)_{i\in\omega})/A)$ if and only if there exists some N_φ such that for any $n_k > ... > n_0 > N_\varphi$, we have that $\mathcal{U} \models \varphi(b_{n_0}, ..., b_{n_k})$.

Notice that a EEM-type of a sequence is always indiscernible in the following sense: if we have indices $i_0, ..., i_k$ and $j_0, ..., j_k$ where $i_0 < ... < i_k$ and $j_0 < ... < j_k$, then $\varphi(x_{i_0}, ..., x_{i_k})$ is in the EEM-type of $(b_i)_{i\in\omega}$ over A if and only if $\varphi(x_{j_0}, ..., x_{j_k})$ is. This follows directly from the definition. Finally, we have one last observation.

Proposition 5.15. *Assume that $(c_i)_{i\in\omega}$ is an eventually indiscernible sequence over A. Then,* $\mathrm{EEM}((\bar{c}_i)_{i\in\omega}/A)$ *is complete.*

Proof. Clear from the definitions. ☐

The next lemma proves the bulk of the main theorem. The proof strategy is as follows: Assume that p is generically stable over a countable model M and let I be a Morley sequence in p over M. Then, we can find a sequence of points in M which converge to $p|_{MI}$ in $S_x(MI)$. After moving to an eventually indiscernible subsequence, we show that the EEM-type of this eventually indiscernible sequence is $p^\omega|_M$. Now, if this eventually indiscernible subsequence does not converge to p in $S_x(\mathcal{U})$, we use compactness to contradict generic stability.

Lemma 5.16. *Assume that $p \in S_x(\mathcal{U})$ and p is generically stable over M. Assume moreover that $|M| = \aleph_0$. Then, there exists a sequence $(c_i)_{i\in\omega}$ of points in M^x such that $\lim_{i\to\infty} \operatorname{tp}(c_i/\mathcal{U}) = p$.*

Proof. Let $I = (a_i)_{i\in\omega}$ be a Morley sequence in p over M. Since T, M, and I are countable, $|\mathcal{L}_x(MI)|$ is countable. It follows that $\{\varphi(x) : \varphi(x) \in p|_{MI}\}$ is countable and we may enumerate this collection of formulas as $(\varphi_i(x))_{i\in\omega}$. Since p is generically stable over M, p is finitely satisfiable in M. For each natural number i, we choose b_i in M such that $\mathcal{U} \models \bigwedge_{j\leq i} \varphi_j(b_i)$. Now, consider the sequence $(b_i)_{i\in\omega}$. By construction, we have that $\lim_{i\to\infty} \operatorname{tp}(b_i/MI) = p|_{MI}$ in $S_x(MI)$. By Fact 5.13, we may choose a subsequence $(c_i)_{i\in\omega}$ of $(b_i)_{i\in\omega}$ such that $(c_i)_{i\in\omega}$ is eventually indiscernible over MI. We now argue that $\lim_{i\to\infty} \operatorname{tp}(c_i/\mathcal{U}) = p$ in $S_x(\mathcal{U})$. For notation purposes, we will write $(c_i)_{i\in\omega}$ as J.

Claim: $\mathrm{EEM}(J/M) = \mathrm{EM}(I/M) = p^{\omega}|_M$.

We show this by induction on the length of the formula. We begin with the base case. Since $\lim_{i\to\infty} \operatorname{tp}(b_i/MI) = p|_{MI}$, and $(c_i)_{i\in\omega}$ is a subsequence of $(b_i)_{i\in\omega}$, it is clear that $\lim_{i\to\infty} \operatorname{tp}(c_i/M) = p|_M$. Our induction hypothesis is as follows: For any formula $\theta(x_0, ..., x_k)$ in $\mathcal{L}_{x_0,...,x_k}(M)$, we have that $\theta(x_0, ..., x_k) \in \mathrm{EM}(I/M)$ if and only if $\theta(x_0, ..., x_k) \in \mathrm{EEM}(J/M)$.

Towards a contradiction, we assume that $\neg\theta(x_0, ..., x_{k+1}) \in \mathrm{EEM}(J/M)$ and $\theta(x_0, ..., x_{k+1}) \in \mathrm{EM}(I/M)$. Since $\neg\theta(\overline{x}) \in \mathrm{EEM}(J/M)$, there exists some natural number N_{θ_1} such that for any $n_{k+1} > ... > n_0 > N_{\theta_1}$, we have that $\models \neg\theta(c_{n_0}, ..., c_{n_{k+1}})$. Since $\theta(\overline{x}) \in \mathrm{EM}(I/M)$, we conclude that $\models \theta(a_0, ..., a_{k+1})$. Since p is generically stable over M, I is totally indiscernible over M (see [50, Proposition 2.1]). Therefore, $\models \theta(a_{k+1}, a_0..., a_k)$ also holds and so $\theta(x, a_0, ..., a_k) \in p|_{Ma_0,...,a_k}$. Since $\lim_{i\to\infty} \operatorname{tp}(c_i/MI) = p|_{MI}$, there exists some N_{θ_2} such that for every $n > N_{\theta_2}$, we have that $\models \theta(c_n, a_0, ..., a_k)$. Choose $n_* > \max\{N_{\theta_1}, N_{\theta_2}\}$. Then, the formula $\theta(c_{n_*}, x_0, ..., x_k) \in \operatorname{tp}(a_0, ..., a_k/M)$. By our induction hypothesis, we have that $\theta(c_{n_*}, \overline{x}) \in \text{EEM}(J/M)$ and so there exists N_{θ_3} such that for any $m_k > ... > m_0 > N_{\theta_3}$, we have that $\models \theta(c_{n_*}, c_{m_0}, ..., c_{m_k})$. Now consider what happens when $m_0 > \max\{N_{\theta_3}, n_*\}$. Then, $m_k > ... > m_0 > n_* > N_{\theta_1}$ and so $\models \neg\theta(c_{n_*}, ..., c_{m_k})$ by our assumption. However, $m_k > ... > m_0 > N_{\theta_3}$ and therefore $\models \theta(c_{n_*}, ..., c_{m_k})$. This is a contradiction.

Claim: *The sequence* $(\text{tp}(c_i/\mathcal{U}))_{i\in\omega}$ *converges to a type in* $S_x(\mathcal{U})$.

It suffices to argue that for any formula $\psi(x) \in \mathcal{L}_x(\mathcal{U})$, the $\lim_{i\to\infty} \chi_\psi(c_i)$ exists. Assume not. Then, we may choose a subsequence $(c'_i)_{i\in\omega}$ of $(c_i)_{i\in\omega}$ such that $\models \psi(c'_i) \leftrightarrow \neg\psi(c'_{i+1})$. For notational purposes, we also denote $((c'_i)_{i\in\omega})$ as J'. It is clear that $(c'_i)_{i\in\omega}$ is also eventually indiscernible and $\text{EEM}((c'_i)_{i\in\omega}/M) = \text{EEM}((c_i)_{i\in\omega}/M)$. Then, by using this sequence, one can show that the following type is finitely consistent:

$$\Theta_1 = \text{EEM}(J'/M) \cup \bigcup_{i \text{ is even}} \{\psi(x_i) \vee \neg\psi(x_{i+1})\}.$$

If we let $(d_i)_{i\in\omega}$ realize this type, then $(d_i)_{i\in\omega}$ is a Morley sequence in p over M since

$$\text{EEM}(J'/M) = \text{EEM}(J/M) = \text{EM}(I/M) = p^\omega|_M.$$

Then, $\models \psi(d_i)$ if and only if i is even. This contradicts generic stability since $\{i \in \omega :\models \psi(d_i)\}$ is neither a finite or cofinite subset of ω.

Claim: *The sequence* $(\text{tp}(c_i/\mathcal{U}))_{i\in\omega}$ *converges to* p.

Again, assume not. Since $(\text{tp}(c_i/\mathcal{U}))_{i\in\omega}$ converges, there must be a formula $\theta(x) \in \mathcal{L}_x(\mathcal{U})$ such that $\theta(x) \in p$ and there exists an N such that for every $n > N$, we have that $\models \neg\theta(c_n)$. By virtually the same argument, one can show the following type is finitely consistent:

$$\Theta_2 = \text{EEM}(J/M) \cup \bigcup_{i\in\omega} \neg\theta(x_i).$$

If we let $(d_i)_{i\in\omega}$ realize this type, then again $(d_i)_{i\in\omega}$ is a Morley sequence in p over M.

Then, $\lim_{i\to\infty} \text{tp}(d_i/\mathcal{U}) \neq p$ in $S_x(\mathcal{U})$. This contradicts statement 2 of Proposition 4.2 and completes the proof. □

Theorem 5.17. *Assume that $p \in S_x(\mathcal{U})$ and p is generically stable over M. Then, p is strongly sequentially approximated over M.*

Proof. By Proposition 4.2, p is generically stable over a model M if and only if δ_p is FIM over M. Therefore δ_p is FIM over M. By Proposition 2.35, δ_p is FIM over a countable model. So, p is generically stable over a countable model and we may apply Lemma 5.16. □

Corollary 5.18. *Assume that T is countable or uncountable. Let $\mathcal{U}$ be a monster model of T and M a small elementary substructure of $\mathcal{U}$. Assume that p is generically stable over M. Then, for any countable collection of formulas $\Delta = \{\psi_i(x, y_i)\}_{i\in\omega}$ in $\mathcal{L}$, there exists a sequence of points $(c_i)_{i\in\omega}$ each in M^x such that $\lim_{i\to\infty} tp_\Delta(c_i/\mathcal{U}) = p|_\Delta$.*

Proof. Let $\mathcal{L}'$ be a countable sublanguage of $\mathcal{L}$ containing all the formulas in Δ. By Remark 4.3 the corresponding type p' is generically stable over the corresponding model M'. Hence, we may apply Theorem 5.17. □

We end this section by collecting the known examples of strongly sequentially approximated types.

Observation 5.19. Assume that $p \in S_x(\mathcal{U})$ and let M be a small elementary submodel. Then, p is strongly sequentially approximated over M if:

1. If T is stable, and p is invariant over M.
2. If T is NIP, $|M| = \aleph_0$, and p is finitely satisfiable in M.
3. If p is generically stable over M.

We just proved 3. Clearly, 1 follows from 3 (we remark that it also follows from 2). As noted previously, the proof of 2 is precisely Lemma 2.8 of [61].

Question 5.20. Does there exist a global type p which is strongly sequentially approximated and definable over a model M but is not generically stable over M? Under what model theoretic tameness assumptions can this happen (e.g. Simple, NTP_2, NSOP_1)?

5.3 Sequential approximations in NIP theories

Throughout this entire section, we assume that T is a countable NIP theory. We show that measures which are finitely satisfiable in a countable model of a countable NIP theory are sequentially approximated. We begin by discussing a "continuous" analogue of eventually indiscernible sequences.

5.3.1 ϵ-Eventually indiscernible sequences

We fix some notation. If $\varphi(x_0, ..., x_n)$ is a formula in $\mathcal{L}(\mathcal{U})$ and $\overline{a}_0, ..., \overline{a}_n$ is a sequence of tuples in $(\mathcal{U}^x)^{<\omega}$ where each $\overline{a}_i = (a_i^0, ..., a_i^{m_i})$, then we write $\varphi_c(\overline{a}_0, ..., \overline{a}_n)$ to mean,

$$\bigotimes_{i=0}^{n} \mathrm{Av}(\overline{a}_i)_{x_i}(\varphi(x_0, ..., x_n)).$$

We observe that by unpacking the definition of the product measure, our formula can be computed as follows:

$$\varphi_c(\overline{a}_0, ..., \overline{a}_n) = \frac{1}{\prod_{i=0}^{n} m_i} \sum_{j_0=0}^{m_i} ... \sum_{j_n=0}^{m_n} \chi_\varphi(a_0^{j_0}, ..., a_n^{j_n}).$$

Definition 5.21. Let $(\overline{c}_i)_{i\in\omega}$ be a sequence of tuples in $(\mathcal{U}^x)^{<\omega}$ and let $A \subseteq \mathcal{U}$ be a collection of parameters. Then, we say that the sequence $(\overline{c}_i)_{i\in\omega}$ is **ϵ-eventually indiscernible** over A if for any formula $\varphi(x_0, ..., x_n)$ in $\mathcal{L}(A)$ and any $\epsilon > 0$, there exists $N_{\epsilon,\varphi}$ such that for any $n_k > ... > n_0 > N$ and $m_k > .... > m_0 > N$, we have that;

$$|\varphi_c(\overline{a}_{n_0}, ..., \overline{a}_{n_k}) - \varphi_c(\overline{a}_{m_0}, ..., \overline{a}_{m_k})| < \epsilon$$

Proposition 5.22. *Let $(\overline{a}_i)_{i\in\omega}$ be a sequence of tuples in $(\mathcal{U}^x)^{<\omega}$. If A is a countable set of parameters, then there exists some subsequence $(\overline{c}_i)_{i\in\omega}$ of $(\overline{a})_{i\in\omega}$ such that $(\overline{c}_i)_{i\in\omega}$ is ϵ-eventually indiscernible over A.*

Proof. This proof is a standard application of Ramsey's theorem applied to the "continuous" setting. Enumerate all formulas pairs in $\mathcal{L}_{(x_i)_{i\in\omega}}(A) \times \{\frac{1}{n} : n \in \mathbb{N}_{>0}\}$. Let $(\overline{a}_i)_{i\in\omega} = (\overline{a}_i^0)_{i\in\omega}$ and set $B_0 = \{\overline{a}_i^0 : i \in \omega\}$. Now, assume we have constructed the subsequence $(\overline{a}_i^l)_{i\in\omega}$ and B_l. We now construct $(\overline{a}_i^{l+1})_{i\in\omega}$ and B_{l+1}. Assume that $(\varphi(x_0, ..., x_k), \frac{1}{n})$ is the $l+1$ indexed pair in our sequence. Then we define the coloring $r_{l+1} : ((B_l)^k)^2 \to \{0, 1\}$ where $r(\overline{a}_{n_0}^l, ..., \overline{a}_{n_k}^l; \overline{a}_{m_0}^l, ..., \overline{a}_{m_k}^l) = 0$ if and only if,

$$|\varphi_c(\overline{a}_{n_0}^l, ..., \overline{a}_{n_k}^l) - \varphi_c(\overline{a}_{m_0}^l, ..., \overline{a}_{m_k}^l)| < \frac{1}{n}.$$

By Ramsey's theorem, there is a monochromatic subset $B_l^{'}$ of B_l. Since finitely many ball of radius $\frac{1}{n}$ cover $[0, 1]$, it must be the case that this monochromatic subset has color 0. Let $(\overline{a}_i^{l+1})_{i\in\omega}$ be the obvious reindexed subsequence of $(\overline{a}_i^l)_{i\in\omega}$ with the elements only from the monochromatic set $B_l^{'}$. Then, we let $B_{l+1} = \{\overline{a}_i^{l+1} : i \in \omega\}$. By construction, the sequence $(\overline{a}_i^i)_{i\in\omega}$ is ϵ-eventually indiscernible. □

We now present a collection of facts which will help us demonstrate that the associated averaging measures along ϵ-eventually indiscernible sequences always converge (in $\mathfrak{M}_x(\mathcal{U})$). The first fact is elementary and left to the reader as an exercise.

Fact 5.23. *Assume that $(\mu_i)_{i\in\omega}$ is a sequence of measures in $\mathfrak{M}_x(\mathcal{U})$. If for every formula $\varphi(x) \in \mathcal{L}_x(\mathcal{U})$, $\lim_{i\to\infty} \mu(\varphi(x))$ converges, then $(\mu_i)_{i\in\omega}$ converges to a measure.*

The next collection of facts can be found in [33]. In particular, 1 follows immediately from Lemma 2.10 while 2 and 3 are from Corollary 2.14. The proof of Lemma 2.10 is non-trivial and is an interpretation of results in [4]. We suggest the reader review Definition 2.27 and Observation 2.42 before continuing this section.

Fact 5.24 (T is NIP). *Let $\mu, \nu \in \mathfrak{M}_x(\mathcal{U})$ such that μ, ν are invariant over M and suppose that λ is a finitely additive probability measure on $\mathcal{L}_{(x_i:)_{i\in\omega}}(\mathcal{U})$ where $|x_i| = |x_j|$ for each i, j. Recall that a measure λ on $\mathcal{L}_{(x_i)_{i\in\omega}}(\mathcal{U})$ is said to be A-**indiscernible** if for every increasing sequence of indices $i_0, ..., i_n$ and any formula $\varphi(x_{i_0}, ..., x_{i_n})$ in $\mathcal{L}_{(x_i)_{i\in\omega}}(A)$, we have that*

$$\lambda(\varphi(x_{i_0}, ..., x_{i_n})) = \lambda(\varphi(x_0, ..., x_n)).$$

The following statements are true.

1. *if $\lambda \in \mathfrak{M}_\omega(\mathcal{U})$ and λ is $\emptyset$-indiscernible, then for any formula $\varphi(x; b) \in \mathcal{L}_x(\mathcal{U})$, we have that $\lim_{i\to\infty} \lambda(\varphi(x_i, b))$ exists.*
2. *Then μ^ω and ν^ω are M-indiscernible.*
3. *If $\mu^\omega|_M = \nu^\omega|_M$, then $\mu = \nu$.*

Proposition 5.25. *If $(\overline{c}_i)_{i\in\omega}$ is an ϵ-eventually indiscernible sequence over M, then the sequence $(\mathrm{Av}(\overline{c}_i))_{i\in\omega}$ converges in $\mathfrak{M}_x(\mathcal{U})$.*

Proof. Assume not. Then there exists some formula $\psi(x; b)$ in $\mathcal{L}_x(\mathcal{U})$, some $\epsilon_0 > 0$, and some subsequence $(\overline{c}'_i)_{i\in\omega}$ of $(\overline{c}_i)_{i\in\omega}$ such that for each natural number i,

$$|\,\mathrm{Av}(\overline{c}'_i)(\psi(x; b)) - \mathrm{Av}(\overline{c}'_{i+1})(\psi(x; b))| > \epsilon.$$

It is clear that $(\overline{c}'_i)_{i\in\omega}$ is also ϵ-eventually indiscernible over M. We now work towards proving a contradiction to 1 of Fact 5.24 via (topological) compactness of the space $\mathfrak{M}_\omega(\mathcal{U})$. For any formula $\varphi(x_{i_0}, ..., x_{i_k}) \in \mathcal{L}_{(x_i)_{i\in\omega}}(M)$, we let r_φ be the unique real number such that for every $\epsilon > 0$, there exists an $N_{\epsilon,\varphi}$ such that for any $n_k > ...n_0 > N_{\varphi,\epsilon}$ we have

$$|\varphi_c(\overline{c}'_{n_0}, ..., \overline{c}'_{n_k}) - r_\varphi| < \epsilon.$$

Since the sequence $(\overline{c}'_i)_{i\in\omega}$ is ϵ-eventually indiscernible over M, r_φ exists for each

$\varphi(\overline{x}) \in \mathcal{L}_{(x_i)_{i\in\omega}}(M)$. Now, for every $\varphi(\overline{x}) \in \mathcal{L}_{(x_i)_{i\in\omega}}(M)$ and $\epsilon > 0$, we define the following family of closed subsets of $\mathfrak{M}_\omega(\mathcal{U})$;

$$C_{\varphi,\epsilon} = \Big\{\lambda \in \mathfrak{M}_\omega(\mathcal{U}) : r_\varphi - \epsilon \leq \lambda(\varphi(\overline{x})) \leq r_\varphi + \epsilon\Big\}.$$

We now define another family of sets and argue that they are closed.

$$D_i = \Big\{\lambda \in \mathfrak{M}_\omega(\mathcal{U}) : |\lambda(\varphi(x_i, b)) - \lambda(\varphi(x_{i+1}, b))| \geq \frac{\epsilon_0}{2}\Big\}.$$

Notice that D_i is closed since for every natural number i, the evaluation map $E_i : \mathfrak{M}_\omega(\mathcal{U}) \to [0,1]$ via $E_i(\lambda) = \lambda(\psi(x_i, b))$ is continuous. Define $F_i = E_i - E_{i+1}$ and $H_i = E_{i+1} - E_i$. Then we have, $D_i = F_i^{-1}([\frac{\epsilon_0}{2}, 1]) \cup H_i^{-1}([\frac{\epsilon_0}{2}, 1])$. Hence, D_i is a finite union of closed sets and therefore closed. It is not difficult to show using $(\overline{c}_i')_{i\in\omega}$ that the collection $\Phi = \{C_{\epsilon,\varphi} : \epsilon > 0, \varphi(\overline{x}) \in \mathcal{L}_\omega(M)\} \cup \{D_i : i \in \omega\}$ has the finite intersection property. Therefore, there exists some $\lambda \in \mathfrak{M}_\omega(\mathcal{U})$ in the intersection of all the sets in φ. Moreover, λ is indiscernible (even more, indiscernible over M) by construction. Since λ is in D_i for each i, its existence contradicts 1 of Fact 5.24. $\square$

5.3.2 Smooth sequences

In this subsection, we define the notion of a smooth sequence and prove the main theorem. If μ is a global M-invariant measure, then a smooth sequence is a collection of models and measures meant to replicate a Morley sequence. The ideology is the following: A Morley sequence in p over M is to the infinite type $p^\omega|_M$ as a smooth sequence in μ over M is to the measure $\mu^\omega|_M$. The following is the formal definition. We remark that many of the computations in this section make use of Fact 2.34.

Definition 5.26. Let $\mu \in \mathfrak{M}_x(\mathcal{U})$ and assume that μ is invariant over some small model M. Then, a **smooth sequence in μ over M** is a sequence of pairs of small models and measures, $(N_i, \mu_i)_{i\in\omega}$, such that:

1. $M \prec N_0$ and $N_i \prec N_{i+1}$ and each N_i is small.
2. μ_i is smooth over N_i.
3. $\mu_0|_M = \mu|_M$ and for $i > 0$, $\mu_i|_{N_{i-1}} = \mu|_{N_{i-1}}$.

Furthermore, we define $\bigotimes_{i=0}^{\omega} \mu_i = \bigcup_{i=0}^{\omega} \bigotimes_{i=0}^{n} \mu_i$ which is a measure on $\mathcal{L}_{(x_i)_{i\in\omega}}(\mathcal{U})$.

Proposition 5.27 (T NIP). *Assume that $\mu \in \mathfrak{M}_x(\mathcal{U})$ and μ is M-invariant. Let $(N_i, \mu_i)_{i\in\omega}$ be a smooth sequence in μ over M. Then, $\bigotimes_{i=0}^{\omega} \mu_i|_M = \mu^{\omega}|_M$.*

Proof. For our base case, it is true by construction that $\mu_0|_M = \mu|_M$. For our induction hypothesis, we assume that $\mu^{k-1}|_M = \bigotimes_{i=0}^{k-1} \mu_i|_M$. We fix $\lambda = \bigotimes_{i=0}^{k-1} \mu_i$ and show the induction step: Let $\varphi(x_0, ..., x_k)$ be any formula in $\mathcal{L}_{x_0,...,x_k}(M)$. Since the product of smooth measures is smooth (Fact 2.34), we have that λ is generically stable over N_{k-1}. Therefore, λ is invariant over N_{k-1}. We let $\overline{x} = (x_0, ..., x_{k-1})$ and consider the computation:

$$\mu_k \otimes \lambda(\varphi(x_0, ..., x_{k-1}, x_k)) = \int_{S_{\overline{x}}(N_k)} F_{\mu_k}^{\varphi} d(\lambda|_{N_k}) = \int_{S_{x_k}(N_k)} F_{\lambda}^{\varphi^*} d(\mu_k|_{N_k})$$

Since λ is invariant over N_{k-1} and $\mu_k|_{N_{k-1}} = \mu|_{N_{k-1}}$,

$$= \int_{S_{x_k}(N_{k-1})} F_{\lambda}^{\varphi^*} d(\mu_k|_{N_{k-1}}) = \int_{S_{x_k}(N_{k-1})} F_{\lambda}^{\varphi^*} d(\mu|_{N_{k-1}}) = \int_{S_{\overline{x}}(N_{k-1})} F_{\mu}^{\varphi} d(\lambda|_{N_{k-1}})$$

Since μ is invariant over M, we may continue,

$$= \int_{S_{\overline{x}}(M)} F_{\mu}^{\varphi} d(\lambda|_M) = \int_{S_{\overline{x}}(M)} F_{\mu}^{\varphi} d(\mu^{k-1}|_M) = \mu \otimes \mu^n(\varphi(x_0, ..., x_{k-1}, x_k)).$$

$\square$

Proposition 5.28. *If T is a countable NIP theory, $\mu \in \mathfrak{M}_x(\mathcal{U})$, and μ is invariant over M where $|M| = \aleph_0$, then there exists a smooth sequence $(N_i, \mu_i)_{i\in\omega}$ in μ over M such that each N_i is countable.*

Proof. We construct the sequence as follows: At step 1, consider $\mu|_M$. Let μ_0 be a smooth extension of $\mu|_M$ to $\mathcal{U}$. Since T is countable, by the Proposition 2.35 we know that there exists N_0 such that $M \prec N_0 \prec \mathcal{U}$, μ_0 is smooth over N_0, and $|N_0| = \aleph_0$. At step m, we repeat the process. We consider $\mu|_{N_{m-1}}$, choose μ_m a smooth extension of $\mu|_{N_{m-1}}$ to $\mathfrak{M}_x(\mathcal{U})$, and let N_m be a countable model such that $N_{m-1} \prec N_m \prec \mathcal{U}$ and μ_m is smooth over N_m. □

We now begin the proof of our main theorem. Again, the proof is similar to both the generically stable case in the previous section and even more so to the proof of Lemma 2.8 in [61]. Here, however, the major difference is that we replace the Morley sequence in that proof with a countable model, N_ω, which contains a smooth sequence in μ over M. Then, we find a sequence of tuples in M such that the associated average measures converge to $\mu|_{N_\omega}$ in $\mathfrak{M}_x(N_\omega)$. After choosing an ϵ-eventually indiscernible subsequence, we know from our NIP assumption that this new sequence converges to a global measure ν in $\mathfrak{M}_x(\mathcal{U})$. Finally, we demonstrate that $\nu^\omega|_M = \mu^\omega|_M$ which completes the proof.

Theorem 5.29 (T is NIP)**.** *Let μ be finitely satisfiable in a countable model M. Then, there exists a sequence $(\overline{a})_{i\in\omega}$ of elements, each in $(M^x)^{<\omega}$, such that for any $\theta(x) \in \mathcal{L}_x(\mathcal{U})$, we have that,*

$$\lim_{i\to\infty} \text{Av}(\overline{a}_i)(\theta(x)) = \mu(\theta(x)).$$

Proof. Choose a smooth sequence (N_i, μ_i) in μ over M. By Proposition 5.28 we may let N_ω be a countable model such that N_ω contains each N_i. We begin by constructing a sequence of tuples of elements in $(M^x)^{<\omega}$ such that $\text{Av}(\overline{a}_i)_{i\in\omega}$ converges to $\mu|_{N_\omega}$ in

$\mathfrak{M}_x(N_\omega)$. Since N_ω is countable, we let $(\theta_i(x))_{i\in\omega}$ be an enumeration of the formulas in $\mathcal{L}_x(N_\omega)$. Since μ is finitely satisfiable in M, we can find we find $\overline{a}_k \in (M)^{<\omega}$ such that for any $j \leq k$, we have that,

$$|\mu(\theta_j(x)) - \mathrm{Av}(\overline{a}_k)(\theta_j(x))| < \frac{1}{k}.$$

By construction, it is clear that the sequence $(\overline{a}_i)_{i\in\omega}$ converges to $\mu|_{N_\omega}$ in $\mathfrak{M}_x(N_\omega)$. Now, we let $(\overline{c}_i)_{i\in\omega}$ be an ϵ-almost indiscernible subsequence of $(\overline{a}_i)_{i\in\omega}$. Then, the sequence $(\overline{c}_i)_{i\in\omega}$ converges in $\mathfrak{M}_x(\mathcal{U})$ by Proposition 5.25. Assume that $(\overline{c}_i)_{i\in\omega}$ converges to some measure $\nu \in \mathfrak{M}_x(\mathcal{U})$. Hence, ν is finitely satisfiable in M by Proposition 5.7 and therefore ν is invariant over M. We now show that $\nu^\omega|_M = \mu^\omega|_M$. This will conclude the proof by 3 of Fact 5.24.

Since $(\overline{c}_i)_{i\in\omega}$ is a subsequence of $(\overline{a}_i)_{i\in\omega}$, it follows immediately that $\nu|_{N_\omega} = \mu|_{N_\omega}$ and therefore $\nu|_M = \mu|_M$. Now we proceed by induction on n. We assume that $\nu^n|_M = \mu^n|_M$. Fix $\varphi(x_0, ..., x_{n+1})$ in $\mathcal{L}(M)$. Let $\lambda = \bigotimes_{i=0}^n \mu_i$. Then, by the Proposition 5.27, $\mu^n|_M = \lambda|_M$. We let $\overline{x} = (x_0, ..., x_n)$. Each step in the following computation follows from either our induction hypothesis, base case, Proposition 5.27, Fact 2.34 or changing our space of integration.

$$\nu \otimes \nu^n(\varphi(x_0, ..., x_{n+1})) = \int_{S_{\overline{x}}(M)} F_\nu^\varphi d(\nu^n|_M) = \int_{S_{\overline{x}}(M)} F_\nu^\varphi d(\mu^n|_M)$$

$$= \int_{S_{\overline{x}}(M)} F_\nu^\varphi d(\lambda|_M) = \int_{S_{\overline{x}}(N_\omega)} F_\nu^\varphi d(\lambda|_{N_\omega}) = \int_{S_{x_{n+1}}(N_\omega)} F_\lambda^{\varphi^*} d(\nu|_{N_\omega})$$

$$= \int_{S_{x_{n+1}}(N_\omega)} F_\lambda^{\varphi^*} d(\mu|_{N_\omega}) = \int_{S_{\overline{x}}(N_\omega)} F_\mu^\varphi d(\lambda|_{N_\omega}) = \int_{S_{\overline{x}}(M)} F_\mu^\varphi d(\lambda|_M)$$

$$= \int_{S_{\overline{x}}(M)} F_\mu^\varphi d(\mu^n|_M) = \mu \otimes \mu^n(\varphi(x_0, ..., x_{n+1})).$$

□

We now observe that we have another proof of the theorem that global measures in NIP theories which are definable and finitely satisfiable are also finitely approximated.

Corollary 5.30. *If T is a countable NIP theory and μ is* dfs *over M, then μ is finitely approximated over M.*

Proof. By Proposition 2.35, μ is dfs over a countable model, M_0. By the previous result, μ is sequentially approximated in M_0. Since μ is also definable, an application of Proposition 5.8 yields the result. □

Observation 5.31. Assume that $\mu \in \mathfrak{M}_x(\mathcal{U})$ and let M be a small elementary submodel. Then, μ is sequentially approximated in M if:

1. T is stable, and μ is invariant over M.
2. T is NIP, $|M| = \aleph_0$, and μ is finitely satisfiable in M.
3. μ is finitely approximated in M.

5.4 Examples

In this section, we exhibit some concrete examples of types which are not strongly sequentially approximated. We begin by describing a type in an NIP theory which is finitely satisfiable in a small model but not strongly sequentially approximated (and it's associated Keisler measure is not sequentially approximated). Before we begin, we remind the reader of the following observation: If a sequence in a topological space converges, then every every subsequence converges (to the same limit).

Proposition 5.32. *Let $M = (\omega_1; <)$ with the usual ordering and let $T_<$ be the theory of M in the language $\{<\}$. Recall that $T_<$ is NIP. Let $p \in S_x(\omega_1)$ be any complete type extending $\{\alpha < x : \alpha < \omega_1\}$. Let $\mathcal{U}$ be a monster model of $T_<$ such that $M \prec \mathcal{U}$ and let $p_* \in S_x(\mathcal{U})$ be any global coheir of p. Then, p_* is not strongly sequentially approximated.*

Proof. Assume for contradiction that p_* is strongly sequentially approximated over some model N. Then there exists a sequence of points $(b_i)_{i \in \omega}$ in N such that $\lim_{i \to \infty} \mathrm{tp}(b_i/\mathcal{U}) = p_*$ in $S_x(\mathcal{U})$. By Ramsey's theorem, there is either an infinite strictly increasing or an infinite strictly decreasing subsequence. Assume that there is an infinite increasing subsequence, $(c_i)_{i \in \omega}$ of $(b_i)_{i \in \omega}$. Notice that for every c_i, it must be the case that $c_i < x \in p_*$. Since p_* is a coheir of p, p_* is finitely satisfiable in ω_1. So, for each c_i there exists α in ω_1 such that $c_i < \alpha$. Now, for each c_i, we let $\alpha_i = \min\{\alpha \in \omega_1 : \mathcal{U} \models c_i < \alpha\}$. Since ω_1 is well-ordered, α_i is well-defined. Now, we let β be the supremum (in ω_1) of $\{\alpha_i : i \in \omega\}$. Then, for each i we have that $c_i < \alpha_i < \beta$. Moreover, we know that $\beta < x \in p$ and so $\beta < x \in p_*$. Now, we consider the formula $\psi(x) = x < \beta$. For each $i \in \omega$, we have that $\models \psi(c_i)$, and so there exists some N, such that for every $n > N$, $\models \psi(b_i)$. This is a contradiction since $\neg\psi(x) \in p_*$.

Now we assume that $(c_i)_{i \in \omega}$ is a decreasing subsequence. Notice that for each i, $c_i > x \in p_*$. By saturation, there exists c_∞ such that $c_\infty < c_i$ for all i and $c_\infty > x \in p_*$. Setting $\psi(x) = c_\infty < x$ we see again that there exists some natural number N such that for every $n > N$, $\models \psi(b_i)$. This again is a contradiction since $\neg\psi(x) \in p_*$. □

Proposition 5.33. *Let p_* be as in Proposition 5.32. Then the associated Keisler measure δ_{p_*} is not sequentially approximated.*

Proof. Clear from 4 of Proposition 5.7. □

The following example was demonstrated to us in conversations with Gabriel Conant.

Proposition 5.34. *Recall Theorem 4.21. Let T_s^2 be the theory of the random K_s-free graph. Let p be the unique global complete type extending the formulas $\{\neg E(x,b) : b \in \mathcal{U}\}$. Then, δ_p is sequentially approximated (even finitely satisfiable) but p is not* *strongly sequentially approximated. Moreover, T_s^2 admits no (non-realized) strongly sequentially approximated types.*

Proof. The proof that δ_p is finitely approximated can be found in Theorem 4.21. By 3 of Proposition 5.7, δ_p is sequentially approximated. By 5 of Proposition 5.7, it suffices to show that there are no non-realized types in one variable which are strongly sequentially approximated. Let p be any type in $S_x(\mathcal{U})$ and assume that $(b_i)_{i\in\omega}$ is a sequence of points in M^x converging to p. Since p is non-realized, we may assume that the points in $(b_i)_{i\in\omega}$ are distinct. Then, by Ramsey's theorem, there is a subsequence which is either independent or complete. It cannot be complete, because that would violate K_s-freeness. Therefore, $(b_i)_{i\in\omega}$ contains an independent subsequence, call it $(c_i)_{i\in\omega}$. By compactness, there exists a $a \in \mathcal{U}$ such that $\models \varphi(c_i, a)$ for every even i. Then, $(c_i)_{i\in\omega}$ does not converge in $S_x(\mathcal{U})$ and so $(b_i)_{i\in\omega}$ does not converge in $S_x(\mathcal{U})$. □

Proposition 5.35. *Let $\mathcal{U} \models T^*_{feq2}$ and $q \in S_2(\mathcal{U})$ (see Corollary 4.18). Then, δ_q is finitely approximated, but not strongly sequentially approximated.*

Proof. Again, see Corollary 4.18 to see why δ_q is finitely approximated. Assume that q is strongly sequentially approximated over M. Then, there exists a sequence $(\bar{c}_i)_{i\in\omega} = ((a_i^1, a_i^2))_{i\in\omega}$ such that each $(a_i^1, a_i^2) \in O(M) \times O(M)$ and $(\bar{c}_i)_{i\in\omega}$ converges to q in $S_x(\mathcal{U})$. Consider the formula $\varphi(x_1, x_2, y) = \neg E(x_1, x_2, y)$. Notice for every $b \in P(\mathcal{U})$, $\varphi(x_1, x_2; b) \in p$.

Case 1: There exists an infinite disjoint subsequence, i.e. there exists a subsequence $(\bar{c}'_i)_{i\in\omega}$ of $(\bar{c}_i)_{i\in\omega}$ such that $\{a_j^{1'}, a_j^{2'}\} \cap \{a_k^{1'}, a_k^{2'}\} = \emptyset$. Then, there exists $b \in P(\mathcal{U})$ such that $\models E(a_j^{1'}, a_j^{2'}, b)$ for every $\bar{c}'_j$ in $(\bar{c}'_i)_{i\in\omega}$. Then $(\bar{c}'_i)_{i\in\omega}$ does not converge to q and so $(\bar{c})_{i\in\omega}$ cannot converge to q.

Case 2: There does not exist an infinite disjoint subsequence. Then, there exists a finite set $B \subset O(M)$ such that for each $\bar{c}_i$, $\{a_i^1, a_i^2\} \cap B \neq \emptyset$. By the infinite pigeon hole principle, there exists an infinite subsequence $(\overline{c}'_i)_{i\in\omega}$ of $(\overline{c}_i)$ and an element $d \in B$ such that for each $\overline{c}'_j$, $d \in \{a_j^{1'}, a_j^{2'}\}$. So, consider the formula $\theta(x_1, x_2) \equiv x_1 = d \vee x_2 = d$. Then, for each i in our subsequence $\mathcal{U} \models \theta(\overline{c}'_i)$ and so $(\overline{c}'_i)_{i\in\omega}$ does not converge to q in $S_2(\mathcal{U})$ (since clearly $\neg\theta(x_1, x_2) \in q$). Since $(\overline{c}'_i)_{i\in\omega}$ does not converge, neither does $(\overline{c}_i)_{i\in\omega}$. □

We leave the following exercise for the curious reader.

Exercise 5.36. Let T_R be the theory of the Random Graph and let $\mathcal{U}$ a monster model of T_R. Then $S_x(\mathcal{U})$ has no strongly sequentially approximated types and $\mathfrak{M}_x(\mathcal{U})$ no sequentially approximated measures.

CHAPTER 6

CONVOLUTION ALGEBRAS OF KEISLER MEASURES

This chapter is joint work with Artem Chernikov and is a modified portion of our preprint *Definable convolution and idempotent Keisler measures* which is currently in preparation [6]. The connection between model theory and topological dynamics began with the work of Newelski [45, 47]. Newelski introduced various notions and ideas from topological dynamics into the model-theoretic study of definable groups. A fundamental observation of this research is that certain spaces of types over a definable group natrually carry the algebraic structure of a compact left-continuous semigroup (see Fact 6.11). In a broad sense, this operation on types (over a definable group) can be extended to a large class of Keisler measures (over the same group), where this operation corresponds to *convolution*.

Before moving to the model theory context, let us first recall the classical setting. Let G be a locally-compact topological group. The space of bounded, regular, Borel probability measures on G carries an algebraic structure, namely the convolution operation. If μ and ν are measures on G, then the convolution product of these two measures is defined via:

$$\mu \star \nu(A) = \int_{y \in G} \int_{x \in G} \chi_A(x \cdot y) d\mu(x) d\nu(y),$$

where A is any Borel subset of G and χ_A is the characteristic function of A. Moreover, we say that a measure μ is idempotent if $\mu \star \mu = \mu$. The first major connection between the convolution product and topological group theory begins with the following theorem of Wendel [67, Theorem 1].

Theorem 6.1 (Wendel)**.** *Let G be a compact topological group and μ a regular Borel probability measure on G. Then μ is idempotent if and only if the support of μ is a closed subgroup of G and the restriction of μ to this subgroup is the unique (bi-invariant) normed Haar measure.*

Conceptually, this result links the existence of an idempotent measure on a compact group with the existence of an algebraic substructure of the same group. We remark that this line of research was extended to the class of locally compact abelian groups by Rudin [55] and Cohen [12]. Fortunately for model theory, individuals such as Glicksberg [28, 29] and Pym [52, 53] continued this research into the category of (semi-)topological semigroups. With these historical connections in mind, we investigate the algebraic structure of Keisler measures under *definable convolution*. The following three questions should act as a guide for this chapter.

1. Under which conditions can the convolution product of two Keisler measures be defined?
2. What structural properties arise from the existence of an idempotent Keisler measure?
3. Is there a connection between convolution algebras and Ellis semigroups?

In Section 6.1, we extend the usual product $\otimes$ on Borel-definable measures to a slight larger context. We will define the product $\tilde{\otimes}$ which only requires the fiber function of the measure on the left hand side of the product to be Borel when *restricted to the support of the right hand side* (Definition 6.3). We will see that this product both extends the usual product on invariant types (see Definition 1.2 for definition) and well as the product on Borel-definable measures (see Proposition 6.5 for the proofs of both claims). In response to Question 1, we define the convolution product on $*$-*Borel pairs* of Keisler measures in terms of $\tilde{\otimes}$ (Definition 6.9). We then observe some basic properties, e.g. this convolution product extends the coheir product studied by Newelski and others (Proposition 6.11).

Following Wendel, the expectation is that there is a connection (in tame contexts) between idempotent measures and group objects in the definable category. In the Section 6.2, we investigate idempotent Keisler measures (Definition 6.15). We observe that right invariant measures of type-definable subgroup are idempotent (Proposition 6.17). We remark that to show this result, we need to make use of the extended notion of product, namely $\tilde{\otimes}$ (it is not clear, a priori, that right invariant Keisler measures are Borel-definable). As in Wendel's proof, we shift our focus to investigating the supports of idempotent Keisler measures. In general, these remain mysterious and so we restrict ourselves to the dfs context. We prove that the support of an idempotent dfs Keisler measures equipped with the usual coheir product, i.e. $(\sup(\mu), *)$, is a compact, left-continuous semigroup with no proper closed two-sided ideals (Corollary 6.23 and Theorem 6.28). Through understanding the support of idempotent dfs measures, we are able to answer Question 2 in the context of stable groups and give a complete classification akin to Wendel's work. In the stable context, we prove that idempotent Keisler measures are in unique correspondence with type-definable subgroups. More precisely, a measure μ is idempotent if and only if μ is the unique invariant Keisler measures concentrating on its (type-definable) stabilizer (Theorem 6.42).

In the final section, we respond to Question 3. Definable group actions in the NIP context where studied extensively in [9] and [7]. In section 6.3, we demonstrate a concrete connection between Ellis semigroups and our definable convolution semigroup in the NIP context. Newelski observed that the semigroup $(S_x(\mathcal{G}, G), *)$ on the space of global types finitely satisfiable in a small model $G \prec \mathcal{G}$ (with the standard coheir product) is isomorphic to the *Ellis semigroup* $\mathrm{E}(S_x(\mathcal{G}, G), G)$ where G acts on $S_x(\mathcal{G}, G)$ [47]. In the NIP setting, we prove an analogous result for Keisler measures.

The convex hull of Dirac measures concentrating on points in G, $\mathrm{conv}(G)$, acts naturally on the space $\mathfrak{M}_x(\mathcal{G},G)$. We prove that the Ellis semigroup of this action is isomorphic to the convolution semigroup $\left(\mathfrak{M}_x(\mathcal{G},G),*\right)$ (Theorem 6.51).

6.1 Definable convolution

6.1.1 Products revisited

In this subsection, we slightly generalize our standard notion of product ($\otimes$) so that we fix a technical issue which arrises later when we define our convolution product. Consider the following scenario: suppose that $\mathcal{G}$ is a monster model of a group and $\mathcal{H}$ is a type-definable subgroup. If $\mathcal{H}$ admits a right-invariant measure μ, then one can intuitively compute the convolution of μ with itself. Namely, the convolution product of μ with itself *should be equal to* μ. While it is not obvious that measures of this kind are Borel-definable, this computation can be realized by slightly tweaking the domain of the integral in Definition 2.23. Moreover, this *newish* notion of product (which we denote as $\tilde{\otimes}$) extends both the Morley product of invariant types and the standard product on Borel-definable measures (Proposition 6.5).

Definition 6.2. Let $\mu \in \mathfrak{M}_x(\mathcal{U})$, $\nu \in \mathfrak{M}_y(\mathcal{U})$, and $\varphi(x;y) \in \mathcal{L}_{xy}(\mathcal{U})$. Then, we say that the triple (μ,ν,φ) is Borel if there exists $N \prec \mathcal{U}$ such that;

1. N contains all the parameters from $\varphi(x;y)$.
2. For any $q \in \sup(\nu|_N)$ and $d, d' \in \mathcal{U}^y$ where $d, d' \models q$, we have that $\mu(\varphi(x;d)) = \mu(\varphi(x;d'))$. Intuitively, μ is invariant over the support of $\nu|_N$.
3. The map $F^{\varphi}_{\mu,N} : \sup(\nu|_N) \to [0,1]$ is Borel (with respect to the topological space $\sup(\nu|_N)$), where $F^{\varphi}_{\mu,N}(q) = \mu(\varphi(x;d))$ and d is some/any realization of q. This is well defined by requirement 2.

Moreover, if N satisfies the hypothesis above for the triple (μ,ν,φ), then we say that (μ,ν,φ) is Borel over N.

Definition 6.3. Let $\mu \in \mathfrak{M}_x(\mathcal{U})$, $\nu \in \mathfrak{M}_y(\mathcal{U})$, and $\varphi(x;y) \in \mathcal{L}_{xy}(\mathcal{U})$. Assume that (μ, ν, φ) is Borel. Then, we compute the product as follows;

$$\mu \tilde{\otimes} \nu(\varphi(x;y)) = \int_{\sup(\nu|_N)} F_{\mu,N}^{\varphi} d\nu_N,$$

where (μ, ν, φ) is Borel over N and the measure ν_N is the restriction of the regular Borel measure $\nu|_N$ on $S_y(N)$ to the compact subset $\sup(\nu|_N)$. As usual, when there is no possibility for confusion, we will write $F_{\mu,N}^{\varphi}$ simply as F_{μ}^{φ}.

We check that our definition is well-defined.

Proposition 6.4. *Assume that (μ, ν, φ) is Borel. Then, $\mu \tilde{\otimes} \nu(\varphi(x;y))$ does not depend on the choice of N.*

Proof. This proof is practically identical to the method used in [59, Proposition 7.19]. Assume that (μ, ν, φ) is Borel over M and N. It suffices assume that $M \subseteq N$ since we may always choose a common extension. Let $r : \sup(\nu|_N) \to \sup(\nu|_M)$ be the natural restriction map. Then $F_{\mu,M}^{\varphi} \circ r = F_{\mu,N}^{\varphi}$ and the pushforward of the measure ν_N, namely $r_*(\nu_N)$, is equal to ν_M by Remark 2.12. Hence, the following sequence of equations hold:

$$\int_{\sup(\nu|_M)} F_{\mu,M}^{\varphi} d\nu_M = \int_{\sup(\nu|_M)} F_{\mu,M}^{\varphi} dr_*(\nu_N)$$

$$= \int_{\sup(\nu|_N)} \left(F_{\mu,M}^{\varphi} \circ r\right) d\nu_N = \int_{\sup(\nu|_N)} F_{\mu,N}^{\varphi} d\nu_N.$$

□

Proposition 6.5. *The product $\tilde{\otimes}$ extends both the Morely product on invariant types as well as the product of Borel-definable Keisler measures. In particular,*

1. *Let $p \in S_x(\mathcal{U})$ and $q \in S_y(\mathcal{U})$. Assume that p is invariant. Then for any formula $\varphi(x;y) \in \mathcal{L}_{xy}(\mathcal{U})$, we have that $\varphi(x;y) \in p \otimes q$ if and only if $\delta_p \tilde{\otimes} \delta_q(\varphi(x;y)) = 1$. In other words, $\delta_{p \otimes q} = \delta_p \tilde{\otimes} \delta_q$.*

2. *Let* $\mu \in \mathfrak{M}_x(\mathcal{U})$ *and* $\nu \in \mathfrak{M}_y(\mathcal{U})$*. Assume that* μ *is Borel-definable. Then for any formula* $\varphi(x;y) \in \mathcal{L}_{xy}(\mathcal{U})$*, we have that* $\mu \otimes \nu(\varphi(x;y)) = \mu \tilde{\otimes} \nu(\varphi(x;y))$*. In other words,* $\mu \otimes \nu = \mu \tilde{\otimes} \nu$*.*

Proof. We first prove statement 1. Fix a $\mathcal{L}(\mathcal{U})$-formula $\varphi(x;y)$ and assume that p is invariant over N where N contains all the parameters from φ. Since q is a type, $\sup(\delta_q|_N)$ is a single point. Therefore, any map from this space to the reals is Borel. Let $b \in \mathcal{U}^y$ and $b \models q|_N$. Then

$$\delta_p \tilde{\otimes} \delta_q(\varphi(x;y)) = \int_{\sup(\delta_q|_N)} F_{\delta_p}^{\varphi} d(\delta_q)_N = F_{\delta_p}^{\varphi}(q|_N) = \begin{cases} 1 & \varphi(x;b) \in p, \\ 0 & \neg\varphi(x;b) \in p. \end{cases}$$

By above, $\delta_p \tilde{\otimes} \delta_q(\varphi(x;y)) = 1$ if and only if $\varphi(x;y) \in \operatorname{tp}(a,b/M)$ where $b \models q$ and $a \models q|_{Nb}$.

Now we prove statement 2. This follows from the fact that integrals in our context depend only on their support. In particular,

$$\mu \otimes \nu(\varphi(x;y)) = \int_{S_y(N)} F_{\mu}^{\varphi} d(\nu|_N) = \int_{\sup(\nu|_N)} F_{\mu}^{\varphi} d\nu_N = \mu \tilde{\otimes} \nu(\varphi(x;y)).$$

□

Remark 6.6. To be pedantic, let $p \in S_x(\mathcal{U})$ and $q \in S_y(\mathcal{U})$ and assume p is invariant (say, over N). Then, the product $\delta_p \otimes \delta_p$ is not always well-defined since the fiber maps of the form $F_{\delta_p}^{\varphi} : S_y(N) \to [0,1]$ may not be Borel. However, $F_{\delta_p}^{\varphi}|_{\sup(\delta_q|_N)}$ is a Borel map from $\sup(\delta_q|_N)$ to $\mathbb{R}$ since $\sup(\delta_q|_N)$ is a single point.

Notation 6.7. For the rest of the chapter, we will identify $\tilde{\otimes}$ with $\otimes$.

6.1.2 Basic properties of convolution

Throughout the rest of this chapter, we let $\mathcal{L} = \{e, \cdot, ...\}$ be a language extending the language of groups and T be an $\mathcal{L}$-theory where 'e' is interpreted as the identity and '$\cdot$' as multiplication. We let $\mathcal{G}$ be a sufficiently saturated model of T and we now use the letters 'x, y' to denote tuples of variables of length 1. We begin by introducing our convolution product, fittingly called the *definable convolution*. Similar to the classical case, this operation takes in pairs of measures in $\mathfrak{M}_x(\mathcal{G})$ and returns a measure in the same space.

Notation 6.8. For any formula $\varphi(x) \in \mathcal{L}_x(\mathcal{G})$, we let $\varphi'(x;y) = \varphi(x \cdot y)$.

Definition 6.9. Suppose $\mu, \nu \in \mathfrak{M}_x(\mathcal{G})$ and let ν_y denote the measure in $\mathfrak{M}_y(\mathcal{U})$ such that for any $\psi(y) \in \mathcal{L}_y(\mathcal{U})$, $\nu_y(\psi(y)) = \nu(\psi(x))$.

1. We say that (μ, ν) is **$*$-Borel** if for every formula $\varphi(x) \in \mathcal{L}_x(\mathcal{U})$, the triple (μ, ν_y, φ') is Borel.
2. If (μ, ν) is $*$-Borel, we define the **definable convolution product of μ with** ν as follows,

$$\mu * \nu(\varphi(x)) = \mu \tilde{\otimes} \nu_y(\varphi'(x;y)) = \int_{\sup(\nu_y|_G)} F_{\mu,G}^{\varphi'} d\nu_G(y),$$

   where $\varphi(x)$ is any formula in $\mathcal{L}_x(\mathcal{U})$, G is some/any small submodel of $\mathcal{G}$ such that (μ, ν, φ) is Borel over G, and $\nu_G(y)$ is the Borel measure ν_y on $\sup(\nu_y|_G)$ (as in Definition 6.3). We will usually write this product simply as $\int_{\sup(\nu|_G)} F_{\mu}^{\varphi'} d\nu_G$ or even $\int F_{\mu}^{\varphi'} d\nu$ when there is no possibility of confusion.

While it is tradition to give privilege to left invariant measures, and one can certainly define convolution to give guarantee that left invariant measures are $*$-Borel, we will see that our definition correctly extends Newelski's notion of product. But first, we need to check that our operation returns a measure.

Proposition 6.10. *Let $\mu, \nu \in \mathfrak{M}_x(\mathcal{G})$. If (μ, ν) is $*$-Borel, then $\mu * \nu$ is a Keisler measure.*

Proof. It is easy to check that $\mu * \nu(x = x) = 1$ and $\mu * \nu(\neg\varphi(x)) = 1 - \mu * \nu(\varphi(x))$. Fix $\psi_1(x), \psi_2(x) \in \mathcal{L}_x(\mathcal{G})$ such that $\psi_1(x) \wedge \psi_2(x) = 0$. To demonstrate that $\mu * \nu$ is a Keisler measure, it suffices to check that

$$\mu * \nu(\psi_1(x) \vee \psi_2(x)) = \mu * \nu(\psi_1(x)) + \mu * \nu(\psi_2(x)).$$

Let $\theta(x; y) = \psi_1(x \cdot y) \vee \psi_2(x \cdot y)$ and let G be a small model such that (μ, ν, ρ) is Borel over G for $\rho \in \{\psi_1, \psi_2, \theta\}$. Then for any $q \in \sup(\nu|_G)$, $F_\mu^\theta(q) = \mu(\theta(x; b)) = \mu(\psi_1(x \cdot b) \vee \psi_2(x \cdot b))$ where $b \models q$. Since $\psi_1(x) \wedge \psi_2(x) = \emptyset$, we have that their translates are also empty. Therefore,

$$F_\mu^\theta(q) = \mu(\theta(x; b)) = \mu(\psi_1(x \cdot b)) + \mu(\psi_2(x \cdot b)) = F_\mu^{\psi_1'}(q) + F_\mu^{\psi_2'}(q).$$

By linearity of integration and the observation that our fiber functions only return positive values,

$$(\mu * \nu)(\psi_1(x) \vee \psi_2(x)) = \int_{\sup(\nu|_G)} F_\mu^\theta d\nu_G = \int_{\sup(\nu|_G)} F_\mu^{\psi_1'} + F_\mu^{\psi_2'} d\nu_G$$

$$= \int_{\sup(\nu|_G)} F_\mu^{\psi_1'} d\nu_G + \int_{\sup(\nu|_G)} F_\mu^{\psi_2'} d\nu_G = (\mu * \nu)(\psi_1(x)) + (\mu * \nu)(\psi_2(x)).$$

□

We now take the opportunity to demonstrate that this notion of convolution extends the coheir product extensively studied by Newelski [45, 47] and other from the point of view of topological dynamics (see e.g. [49] for the following fact).

Fact 6.11. *Fix $G \prec \mathcal{G}$. Given $p, q \in S_x(\mathcal{G}, G)$, we define the operation $p * q := \mathrm{tp}(a \cdot b/\mathcal{G}) \in S_x(\mathcal{G}, G)$, for some/any $(a, b) \models p \otimes q$. Then, $(S_x(\mathcal{G}, G), *)$ is a left continuous semigroup: for each $q \in S_x(\mathcal{G}, G)$, the map $- * q : S_x(\mathcal{G}, G) \to S_x(\mathcal{G}, G)$ is continuous.*

Proposition 6.12. *Let $\delta : S_x(\mathcal{G}, G) \to \mathfrak{M}_x(\mathcal{G}, G)$ via $\delta(p) = \delta_p$. Then, δ is a topological embedding and for any $\varphi(x) \in \mathcal{L}_x(\mathcal{G})$, we have that $\varphi(x) \in p * q$ if and* *only if* $\delta_p * \delta_q(\varphi(x)) = 1$. *In other words,* $\delta_{p*q} = \delta_p * \delta_q$.

Proof. It can be easily checked that this map is an embedding. Fix $\varphi(x) \in \mathcal{L}_x(\mathcal{U})$. By Proposition 6.5, we have the following equalities:

$$\delta_{p*q}(\varphi(x)) = \delta_{p_x \otimes q_y}(\varphi(x \cdot y)) = \delta_{p_x} \tilde{\otimes} \delta_{q_y}(\varphi(x \cdot y)) = \delta_p * \delta_q(\varphi(x)).$$

□

The following computations are straight forward and left to the reader as an exercise. These facts demonstrate that definable convolution behaves reasonably.

Fact 6.13. *Let* $\mu, \mu_1, ..., \mu_n, \nu_1, ..., \nu_m \in \mathfrak{M}_x(\mathcal{G})$, *and assume the pairs* (μ_i, ν_j) *are* $*$-*Borel for* $i \leq n$ *and* $j \leq m$. *Let* $a, b, a_1, ..., a_n \in \mathcal{G}$, *and* $r_1, ..., r_n, s_1, ..., s_m \in \mathbb{R}_{\geq 0}$ *such that* $\sum_{i=1}^{n} r_i = \sum_{j=1}^{m} s_j = 1$. *Then, the following hold.*

1. $\mu * \delta_e = \delta_e * \mu = \mu$.
2. $\delta_a * \delta_b = \delta_{ab}$.
3. *For any formula* $\varphi(x) \in \mathcal{L}_x(\mathcal{U})$, *we have* $(\delta_a * \mu)(\varphi(x)) = \mu(\varphi(a \cdot x))$.
4. $\left(\sum_{i=1}^{n} r_i\mu_i\right) * \left(\sum_{j=1}^{m} s_j\nu_j\right) = \sum_{i,j}^{n,m} r_i \cdot s_j(\mu_i * \nu_j)$.
5. *For any formula* $\varphi(x) \in \mathcal{L}_x(\mathcal{U})$, *we have*

$$\Big(\Big(\sum_{i=1}^{n} r_i\delta_{a_i}\Big) * \mu\Big)(\varphi(x)) = \sum_{i=1}^{n} r_i\mu(\varphi(a_i \cdot x)).$$

We now observe that the following model theoretic properties are preserved under convolution. Most of the following propositions are slight variations of proofs already found in this dissertation.

Proposition 6.14. *Let* $\mu, \nu \in \mathfrak{M}_x(\mathcal{G})$. *Assume that* (μ, ν) *is* $*$-*Borel. Then,*

1. *If* μ, ν *are definable over* G, *then* $\mu * \nu$ *is definable over* G.
2. *If* μ, ν *are finitely satisfiable in* G, *then* $\mu * \nu$ *is finitely satisfiable in* G.

3. *If* μ, ν *are finitely approximated over* G*, then* $\mu * \nu$ *is finitely approximated over* G*.*

4. *If* $\mu(x = b) = 0$ *for every* b *in* $\mathcal{G}$*, then* $\mu * \nu(x = b) = 0$ *for every* $b \in \mathcal{G}$*.*

Proof. The proofs of 1, 2, and 3 are slight variations on Proposition 2.24, Proposition 2.25, and Lemma 2.31, respectively. We only need to prove 4. Fix $b \in \mathcal{G}$ and let $\varphi(x) \equiv x = b$. Notice that;

$$\mu * \nu(x = b) = \mu \tilde{\otimes} \nu_y (x \cdot y = b) = \int_{\sup(\nu|_G)} F_{\mu}^{\varphi'} d\nu_G.$$

We have that $F_{\mu}^{\varphi'}(q) = \mu(x \cdot c = b)$ where $c \models q$. Then, $\mu(x \cdot c = b) = \mu(x = bc^{-1}) = 0$ by assumption. Therefore, $\int F_{\mu}^{\varphi'} d\nu = \int 0 d\nu_G = 0$. □

6.2 Idempotent measures

Idempotent measures are an important class of measures. In the compact group setting, the existence of an idempotent implies the existence of a closed subgroup (and vice versa). Glicksberg demonstrates this result also holds in the abelian semi-topological semigroup setting [29] (which is essentially the stable abelian group case). Here, we define idempotent Keisler measures. We then translate some of Glicksberg's results into the definable convolution setting. At the end of this section, we use these results to prove that in stable groups, idempotent measures are in unique correspondence with type-definable subgroups. We begin by giving a few measure-theoretic definitions in the group context. We continue working with a monster model $\mathcal{G}$ of a theory T expanding a group.

6.2.1 Basic facts and definitions

Definition 6.15. Let $\mu \in \mathfrak{M}_x(\mathcal{G})$.

1. We say that μ is $*$**-Borel** if (μ, μ) is $*$-Borel.

2. We say that μ is **idempotent** if μ is $*$-Borel and $\mu * \mu = \mu$.
3. We say that μ is **right-invariant** if for any formula $\varphi(x) \in \mathcal{L}_x(\mathcal{G})$ and any $a \in \mathcal{G}$, $\mu(\varphi(x)) = \mu(\varphi(x \cdot a))$.

Definition 6.16. Let $\mathcal{H}$ be a type-definable subgroup of $\mathcal{G}$, where $H(x)$ is the partial type defining the domain of $\mathcal{H}$. Then $\mathcal{H}$ is **definably amenable** if there exists a measure $\mu \in \mathfrak{M}_x(\mathcal{G})$ such that $\mu(H(x)) = 1$ and for any formula $\varphi(x) \in \mathcal{L}_x(\mathcal{G})$ and any $a \in \mathcal{H}$, $\mu(\varphi(x)) = \mu(\varphi(x \cdot a))$. If this is the case, we call μ **right $\mathcal{H}$-invariant**.

Proposition 6.17. *Let $\mathcal{H}$ be a type-definable definably amenable subgroup of $\mathcal{G}$, with definition $H(x)$. Suppose that $\mu \in \mathfrak{M}_x(\mathcal{G})$ is right $\mathcal{H}$-invariant. Then the pair (μ, μ) is $*$-Borel and μ is idempotent. Moreover, if ν is another measure such that $\nu(H(x)) = 1$, then (μ, ν) is $*$-Borel and $\mu * \nu = \mu$.*

Proof. We show that for any measure $\nu \in \mathfrak{M}_x(\mathcal{G})$ such that $\nu(H(x)) = 1$, (μ, ν) is $*$-Borel and $\mu * \nu = \mu$. For ease of notation, we will identify ν with ν_y. Fix a formula $\varphi(x)$ in $\mathcal{L}_x(\mathcal{G})$. Let G be an elementary submodel of $\mathcal{G}$ such that G contains the parameters from both in $H(x)$ and φ. Fix some $q \in \sup(\nu|_G) \subseteq S_y(G)$. We now observe that q is in $H(y)$ (i.e. $q \vdash H(y)$). Why? If not, then $q \in S_y(G) \backslash H(y)$. Since $H(y)$ is closed, $S_y(G) \backslash H(y)$ is open. Therefore, $S_x(G) \backslash H(y) = \bigvee_{i \in I} \psi_i(y)$. Then $\psi_i(y) \in q$ for some i and since $q \in \sup(\nu|_G)$, we know that $\nu(\psi_i(y)) > 0$. But this is a contradiction since $\nu(H(y)) = 1$ and $\psi_i(y)$ is disjoint from $H(y)$. Therefore, if $b \in \mathcal{G}$ and $b \models q$, then $b \in \mathcal{H}$. Now, we notice that the function $F_{\mu,G}^{\varphi'}$ is constant on $\sup(\nu|_G)$ since $F_{\mu,G}^{\varphi'}(q) = \mu(\varphi(x \cdot b)) = \mu(\varphi(x))$. Therefore, (μ, ν) is $*$-Borel. Now, we compute $\mu * \nu(\varphi(x))$;

$$\mu * \nu(\varphi(x)) = \int_{\sup(\nu|_G)} F_{\mu}^{\varphi'} d\nu_G = \int_{\sup(\nu|_G)} \mu(\varphi(x)) d\nu_G = \mu(\varphi(x)).$$

Notice that this implies that (μ, μ) is Borel and $\mu * \mu = \mu$. Therefore, μ is idempotent.

□

With the previous proposition in mind, we observe the following.

Observation 6.18. Let $\mathcal{H}$ be a type-definable subgroup of $\mathcal{G}$. The following classes of measures are idempotent:

1. δ_e is idempotent.
2. If $\mathcal{G}$ is definably amenable and μ is a right $\mathcal{G}$-invariant measure, then μ is idempotent.
3. If $\mathcal{G}$ is definable amenable, μ is the right invariant measure on $\mathcal{G}$, and $\mathcal{H}$ has finite index in $\mathcal{G}$, then $\mu_H(\varphi(x)) = [\mathcal{G} : \mathcal{H}] \cdot \mu(\varphi(x) \cap H(x))$ is idempotent.
4. If $\mathcal{H}$ is amenable as a discrete group and μ is a right invariant measure on $\mathcal{P}(\mathcal{H})$. Then, the Keisler measure $\mu_H(\varphi(x)) = \mu(\varphi(x) \cap \mathcal{H})$ is idempotent.

We now classify idempotent measures which concentrate on finite subsets of $\mathcal{G}$. Our proof follows from Theorem 1 of [67] (see Theorem 6.1) applied to finite groups. However, we expect there is a much more elementary proof of this result.

Proposition 6.19. *Let $\mathcal{H}$ be a finite subgroup of $\mathcal{G}$ and let μ_H be the Haar measure on this finite group, i.e. $\mu_H = \frac{1}{|\mathcal{H}|}\sum_{a \in H} \delta_a$. Then, μ_H is idempotent. Moreover, if μ is any idempotent measure whose support is a finite collection of realized types, then $\mu = \frac{1}{|H|}\sum_{a \in H} \delta_a$ for some finite subgroup H of $\mathcal{G}$.*

Proof. If $\mu_H = \frac{1}{|H|}\sum_{a \in \mathcal{H}} \delta_a$, then

$$\mu_H * \mu_H = \Big(\frac{1}{|H|}\sum_{a \in \mathcal{H}} \delta_a\Big) * \Big(\frac{1}{|H|}\sum_{a \in \mathcal{H}} \delta_a\Big) = \frac{1}{|H|^2} \sum_{(a,b) \in \mathcal{H} \times \mathcal{H}} \delta_{ab}$$

$$= \frac{1}{|H|^2}|\mathcal{H}| \sum_{c \in \mathcal{H}} \delta_c = \frac{1}{|H|}\sum_{c \in \mathcal{H}} \delta_c = \mu_H$$

The other direction is elementary modulo Wendel's Theorem 1 applied to finite groups. Assume that $\sup(\mu) = \{a_1, ..., a_n\} = A$. Notice that if μ is idempotent, then $\sup(\mu)$ is closed under multiplication. If not, then there exists $c \in \mathcal{G}$ such that $c = a_i \cdot a_j$ and c is not in A. Notice that $\mu(x = c) = 0$, but $\mu * \mu(x = c) > 0$. Therefore, we may assume A is closed under products. Furthermore, any finite subset of a group closed under products is itself a group. If $|A| = n$, then for any $a_i \in A$, we claim that $\{a_i, a_i^2 ..., a_i^{n+1}\}$ contains a_i^{-1}. Therefore, A is a compact group and $\mu|_A$ is an idempotent measure on A. By Theorem 6.1, $\mu|_A$ is the unique Haar measure on a subgroup of A. Since $\sup(\mu) = A$, we may conclude that $\mu = \frac{1}{n}\sum_{a \in A} \delta_a$. □

Finally, we show that in the NIP abelian context, the class of idempotent dfs measures is preserved under definable convolution.

Proposition 6.20. *Assume that T is abelian and NIP. Assume that $\mu, \nu \in \mathfrak{M}_x(\mathcal{G})$ and both μ, ν are* dfs *and idempotent. Then, $\mu * \nu$ is* dfs *and idempotent.*

Proof. By Proposition 6.14, $\mu * \nu$ is dfs. Fix a formula $\varphi(x) \in \mathcal{L}(\mathcal{G})$ and fix $G \prec \mathcal{G}$ such that G contains all the parameters from $\varphi(x)$ and both μ, ν are dfs over G. By NIP, μ, ν are finitely approximated and so their products commute (see Theorem 2.39 and Proposition 2.32). Hence, we have the following

$$\mu * \nu(\varphi(x)) = \mu_x \otimes \nu_y(\varphi(x \cdot y)) = \nu_y \otimes \mu_x(\varphi(x \cdot y)).$$

By change of variables and commutativity, we can continue

$$= \nu_x \otimes \mu_y(\varphi(y \cdot x)) = \nu_x \otimes \mu_y(\varphi(x \cdot y)) = \nu * \mu(\varphi(x)).$$

Now, let $\lambda = \mu * \nu$. By associativity (similar to Proposition 2.41), we notice

$$\lambda * \lambda = \mu * \nu * \mu * \nu = \mu * \mu * \nu * \nu = \mu * \nu = \lambda.$$

Therefore, $\mu * \nu$ is dfs and idempotent. □

6.2.2 Supports and convolution

In the proof of Wendel's theorem (as well as Glicksberg's proof in the abelian semitopological semigroup setting [29]), an idempotent regular Borel measure μ is associated to a closed subgroup via the support of the measure. In particular, $\sup(\mu)$ is a closed group and $\mu|_{\sup(\mu)}$ is its associated (normed, bi-invariant) Haar measure. In the general model theory context, our situation is not as nice. We begin this section by exploring two concrete model theoretic examples of idempotent measures. In the next few sections, we will become increasingly interested in the semigroup $(\sup(\mu), *)$ where μ is a global dfs measures and $*$ is the usual coheir product restricted to the support. This is well defined since if μ is dfs, then every element in the support of μ is finitely satisfiable over a fixed small model. By adapting some of Glicksberg's work to our context, we will show that the support of an idempotent dfs measures has a relatively nice semigroup structure (i.e. compact, left-continuous, no closed two-sided ideals). We now consider the supports of a few idempotent Keisler measures.

Example 6.21. Let T_{doag} be the complete theory of an infinite divisible ordered abelian group in the language $\{+, <, 0, 1\}$. Let $\mathcal{G}$ be a monster model of T and consider $\mathbb{Q}$ as an elementary substructure in the natural way. Let p_∞ be the unique global coheir extending the type $\{x > a : a \in \mathbb{Q}\} \in S_x(\mathbb{Q})$. Similarly let $p_{-\infty}$ be the unique global coheir extending the type $\{x < a : a \in \mathbb{Q}\} \in S_x(\mathbb{Q})$. We let $\mu = \frac{1}{2}\delta_{p_{-\infty}} + \frac{1}{2}\delta_{p_\infty}$ and we claim that μ, δ_{p_∞}, and $\delta_{p_{-\infty}}$ are idempotent. By Proposition 6.14, the products $\delta_\alpha * \delta_\beta$ for $\alpha, \beta \in \{\infty, -\infty\}$ are finitely satisfiable in $\mathbb{Q}$. From this observation, and Fact 6.13 it is not difficult to demonstrate the following computation:

$$\mu * \mu = \Big(\frac{1}{2}\delta_{p_{-\infty}} + \frac{1}{2}\delta_{p_\infty}\Big) * \Big(\frac{1}{2}\delta_{p_{-\infty}} + \frac{1}{2}\delta_{p_\infty}\Big)$$

$$= \frac{1}{4}\Big(\delta_{p_{-\infty}} * \delta_{p_{-\infty}}\Big) + \frac{1}{4}\Big(\delta_{p_{-\infty}} * \delta_{p_\infty}\Big) + \frac{1}{4}\Big(\delta_{p_\infty} * \delta_{p_{-\infty}}\Big) + \frac{1}{4}\Big(\delta_{p_\infty} * \delta_{p_\infty}\Big)$$

$$= \frac{1}{4}\delta_{p_{-\infty}} + \frac{1}{4}\delta_{p_{\infty}} + \frac{1}{4}\delta_{p_{-\infty}} + \frac{1}{4}\delta_{p_{\infty}} = \frac{1}{2}\delta_{p_{-\infty}} + \frac{1}{2}\delta_{p_{\infty}} = \mu.$$

We observe that the semigroups $(\sup(\delta_{p_\infty}), *)$ and $(\sup(\delta_{p_{-\infty}}), *)$ are groups (with a single element) and $(\sup(\mu), *)$ is not a group since it does not contain an identity.

Example 6.22. Let $G = (S^1, \cdot, C(x, y, z))$ be the standard circle group over $\mathbb{R}$ with the clockwise ordering. Let T_O be the corresponding theory. Let μ be the Keisler measure on this structure which corresponds to the normed Haar measure on S^1. Let $\mathcal{G}$ be a monster model of T_O such that $G \prec \mathcal{G}$. We know that μ is smooth over S^1 and admits a unique global extension $\tilde{\mu}$. We remark that $\tilde{\mu}$ is right invariant and so $\tilde{\mu}$ is idempotent (Proposition 6.17). Let $\mathrm{St} : S_x(\mathcal{G}) \to S^1$ be the standard part map and assume that $p \in \sup(\tilde{\mu})$ and $\mathrm{St}(p) = a$. Then $\varphi_\xi := C(a - \xi, x, a + \xi) \notin p$ for every infinitesimal $\xi \in \mathcal{G}$ (follows directly from the fact that p is finitely satisifable in S^1). As the types are determined by their cut in the circular order, it follows that for every $a \in S^1$ there are exactly two types $a_+(x), a_-(x) \in \sup(\mu)$ such that $\mathrm{St}(a_+(x)) = \mathrm{St}(a_-(x)) = a$. These types are determined by whether $C(a + \xi, x, b)$ holds for every infinitesimal ξ and element $b \in S^1$ or $C(b, x, a - \xi)$ holds for every infinitesimal ξ and element $b \in S^1$, respectively. It follows that $(\sup(\tilde{\mu}), *) \cong S^1 \times \{+, -\}$ with multiplication defined as follows;

$$a_\delta(x) * b_\epsilon(x) = (a \cdot b)_\delta(x),$$

where $\delta, \epsilon \in \{+, -\}$. Again, $(\sup(\mu), *)$ is not a group.

We now start our investigation of $(\sup(\mu), *)$ where μ is an idempotent dfs measure. As stated previously, we demonstrate that if μ is both dfs and idempotent, then $(\sup(\mu), *)$ is a compact, left-continuous, semigroup with no closed two-side ideals.

Proposition 6.23. *Let $\mu, \nu \in \mathfrak{M}_x(\mathcal{G})$. Assume that μ is* dfs *and ν is finitely satisfiable in a small model. Then, we have that* $\sup(\mu) * \sup(\nu) \subseteq \sup(\mu * \nu)$.

Proof. Assume that $p \in \sup(\mu)$ and $q \in \sup(\nu)$ and let $\varphi(x) \in p * q$. Choose G such that μ is dfs over G, ν is finitely satisfiable in G, and G contains all the parameters from φ. We need to show that $\mu * \nu(\varphi(x)) > 0$. Recall,

$$\mu * \nu(\varphi(x)) = \int_{\sup(\nu|_G)} F_\mu^{\varphi'} d\nu_G.$$

Since μ is dfs, the map $F_\mu^{\varphi'} : \sup(\nu|_G) \to [0,1]$ is continuous. Therefore, it suffices to find some $r \in \sup(\nu|_G)$ such that $F_\mu^{\varphi'}(r) > 0$. Consider $r = q|_G$. Then, $F_\mu^{\varphi'}(q|_G) = \mu(\varphi(x \cdot b))$ where $b \models q|_G$. Then, $\varphi(x \cdot b) \in p$ and since $p \in \sup(\mu)$, we have that $\mu(\varphi(x \cdot b)) > 0$. Hence, $F_\mu^{\varphi'}(q|_G) > 0$ and so $\mu * \nu(\varphi(x)) > 0$. □

Corollary 6.24. *Assume that μ is* dfs *and idempotent. Then $(\sup(\mu), *)$ is a compact Hausdorff semigroup which is left-continuous, i.e. the map $- * q : \sup(\mu) \to \sup(\mu)$ is continuous for each $q \in \sup(\mu)$.*

Proof. By Proposition 2.10, we conclude that $\sup(\mu)$ is a compact Hausdorff space. By Proposition 6.23, we have that $\sup(\mu) * \sup(\mu) \subseteq \sup(\mu * \mu) = \sup(\mu)$. Now, choose some $G \prec \mathcal{G}$ such that μ is dfs over G. Then, every element in $\sup(\mu)$ is finitely satisfiable in G. So, $(\sup(\mu), *)$ is a subsemigroup of $(S_x(\mathcal{G}, G), *)$. Since $(S_x(\mathcal{G}, G), *)$ is a left-continuous semigroup (Fact 6.11), we conclude that $(\sup(\mu), *)$ is also left-continuous. □

Proposition 6.25. *Let $\mu, \nu \in \mathfrak{M}_x(\mathcal{G})$. Assume that μ is* dfs *and ν is finitely satisfiable in some small model. Then, $\sup(\mu) * \sup(\nu)$ is a dense subset of $\sup(\mu * \nu)$.*

Proof. By Proposition 6.23, we already know that $\sup(\mu) * \sup(\nu) \subseteq \sup(\mu * \nu)$. We only need to demonstrate the density claim. Fix some $r \in \sup(\mu * \nu)$ and a formula $\varphi(x) \in \mathcal{L}_x(\mathcal{G})$. Assume that $\varphi(x) \in r$. We need to find $p \in \sup(\mu)$ and $q \in \sup(\nu)$ such that $\varphi(x) \in p * q$. Choose G such that μ is dfs over G, ν is finitely satisfiable in G, and G contains all the parameters from $\varphi(x)$. Since $\varphi(x) \in r$ and r is in the support of $\mu * \nu$, we know that $\mu * \nu(\varphi(x)) > 0$. Therefore, $\int F_{\mu}^{\varphi'} d\nu_G > 0$ and so there exists $t \in \sup(\nu|_G)$ such that $F_{\mu}^{\varphi'}(t) > 0$. If $c \models t$, then $\mu(\varphi(x \cdot c)) > 0$. So by Proposition 2.9 there exists $p \in \sup(\mu)$ such that $\varphi(x \cdot c) \in p$. By Proposition 2.11, we let $q \in \sup(\nu)$ such that $q|_G = t$. By construction, we then observe that $\varphi(x) \in p * q$. □

We now define a family of global functions which mimic the map $y \to \int f(xy)d\mu$.

Definition 6.26. Let μ be dfs and fix $\varphi(x) \in \mathcal{L}_x(\mathcal{G})$. We then define the global function $D_{\mu}^{\varphi'} : S_y(\mathcal{G}) \to [0,1]$ as $\mu(\varphi(x \cdot c))$, where $c \models p|_G$ and G is a small elementary substructure of $\mathcal{G}$ such that μ is dfs over G and G contains all the parameters from $\varphi(x)$.

Notice that for any formula $\varphi(x) \in \mathcal{L}_x(\mathcal{G})$, the map $D_{\mu}^{\varphi'}$ is continuous since μ is dfs. We remark that $D_{\mu}^{\varphi'} = F_{\mu,G}^{\varphi'} \circ r$ where r is the standard restriction map from $S_x(\mathcal{G})$ to $S_x(G)$. The next two results are adapted from Glicksberg's work on semitopological semigroups into the general model theory context (we refer the interested reader to [28, 29]).

Proposition 6.27. *Let $\mu \in \mathfrak{M}_x(\mathcal{G})$ and assume μ is* dfs *and idempotent. Assume that $D_{\mu}^{\varphi'}|_{\sup(\mu)}$ attains a maximum at q. Then for any $p \in \sup(\mu)$, we have that $D_{\mu}^{\varphi'}(q) = D_{\mu}^{\varphi'}(p * q)$.*

Proof. Fix G_0 a submodel where μ is dfs over G_0 and contains all the parameters from $\varphi(x)$. Let $b \models q|_{G_0}$ and let $\theta(x; y) = \varphi((x \cdot y) \cdot b)$. Now fix a larger submodel G such that $G_0 b \subset G$. We also let $\delta = \mu(\varphi(x \cdot b))$. Then,

$$D_{\mu}^{\varphi'}(q) = \mu(\varphi(x \cdot b)) = \mu * \mu(\varphi(x \cdot b)) = \mu_x \tilde{\otimes} \mu_y(\theta(x; y))$$

$$= \int_{\sup(\mu|_G)} F_{\mu}^{\theta} d\mu_G \leq \int_{\sup(\mu|_G)} \delta d\mu_G = D_{\mu}^{\varphi'}(q)$$

Observe that for any $t \in \sup(\mu|_G)$, $a \models t$, and $\hat{t}$ in $\sup(\mu)$ such that $\hat{t}|_G = t$, we have $F_\mu^\theta(t) = \mu(\varphi(x \cdot a) \cdot b) = \mu(\varphi(x \cdot ab)) = D_\mu^{\varphi'}(\tilde{t} * q) \leq D_\mu^{\varphi'}(q) = \delta$. We conclude that for any $t \in \sup(\mu|_G)$, $F_\mu^\theta(t) \leq \delta$. Therefore, $F_\mu^\theta = \delta$ almost everywhere (with respect to μ_G). Since both maps are continuous, they are equal over $\sup(\mu|_G)$. Finally, for any $p \in \sup(\mu)$, we notice;

$$D_\mu^{\varphi'}(q) = \delta = F_\mu^\theta(p|_G) = \mu(\varphi((x \cdot a) \cdot b)) = \mu(\varphi(x \cdot (a \cdot b))) = D_\mu^{\varphi'}(p * q).$$

□

Theorem 6.28. *Let $\mu \in \mathfrak{M}_x(\mathcal{G})$ and assume that μ is* dfs *and idempotent. Let $I \subseteq \sup(\mu)$ be a closed two-sided ideal, i.e. I is a closed subset of* $\sup(\mu)$ *and $p * I = I * p = I$ for any $p \in \sup(\mu)$. Then, $I = \sup(\mu)$.*

Proof. Let I be a closed two-sided ideal of $\sup(\mu)$. Notice that if I is a dense subset $\sup(\mu)$, then $I = \sup(\mu)$. Hence, we may assume that I is not dense and so there exists some $\varphi(x) \in \mathcal{L}_x(\mathcal{G})$ such that $\varphi(x) \cap \sup(\mu)$ is nonempty and $\varphi(x) \subset \sup(\mu) \backslash I$. Let $G \prec \mathcal{G}$ contain the parameters for φ and such that μ is dfs over G. We claim that there exists some $q \in \sup(\mu)$ such that $D_\mu^{\varphi'}(q) > 0$. Assume not. Let $p, q \in \sup(\mu)$ be arbitrary. Let $b \models q|_G$. Then, $\mu(\varphi(x \cdot b)) = D_\mu^{\varphi'}(q) = 0$ by assumption. Since $p \in \sup(\mu)$ and $\mu(\varphi(x \cdot b)) = 0$, we have that $\neg\varphi(x \cdot b) \in p$ which implies that $\neg\varphi(x) \in p * q$. Consider the continuous characteristic function $\chi_\varphi : \sup(\mu) \to \{0, 1\}$. Since p, q were arbitrary, we have that χ_φ vanishes on $\sup(\mu) * \sup(\mu)$. By Proposition 6.25, χ_φ vanishes on the dense subset $\sup(\mu) * \sup(\mu) \subseteq \sup(\mu)$, and so χ_φ vanishes on $\sup(\mu)$. But this contradicts the choice of φ.

So, there exists a $q \in \sup(\mu)$ such that $D_\mu^{\varphi'}(q) > 0$. Since $D_\mu^{\varphi'}$ is continuous, it attains a maximum on $\sup(\mu)$. Let r be a maximum. We claim for any $h \in I$, we have that $D_\mu^{\varphi'}(h) = 0$. Notice that $D_\mu^{\varphi'}(h) = \mu(\varphi(x \cdot b))$ where $b \models h|_G$. Now,

$$\mu(\varphi(x \cdot b)) = \mu(\{p \in \sup(\mu) : \varphi(x \cdot b) \in p\}) = \mu(\{p \in \sup(\mu) : \varphi(x) \in p * h\}).$$

However, I is a (left) ideal and so $\sup(\mu) * h \subseteq I$. By assumption, $\varphi(x) \cap I = \emptyset$, and so we have $\{p \in \sup(\mu) : \varphi(x) \in p * h\} = \emptyset$. Therefore, $D_\mu^{\varphi'}(h) = 0$. Since I is a (right) ideal, we have that $h * r \in I$. Therefore,

$$0 < D_\mu^{\varphi'}(r) = D_\mu^{\varphi'}(h * r) = 0.$$

We have obtained a contradiction. □

Corollary 6.29. *Assume that μ is* dfs *and idempotent. Suppose that $|\sup(\mu)| > 1$, i.e. μ is not a type. Then,* $\sup(\mu)$ *contains no zero elements, i.e., there is no element $p \in \sup(\mu)$ such that for any q in* $\sup(\mu)$*, $p * q = q * p = p$.*

Proof. If p is a zero-element, then $\{p\}$ is a closed two-sided ideal. □

As is true in most mathematics, *if we buy more, we get more.* If we add some additional assumption about the structure of $\sup(\mu)$, we can prove more things. We now add the additional assumption that our semigroup is somewhat minimal. We notice that in Example 6.22, the support of the measure $\tilde{\mu}$ is equal to its own minimal left ideal. Let us make that our assumption. We first recall the following structural theorem due to Ellis (see [21, Proposition 4.2]).

Fact 6.30. *Assume that $(S, \cdot)$ is a compact Hausdorff semigroup such that for any $a \in S$, the map $- \cdot a : S \to S$ is continuous. Then, there exists a minimal left ideal, I. Moreover, if $J(I) = \{i \in I : i^2 = i\}$ is the set of idempotents in I, then the following hold.*

1. *$J(I)$ is non-empty.*

2. *For every $p \in I$ and $i \in J(I)$, we have that $p \cdot i = p$.*

3. *$I = \bigcup\{i \cdot I : i \in J(I)\}$ where the union is disjoint one and each set $i \cdot I$ is a group with identity i.*

Definition 6.31. Let $\mu \in \mathfrak{M}_x(\mathcal{G})$. Assume that μ is dfs and idempotent. Then, we say that μ is **minimal** if $(\sup(\mu), *)$ has no proper left ideals.

We now use the minimality assumption to prove a few stronger result.

Proposition 6.32. *Assume that μ is* dfs*, idempotent, and minimal. Let $\varphi(x) \in \mathcal{L}_x(\mathcal{G})$ be any formula. Then, for any $p, q \in \sup(\mu)$, we have that $D_\mu^{\varphi'}(p) = D_\mu^{\varphi'}(q)$.*

Proof. By Fact 6.30, $\sup(\mu) = \bigcup\{i_j * \sup(\mu) : i_j \in J(\sup(\mu))\}$. Assume that $D_\mu^{\varphi'}$ attains a maximum at p and $p \in i_1 * \sup(\mu)$. In particular, $p = i_1 * p$ since $i_1 * \sup(\mu)$ is a group. We know that $q = i_2 * q$ for some $i_2 \in J(\sup(\mu))$. By Proposition 6.27, we know

$$D_\mu^{\varphi'}(p) = D_\mu^{\varphi'}(i_1 * p) = D_\mu^{\varphi'}(i_2 * (i_1 * p)) = D_\mu^{\varphi}((i_2 * i_1) * p) = D_\mu^{\varphi'}(i_2 * p).$$

Notice that q and $i_2 * p$ are elements of the group $i_2 * \sup(\mu)$. Therefore there exists some $r \in i_2 * I$ such that $r * (i_2 * p_1) = q$. Hence $D_\mu^{\varphi'}(i_2 * p_1) = D_\mu^{\varphi'}(r * (i_2 * p_1)) = D_\mu^{\varphi'}(q)$. ☐

Proposition 6.33. *Assume that μ is* dfs*, idempotent, and minimal. Then for every $\varphi(x) \in \mathcal{L}_x(\mathcal{G})$, $\mu(\varphi(x)) = D_\mu^{\varphi'}(p)$ for any $p \in \sup(\mu)$.*

Proof. By the Proposition 6.32, we may assume towards a contradiction that $\mu(\varphi(x)) > D_\mu^{\varphi'}(i)$ where i is an idempotent in $\sup(\mu)$. Choose G such that μ is dfs over G and G contains the parameters from φ. Then, $\mu(\varphi(x) \wedge \neg\varphi(x \cdot b)) > 0$ where $b \models i|_G$. So there exists $q \in \sup(\mu)$ such that $\varphi(x) \wedge \neg\varphi(x \cdot b) \in q$. Notice that $\varphi(x) \in q$ and $\neg\varphi(x \cdot b) \in q \implies \neg\varphi(x) \in q * i$. However, by Fact 6.30, $q * i = q$ and so now we have that $\varphi(x), \neg\varphi(x) \in q$ which is a contradiction. ☐

A direct translation of the previous proposition demonstrates that all dfs, idempotent, minimal Keisler measures are *generically right invariant over their support.* Notice the following:

Corollary 6.34. *Assume that μ is* dfs*, idempotent, and minimal. Let $\varphi(x;b) \in \mathcal{L}_x(\mathcal{G})$ and let μ be* dfs *over G. Then, for any $a \in \mathcal{G}$ such that* $\mathrm{tp}(a/Gb) \in \sup(\mu|_{Gb})$*, we have that*

$$\mu(\varphi(x)) = \mu(\varphi(x \cdot a)).$$

6.2.3 Idempotent measures on stable groups

In this subsection, we classify idempotent measures on stable groups. We prove that each idempotent measure is the unique Keisler measure which witnesses the definable amenability of its own stabilizer. Our proof relies on the results of the previous section (namely Theorem 6.28 and Corollary 6.34) and a variant of Hrushovski's group chunk theorem due to Newelski [46]. This proof is dependent on the existence of ranks and so a generalization to the arbitrary dfs-idempotent (even NIP) context does not directly follow. We assume some familiarity with stable group theory (see [51] or [66]). Throughout this section, we will clearly mark the theorems which assume global stability. As usual, T is a first order theory expanding a group and $\mathcal{G}$ is a monster model of T. We now recall the definition of the stabilizer of a measure.

Definition 6.35. Let $\mu \in \mathfrak{M}_x(\mathcal{G})$. Then the stabilizer of μ is defined as follows:

$$\mathrm{Stab}(\mu) = \{g \in \mathcal{G} : \text{for any } \varphi(x) \in \mathcal{L}_x(\mathcal{G}),\ \mu(\varphi(x)) = \mu(\varphi(g \cdot x))\}.$$

The next result follows from continuity and compactness and is left as an exercise.

Fact 6.36. *Let $G \prec \mathcal{G}$. If μ is definable over G, then* $\mathrm{Stab}(\mu)$ *is type-definable over G. In particular, there exists a small collection of formulas $\{\psi_i(x) : i \in I\}$ with parameters only from G such that* $\mathrm{Stab}_\mu(x) = \bigwedge_{i \in I} \psi(x)$ *and* $\mathrm{Stab}_\mu(\mathcal{G}) = \mathrm{Stab}(\mu)$.

We now restrict ourselves to the stable group theory context. Let us recall an important observation about Keisler measures in stable theories which follows from Keisler's original work (see [36] or [10, Lemma 4.3])

Fact 6.37 (T is stable). *Every measure in* $\mathfrak{M}_x(\mathcal{G})$ *is* dfs.

The next collections of facts follow from [49, Fact 1.8] and [9].

Fact 6.38 (T is stable). *Let* $\mathcal{H}$ *be a type-definable subgroup of* $\mathcal{G}$*. For notation, if* $H(x) = \bigwedge_{j\in J}\psi_j(x)$ *is a definition of* $\mathcal{H}$*, then we let* $S_H(\mathcal{G})$ *be the collection of type in* $S_x(\mathcal{G})$ *which concentrate on* $\mathcal{H}$*, i.e.* $p \in S_H(\mathcal{G})$ *if and only if* $\psi_j(x) \in p$ *for each* $j \in J$*.*

1. *For* $p, q \in S_H(\mathcal{G})$*, we have that* $p * q$ *is equal to* $\mathrm{tp}(a \cdot b/G)$*, where* $a \models p$ *and* $b \models q$ *and* $a \mathop{\smile\!\!\!\!\!\!\mid}_G b$ *(in the sense for forking independence). By stability, this is well defined and corresponds to the standard product on* $S_x(\mathcal{G})$*.*
2. $S_H(\mathcal{G})$ *has a unique minimal closed left ideal* I *(which is also the unique minimal closed right ideal) which is already a subgroup of* $(S_H(\mathcal{G}), *)$*.*
3. I *is precisely the generic types of* $S_H(\mathcal{G})$ *and, with the induced topology,* $(I, *)$ *is a compact Hausdorff topological group (isomorphic to* $\mathcal{H}/\mathcal{H}^0$*).*
4. $\mathcal{H}$ *is definably amenable. Moreover, there exists a measure* $\mu \in \mathfrak{M}_x(\mathcal{G})$ *such that* μ *is both the unique right* $\mathcal{H}$*-invariant Keisler measure and unique left* $\mathcal{H}$*-invariant Keisler measure with* $\sup(\mu) = I$*. If one views this measure as a regular Borel measure on* $S_H(\mathcal{G})$*, then after restricting* μ *to the closed set* I*,* $\mu|_{\sup(\mu)}$ *coincides with the Haar measure.*

Definition 6.39 (T is stable). Let $\hat{\mathcal{G}}$ be a larger monster model of T such that $\mathcal{G} \prec \hat{\mathcal{G}}$.

1. The symbol Δ will denote a finite *invariant set of formulas*, i.e. formulas of the form $\varphi(u \cdot x \cdot y, \overline{y}) \in \mathcal{L}$ (so any right or left translative of an instance of φ is also an instance of φ).
2. We write R_Δ to denote Shelah's Δ-rank, note that it is invariant under two-sided translations since Δ is.
3. For $P \subseteq S_x(\mathcal{G})$, we let $\mathrm{cl}(P)$ denote the topological closure of P and $*P$ denote the closure of P under $*$.

4. For $P \subseteq S_x(\mathcal{G})$, we let $\mathrm{gen}(P)$ denote the set of $r \in \mathrm{cl}(*P)$ such that there is no $q \in \mathrm{cl}(*P)$ with $R_\Delta(r) \leq R_\Delta(q)$ for all Δ and $R_\Delta(r) < R_\Delta(q)$ for some Δ.

5. For $P \subseteq S_x(\mathcal{G})$, we let $\langle P \rangle$ denote the smallest $\mathcal{G}$-type-definable subgroup of $\hat{\mathcal{G}}$ containing $P(\hat{\mathcal{G}})$ where $P(\hat{\mathcal{G}}) = \{b \in \hat{\mathcal{G}} : b \models p : \text{for some } p \in P\}$.

In the next result, we will see that the interaction between the $*$-product on types and local ranks Δ ranks is extremely tame (in stable groups setting).

Fact 6.40 (T is stable)**.** *Let $\hat{\mathcal{G}}$ be a larger monster model of T such that $\mathcal{G} \prec \hat{\mathcal{G}}$.*

1. *[46, Fact 2.1] If $P \subseteq S_x(\mathcal{G})$ is non-empty, then* $\mathrm{gen}(P)$ *is a non-empty closed subset of $S_x(\mathcal{G})$.*

2. *[46, Lemma 2.2] $R_\Delta(p*q) \geq R_\Delta(p), R_\Delta(q)$ for any $p, q \in S_x(\mathcal{G})$ and Δ (this follows by the symmetry of forking, invariance of R_Δ under two-sided translations, and the fact that forking is characterized by drop in rank).*

The following fact is [46, Theorem 2.2]. It is stated there for strong types over $\emptyset$, which implies our statement after naming the elements of $\mathcal{G}$ with constants (call this theory $T_{\mathcal{G}}$) and viewing $\hat{\mathcal{G}}$ as a monster model of $T_{\mathcal{G}}$.

Fact 6.41 (T is stable)**.** *Let $\hat{\mathcal{G}}$ be a larger monster model of T such that $\mathcal{G} \prec \hat{\mathcal{G}}$. Suppose that $P \subseteq S_x(\mathcal{G})$ is a non-empty set of types. Then,*

$$\langle P \rangle = \{a \in \hat{\mathcal{G}} : \mathrm{tp}(a/\mathcal{G}) * \mathrm{gen}(P) = \mathrm{gen}(P) \ \textit{setwise}\},$$

is a $\mathcal{G}$ type-definable subgroup of $\hat{\mathcal{G}}$ and $\mathrm{gen}(P)$ *is precisely the set of generic types of $\langle P \rangle$ over $\mathcal{G}$.*

Theorem 6.42 (T is stable)**.** *Let $\mathcal{G}$ be a monster model of T. Let $\mu \in \mathfrak{M}_x(\mathcal{G})$. Then the following are equivalent:*

1. *μ is idempotent.*

2. *μ is the unique right* $\mathrm{Stab}(\mu)$*-invariant (and also unique left* $\mathrm{Stab}(\mu)$*-invariant) Keisler measure.*

Proof. Notice that 2 $\implies$ 1 follows directly from Proposition 6.17. It suffices to show 1 $\implies$ 2. Let $\mu \in \mathfrak{M}_x(\mathcal{G})$ be an idempotent measure. By Fact 6.37 μ is dfs over some small model $G \prec \mathcal{G}$. By Corollary 6.24, $\sup(\mu)$ is a closed subset of $S_x(\mathcal{G})$ and closed under $*$, therefore $\mathrm{cl}(*\sup(\mu)) = \sup(\mu)$ and $\mathrm{gen}(\sup(\mu)) \subseteq \sup(\mu)$.

We claim that $\mathrm{gen}(\sup(\mu))$ is a two-sided ideal in $(\sup(\mu), *)$. Let $r \in \mathrm{gen}(\sup(\mu))$ and $q \in \sup(\mu)$. If $r * q$ is not in $\mathrm{gen}(\sup(\mu))$, then there exists some $p \in \sup(\mu)$ such that $R_\Delta(p) \geq R_\Delta(r * q) \geq R_\Delta(r)$ for all Δ and some inequality strict (by Fact 6.40), which contradicts the assumption that $r \in \mathrm{gen}(\sup(\mu))$. But also, if $q * r$ is not in $\mathrm{gen}(\sup(\mu))$, there is some $p \in \sup(\mu)$ with $R_\Delta(p) \geq R_\Delta(q * r) \geq R_\Delta(r)$ and some inequality strict, again by Fact 6.40, contradicting $r \in \mathrm{gen}(\sup(\mu))$. Therefore, by Theorem 6.28, we conclude that $\mathrm{gen}(\sup(\mu)) = \sup(\mu)$.

We now fix a larger monster, $\hat{\mathcal{G}}$ and think of $\mathcal{G}$ as a small elementary submodel, i.e. $\mathcal{G} \prec \hat{\mathcal{G}}$. Then, by Fact 6.41, we have that $\hat{\mathcal{H}} := \langle \sup(\mu) \rangle = \{a \in \hat{\mathcal{G}} : a \models p, p \in \sup(\mu)\}$ is a $\mathcal{G}$-type-definable subgroup of $\hat{\mathcal{G}}$ and $\sup(\mu) = \mathrm{gen}(\sup(\mu))$ is precisely the set of generic types on $\hat{\mathcal{H}}$ restricted to $\mathcal{G}$. Since $\hat{\mathcal{H}}$ possibly uses all the parameters in $\mathcal{G}$, we have to be careful. Let $H(x)$ be a definition of $\hat{\mathcal{H}}$ over with parameters only from $\mathcal{G}$ such that $H(\hat{\mathcal{G}}) = \hat{\mathcal{H}}$. Given $p \in S_x(\mathcal{G})$, we let $\hat{p} \in S_x(\hat{\mathcal{G}})$ be its unique $\mathcal{G}$-definable extension, and let $\hat{\mu}$ be the unique $\mathcal{G}$-definable extension of μ. We will argue that $\hat{\mathcal{H}} = \mathrm{Stab}_\mu(\hat{\mathcal{G}})$. We have the following observations.

1. $p * q = r \iff \hat{p} * \hat{q} = \hat{r}$ for any $p, q, r \in S_x(\mathcal{G})$.
2. The same holds for measures, in particular $\hat{\mu}$ is an idempotent of $\big(\mathfrak{M}_x(\hat{\mathcal{G}}) *\big)$.
   Assume that $\mu, \nu \in \mathfrak{M}_x(\mathcal{G})$ are definable over G. Then $\hat{\mu} * \hat{\nu}$ is also definable over G (Proposition 6.14) and extends $\mu * \nu$, hence $\hat{\mu} * \hat{\nu} = \widehat{\mu * \nu}$ by uniqueness of definable extensions.
3. $\mathrm{Stab}_\mu(\hat{\mathcal{G}}) = \mathrm{Stab}(\hat{\mu})$ by definability (Fact 6.36).
4. $\sup(\hat{\mu}) = \{\hat{p} : p \in \sup(\mu)\}$.
   Assume that there exists some $p \in \sup(\mu)$ such that $\hat{p} \notin \sup(\hat{\mu})$. Then, there exists a formula $\psi(x; b) \in \hat{p}$ such that $\hat{\mu}(\psi(x; b)) = 0$. Since μ is definable over some small submodel G, $\hat{\mu}$ is also definable over G and in particular,

$\hat{\mu}$ is G-invariant. Since $\mathcal{G}$ is $|G|^+$ saturated, there exists $a \in \mathcal{G}$ such that $\text{tp}(a/G) = \text{tp}(b/G)$. Since $\hat{p}$ is the unique definable extension of p, we know that $\psi(x; a) \in p$. Then, $\mu(\psi(x; a)) > 0$ since $p \in \sup(\mu)$. Then $\hat{\mu}(\psi(x; a)) > 0$ and $\hat{\mu}(\psi(x; b)) = 0$. By G-invariant of $\hat{\mu}$, we have a contradiction.

Conversely, if $p \in S_x(\mathcal{G})$ and $\hat{p} \in \sup(\hat{\mu})$ it is clear that $p \in \sup(\mu)$ since $\hat{\mu}|_{\mathcal{G}} = \mu$.

5. The generics of $H(x)$ over $\hat{\mathcal{G}}$ are precisely $\{\hat{p} : p \text{ is a generic of } H(x) \text{ over } \mathcal{G}\}$.

   By stability, every generic r of $H(x)$ over $\hat{\mathcal{G}}$ does not fork over $\mathcal{G}$, so it is definable over $\mathcal{G}$ and $r|_{\mathcal{G}}$ is a generic of $H(x)$ over $\mathcal{G}$. Hence, $r = \widehat{(r|_{\mathcal{G}})}$. Conversely, a definable (non-forking) extension of a generic type is generic.

6. Hence $\sup(\hat{\mu})$ is precisely the set of generics of $H(x)$ over $\hat{\mathcal{G}}$, in particular $(\sup(\hat{\mu}), *)$ is a topological group by Fact 6.38.

7. Then $\hat{\mu}$ restricted to $(\sup(\hat{\mu}), *)$ (viewed as a regular Borel measure) is right $*$-invariant.

   By (6), $(\sup(\hat{\mu}), *)$ is a group and so for any $p \in \sup(\hat{\mu})$, p^{-1} is well-defined. By regularity, it suffices to check $*$-invariance for formulas. Let $\varphi(x; b) \in \mathcal{L}_x(\hat{\mathcal{G}})$. Then, for any $p \in \sup(\hat{\mu})$, we have that,

   $$\hat{\mu}(\varphi(x; b)*p) = \hat{\mu}(\{q*p : \varphi(x; b) \in q\}) = \hat{\mu}(\{q : \varphi(x; b) \in q*p^{-1}\}) = \hat{\mu}(\varphi(x\cdot c, b)),$$

   where $c \models p^{-1}|_{Gb}$. By Corollary 6.34, $\hat{\mu}(\varphi(x \cdot c, b)) = \hat{\mu}(\varphi(x; b))$ and so $\hat{\mu}$ is right $*$-invariant.

8. By Fact 6.38 for $\hat{\mathcal{H}}$, there is a unique right $\hat{\mathcal{H}}$-invariant Keisler measure ν in $\mathfrak{M}_x(\hat{\mathcal{G}})$ such that $\nu(H(x)) = 1$, $\sup(\nu)$ is the set of generics of $H(x)$ over $\hat{\mathcal{G}}$, and $\nu|_{\sup(\nu)}$ (viewed as a Borel measure) is the Haar measure on the compact topological group $(\sup(\nu), *)$.

9. Thus $\sup(\hat{\mu}) = \sup(\nu)$. Since both μ and ν are right $*$-invariant and by uniqueness of the Haar measure, we have that $\hat{\mu}|_{\sup(\mu)} = \nu|_{\sup(\nu)}$ and hence $\hat{\mu} = \nu$.

10. In particular, $\hat{\mathcal{H}} = \text{Stab}(\nu) = \text{Stab}(\hat{\mu}) = \text{Stab}_\mu(\hat{\mathcal{G}})$, so μ is the right invariant measure on the G-type definable group $\text{Stab}_\mu(x)$.

From stability, we know that there exists a unique right $\text{Stab}_\mu(\mathcal{G})$-invariant Keisler measure in $\mathfrak{M}_x(\mathcal{G})$. Notice that for every $a \in \text{Stab}_\mu(\mathcal{G})$, we have that $\mu(\varphi(x \cdot a)) = \mu(\varphi(x))$ by definition. Therefore, It suffices to check that $\mu(\text{Stab}_\mu(x)) = 1$. From line 10, we have that,

$$\mu(\text{Stab}_\mu(x)) = \hat{\mu}(\text{Stab}_\mu(x)) = \hat{\mu}(\text{Stab}_{\hat{\mu}}(x)) = \nu(\text{Stab}_\nu(x)) = 1.$$

$\square$

6.3 Ellis Semigroup

In this section, we demonstrate that definable convolution and model theory are connected via the structure of a particular Ellis semigroup. Let us begin by recalling the construction of an Ellis semigroup. Let X be a compact Hausdorff space and S be a semigroup acting on X by homeomorphisms. Therefore we can consider the map $\pi : S \times X \to X$ such that for each $s \in S$, the map $\pi_s : X \to X$ is a homeomorphism. Let X^X be space of functions from X to X equipped with the product topology. Then $\{\pi_s : s \in S\}$ is naturally a subset of X^X. The **Ellis semigroup of the action** (X, S) is the closure of $\big(\{\pi_s : s \in S\}, \circ\big)$ in X^X and we denote this semigroup as $\mathrm{E}(X, S)$.

Let us return to the model theory context. We now fix T, a first order theory expanding a group, $\mathcal{G}$ a saturated model of T, and G a small submodel of $\mathcal{G}$. Recall that we denote the collection of global types finitely satisfiable in G as $S_x(\mathcal{G}, G)$. Then, there is a natural action of G on $S_x(\mathcal{G}, G)$. Newelski showed that the Ellis semigroup of this action has a very natural presentation [45].

Fact 6.43 (Newelski)**.** $\mathrm{E}(S_x(\mathcal{G}, G), G) \cong (S_x(\mathcal{G}, G), *)$.

We will show an analogous result for measures in the context of NIP theories.

6.3.1 Measure case

Recall that $\mathrm{conv}(G)$ is the convex hull of Dirac measures concentrating on G and $\mathfrak{M}_x(\mathcal{G}, G)$ is the collection of global measures in $\mathfrak{M}_x(\mathcal{G})$ which are finitely satisfiable in G. Observe that there is a natural semigroup action of $\mathrm{conv}(G)$ on $\mathfrak{M}_x(\mathcal{G}, G)$ as follows: For any $\sum_{i=1}^{n} r_i\delta_{g_i} \in \text{conv}(G)$ and $\mu \in \mathfrak{M}_x(\mathcal{G}, G)$, we define

$$\Big(\Big(\sum_{i=1}^{n} r_i\delta_{g_i}\Big) \cdot \mu\Big)(\varphi(x)) = \sum_{i=1}^{n} r_i\mu(\varphi(g_i \cdot x)).$$

any formula $\varphi(x) \in \mathcal{L}_x(\mathcal{G})$. Throughout this final section, we will denote elements of $\text{conv}(G)$ as k, the semigroup action described above as π, and the corresponding homeomorphisms of elements in $\text{conv}(G)$ under this action as π_k, i.e.

$$\pi : \text{conv}(G) \times \mathfrak{M}_x(\mathcal{G}, G) \to \mathfrak{M}_x(\mathcal{G}, G),$$

and for $k \in \text{conv}(G)$,

$$\pi_k : \mathfrak{M}_x(\mathcal{G}, G) \to \mathfrak{M}_x(\mathcal{G}, G).$$

It is not difficult to see that for every $k \in \text{conv}(G)$, the map π_k is continuous. Therefore, we can consider the Ellis semigroup of this semigroup action. For notational purposes, we will sometimes write $\text{E}(\mathfrak{M}_x(\mathcal{G}, G), \text{conv}(G))$ as simply $\mathbf{E}$.

We now show that if T is NIP, then $\text{E}(\mathfrak{M}_x(\mathcal{G}, G), \text{conv}(G))$ is isomorphic to the convolution algebra of global measures which are finitely satisfiable in G, i.e. $(\mathfrak{M}_x(\mathcal{G}, G), *)$. We demonstrate that these two semigroups are isomorphic by considering the map $\rho : \mathfrak{M}_x(\mathcal{G}, G) \to \mathfrak{M}_x(\mathcal{G}, G)^{\mathfrak{M}_x(\mathcal{G},G)}$ where $\rho(\nu) = \rho_\nu = \nu * -$, proving that the image of ρ is precisely the Ellis semigroup, and showing that ρ is indeed an isomorphism. Before continuing, we observe that our map ρ is well-defined and that $\mathfrak{M}_x(\mathcal{G}, G)$ is a semigroup by recalling the following facts.

Observation 6.44. Let T be NIP and assume that $\mu \in \mathfrak{M}_x(\mathcal{G}, G)$. Then,

1. μ is Borel-definable over G (Fact 2.36).
2. For any $\nu \in \mathfrak{M}_x(\mathcal{G}, G)$, $\mu * \nu \in \mathfrak{M}_x(\mathcal{G}, G)$ (Proposition 6.14).
3. The operation $*$ on $\mathfrak{M}_x(\mathcal{G}, G)$ is associative and so $(\mathfrak{M}_x(\mathcal{G}, G), *)$ is a semigroup (Similar to the proof that $\otimes$ is associative (Proposition 2.41)).

We do however first need to show that $(\mathfrak{M}_x(\mathcal{G}, G), *)$ is left-continuous. This follows directly from a general continuity result we proved about Keisler measures in NIP theories in Chapter 2 (namely Lemma 2.43). This is necessary to show that our map ρ is surjective.

Proposition 6.45. *Let $\nu \in \mathfrak{M}_x(\mathcal{G}, G)$. Then the map $- * \nu : \mathfrak{M}_x(\mathcal{G}, G) \to \mathfrak{M}_x(\mathcal{G}, G)$ is continuous.*

Proof. Let U be a basic open subset of $\mathfrak{M}_x(\mathcal{G}, G)$. Then, there exists a sequence of formulas $\varphi_1(x), ..., \varphi_n(x)$ in $\mathcal{L}_x(\mathcal{G})$ and real numbers $r_1, ..., r_n, s_1, ..., s_n$ such that,

$$U = \bigcap_{i=1}^{n} \{\mu \in \mathfrak{M}_x(\mathcal{G}, G) : r_i < \mu(\varphi_i(x)) < s_i\}.$$

Then, we have that;

$$\Big(- *\nu\Big)^{-1}(U) = \bigcap_{i=1}^{n} \{\mu \in \mathfrak{M}_x(\mathcal{G}, G) : r_i < \mu * \nu(\varphi_i(x)) < s_i\}$$

$$= \bigcap_{i=1}^{n} \{\mu \in \mathfrak{M}_x(\mathcal{G}, G) : r_i < \mu_x \otimes \nu_y(\varphi_i(x \cdot y)) < s_i\}$$

$$= \bigcap_{i=1}^{n} \Big(- \otimes \nu_y(\varphi_i(x \cdot y))\Big)^{-1} \Big(\{\mu \in \mathfrak{M}_x(\mathcal{G}, G) : r_i < \mu(\varphi_i(x)) < s_i\}\Big).$$

By Lemma 2.43, the preimage of U under $- * \nu$ is the finite intersection of open sets and therefore open. □

6.3.2 The isomorphism

In this subsection, we will show the map $\rho : \mathfrak{M}_x(\mathcal{G}, G) \to \mathrm{E}(\mathfrak{M}_x(\mathcal{G}, G), \mathrm{conv}(G))$ via $\rho(\nu) = \rho_\nu = \nu * -$ is an isomorphism. We begin by recalling the topology on $\mathfrak{M}_x(\mathcal{G}, G)^{\mathfrak{M}_x(\mathcal{G}, G)}$.

Fact 6.46. *If* U *is a basic open subset of* $\mathfrak{M}_x(\mathcal{G},G)^{\mathfrak{M}_x(\mathcal{G},G)}$*, then*

$$U = \bigcap_{i=1}^{k}\{f : \mathfrak{M}_x(\mathcal{G},G) \to \mathfrak{M}_x(\mathcal{G},G) | r_i < f(\nu_i)(\psi_i(x)) < s_i\},$$

where each $r_i, s_i \in \mathbb{R}$*, each* $\psi_i(x)$ *is an element of* $\mathcal{L}_x(\mathcal{G})$*, and* $\nu_i \in \mathfrak{M}_x(\mathcal{G},G)$ *(with possible repetitions in the* ψ_i*'s and* ν_i*'s).*

Proposition 6.47. *The map* ρ *is injective.*

Proof. Notice that for every $\nu \in \mathfrak{M}_x(\mathcal{G},G)$, $\rho_\nu(\delta_e) = \nu$. □

We now show that the image of ρ is a subset of $\mathbf{E}$. This proof relies on the fact that measures in NIP theories have smooth extensions and Lemma 2.43. It is interesting to note that in the countable NIP case, one can replace all instances of smooth measures with results involving Lemma 3.29 (i.e. application of BFT).

Lemma 6.48. *If* $\mu \in \mathfrak{M}_x(\mathcal{G},G)$*, then* $\rho_\mu \in cl(\{\pi_k : k \in \operatorname{conv}(G)\})$*.*

Proof. Let U be a basic open subset of $\mathfrak{M}_x(\mathcal{G},G)^{\mathfrak{M}_x(\mathcal{G}/G)}$ containing π_μ. Then, we may find some $\psi_1(x),...,\psi_n(x)$ in $\mathcal{L}_x(\mathcal{U})$ and $\nu_1,...,\nu_n \in \mathfrak{M}_x(\mathcal{G},G)$ and $B_\epsilon \subset U$ where,

$$B_\epsilon = \bigcap_{i=1}^{n}\{f : |f(\nu_i)(\psi_i(x)) - \pi_\mu(\nu_i)(\psi_i(x))| < \epsilon\}.$$

Now, for each $i \leq n$, we choose a model N_i and a measure $\hat{\nu}_i$ such that $\hat{\nu}_i|_M = \nu|_M$, N_i contains all the parameters from ψ_i, and $\hat{\nu}_i$ is smooth over N_i. Let $\epsilon_0 < \frac{\epsilon}{4}$ and $\operatorname{Av}(\overline{a}_i)$ be a (ψ_i,ϵ_0)-approximation for ν_i where the tuple $\overline{a}_i \in (N_i)^{<\omega}$. In the following computation we associate ψ_i and ψ_i'.

$$\pi_\mu(\nu_i)(\psi_i(x)) = \mu * \nu_i(\psi_i(x)) = \mu \otimes \nu_i(\psi_i(x\cdot y))$$

$$= \int_{S_y(M)} F_\mu^{\psi_i} d(\nu_i|_M) = \int_{S_y(M)} F_\mu^{\psi_i} d(\hat{\nu}_i|_M)$$

$$= \int_{S_y(N)} F_{\mu}^{\psi_i} d(\hat{\nu}_i|_N) = \int_{S_x(N)} F_{\nu_i}^{\psi_i^*} d(\mu|_N)$$

$$\approx_{\epsilon_0} \int_{S_x(N)} F_{\mathrm{Av}(\overline{a}_i)}^{\psi_i^*} d(\mu|_N) = \frac{1}{m} \sum_{j=1}^{m} \mu(\psi_i(x \cdot a_j)).$$

Consider the finite collection of formulas obtained from this computation with each pair (ν_i, ψ_i). In particular, we let $\Psi_i = \bigcup_{j=1}^{m} \{\psi_i(x, a_j)\}$ and we let $\Psi = \bigcup_{i=1}^{n} \Psi_i$ be this collection. Since μ is finitely satisfiable in G, we can find some element k_μ in $\mathrm{conv}(G)$ such that for each $\theta(x) \in \Psi$, we have that $k_\mu(\theta(x)) = \mu(\theta(x))$ (see Proposition 2.19). Now, we claim that π_{k_μ} is in B_ϵ. This follows directly from running the equations above in reverse. In particular, we have that;

$$\frac{1}{m} \sum_{j=1}^{m} \mu(\psi_i(x \cdot a_j)) = \frac{1}{m} \sum_{j=1}^{m} k_\mu(\psi_i(x \cdot a_j))$$

$$= \int_{S_x(N)} F_{\mathrm{Av}(\overline{a})}^{\psi_i^*} d(k_\mu|_N) \approx_{\epsilon_0} \int_{S_x(N)} F_{\hat{\nu}_i}^{\psi_i^*} d(k_\mu|_N)$$

$$= \int_{S_y(N)} F_{k_\mu}^{\psi_i} d(\nu_i|_N) = \int_{S_y(M)} F_{k_\mu}^{\psi_i} d(\hat{\nu}_i|_M)$$

$$= k_\mu \otimes \nu_i(\psi_i(x \cdot y)) = \pi_{k_\mu}(\nu_i)(\psi_i(x)).$$

Now, $\pi_\mu(\nu_i)(\psi_i(x)) \approx_{2\epsilon_0} \pi_{k_\mu}(\nu_i)(\psi_i(x))$ for each pair of ψ_i and ν_i and so $\pi_{k_\mu} \in B_\epsilon$. □

Lemma 6.49. *The map ρ is surjective onto it's image.*

Proof. Let $f \in \mathrm{E}(\mathfrak{M}_x(\mathcal{G}, G), \mathrm{conv}(G))$. Since $f \in cl(\{\pi_k : k \in \mathrm{conv}(G)\})$, there exists a net $(k_i)_{i \in I}$ of elements in $\mathrm{conv}(G)$ such that $\lim_{i \in I} \pi_{k_i} = f$. For each $\psi(x) \in \mathcal{L}_x(\mathcal{G})$ and for every $\nu \in \mathfrak{M}_x(\mathcal{G}, G)$, we have that:

$$\lim_{i \in I} \pi_{k_i}(\nu)(\psi(x)) = f(\nu)(\psi(x)).$$

Consider δ_e. Let $\mu_f = f(\delta_e)$. Then we claim that the net $\lim_{i \in I} k_i$ convergence to μ_f

in $\mathfrak{M}_x(\mathcal{G}, G)$. Notice that for any $\psi(x) \in \mathcal{L}_x(\mathcal{G})$, we have the following:

$$\lim_{i \in I} k_i(\psi(x)) = \lim_{i \in I} \pi_{k_i}(\delta_e)(\psi(x)) = f(\delta_e)(\psi(x)) = \mu_f(\psi(x)).$$

Now, we claim that for any $\nu \in \mathfrak{M}_x(\mathcal{G}, G)$, we have that $f(\nu) = \rho_\mu(\nu)$.

$$f(\nu) = \lim_{i \in I} \pi_{k_i}(\nu) = \lim_{i \in I} [\pi_{k_i} \circ \rho_\nu](\delta_e) = \lim_{i \in I} \rho_{k_i * \nu}(\delta_e) = \lim_{i \in I} [k_i * \nu]$$

Then, since $- * \nu$ is a continuous map from $\mathfrak{M}_x(\mathcal{G}, G) \to \mathfrak{M}_x(\mathcal{G}, G)$, we have that $- * \nu$ commutes with nets. Therefore,

$$\lim_{i \in I} [k_i * \nu] = [\lim_{i \in I} k_i] * \nu = \mu_f * \nu = \rho_{\mu_f}(\nu).$$

Therefore, we conclude that $f = \rho_{\mu_f} = \mu_f * -$. □

Proposition 6.50. *The map* $\rho^{-1}|_{Im(\rho)} : \mathrm{E}(\mathfrak{M}_x(\mathcal{G}, G), \mathrm{conv}(G)) \to \mathfrak{M}_x(\mathcal{G}, G)$ *is well-defined and continuous.*

Proof. The map is well defined since ρ is injective and the image of ρ is precisely the domain. If U is a basic open set in $\mathfrak{M}_x(\mathcal{G}, G)$, then for some $\varphi_1(x), ..., \varphi_n(x) \in \mathcal{L}_x(\mathcal{U})$ and $r_1, ..., r_n, s_1, ..., s_n \in \mathbb{R}$, we have

$$U = \bigcap_{i=1}^{n} \{\mu \in \mathfrak{M}_x(\mathcal{G}, G) : r_i < \mu(\varphi_i(x)) < s_j\}.$$

Then,

$$(\rho^{-1}|_E)^{-1}(U) = \bigcap_{i=1}^{n} \{f : r_i < f(\delta_e)(\varphi_i(x)) < s_i\}.$$

which is a restriction of a basic open subset to $\mathrm{E}(\mathfrak{M}_x(\mathcal{G}, G), \mathrm{conv}(G))$ and thus open.

□

Theorem 6.51. *The map* $\rho : (\mathfrak{M}_x(\mathcal{G}, G), *) \to \mathrm{E}(\mathfrak{M}_x(\mathcal{G}, G), \mathrm{conv}(G))$ *is a homeo-*

morphism which respects the semigroup operation, and therefore an isomorphism.

Proof. ρ is a homeomorphism since $\rho^{-1}|_E$ is a continuous bijection between compact Hausdorff spaces. Now, notice that $\rho(\mu*\nu)(\lambda) = (\mu*\nu)*\lambda = \mu*(\nu*\lambda) = \rho_\mu(\nu*\lambda) = \rho_\mu \circ \rho_\nu(\lambda)$. We conclude that $\rho(\mu*\nu) = \rho_\mu \circ \rho_\nu$. $\square$

BIBLIOGRAPHY

1. Adler, Hans, Enrique Casanovas, and Anand Pillay. "Generic stability and stability." *The Journal of Symbolic Logic* 79.1 (2014): 179-185.

2. Ajtai, Miklós, János Komlós, and Endre Szemerédi. "A note on Ramsey numbers." *Journal of Combinatorial Theory, Series A* 29.3 (1980): 354-360.

3. Baudisch, Andreas. "Generic variations of models of T." *The Journal of Symbolic Logic* 67.3 (2002): 1025-1038.

4. Ben Yaacov, Itaï. "Continuous and random Vapnik-Chervonenkis classes." *Israel Journal of Mathematics* 173.1 (2009): 309-333.

5. Bourgain, Jean, David H. Fremlin, and Michael Talagrand. "Pointwise compact sets of Baire-measurable functions." *American Journal of Mathematics* 100.4 (1978): 845-886.

6. Chernikov, Artem, and Kyle Gannon. "Definable convolution and idempotent Keisler measures." *arXiv preprint* arXiv:2004.10378 (2020).

7. Chernikov, Artem, Anand Pillay, and Pierre Simon. "External definability and groups in NIP theories." *Journal of the London Mathematical Society* 90.1 (2014): 213-240.

8. Chernikov, Artem, and Nicholas Ramsey. "On model-theoretic tree properties." *Journal of Mathematical Logic* 16.02 (2016): 1650009.

9. Chernikov, Artem, and Pierre Simon. "Definably amenable NIP groups." *Journal of the American Mathematical Society* 31.3 (2018): 609-641.

10. Chernikov, Artem, and Sergei Starchenko. "Definable regularity lemmas for NIP hypergraphs." *arXiv preprint* arXiv:1607.07701 (2016).

11. Chernikov, Artem, and Sergei Starchenko. "Regularity lemma for distal structures." *arXiv preprint* arXiv:1507.01482 (2015).

12. Cohen, Paul J. "On a conjecture of Littlewood and idempotent measures." *American Journal of Mathematics* 82.2 (1960): 191-212.

13. Conant, Gabriel. "An axiomatic approach to free amalgamation." *The Journal of Symbolic Logic* 82.2 (2017): 648-671.

14. Conant, Gabriel, and Kyle Gannon. "Remarks on generic stability in independent theories." *Annals of Pure and Applied Logic* 171.2 (2020): 102736.

15. Conant, Gabriel, and Anand Pillay. "Pseudofinite groups and VC-dimension." *arXiv preprint* arXiv:1802.03361 (2018).

16. Conant, Gabriel, Anand Pillay, and Caroline Terry. "A group version of stable regularity." *Mathematical Proceedings of the Cambridge Philosophical Society*. Cambridge University Press, 2018.

17. Conant, Gabriel, Anand Pillay, and Caroline Terry. "Structure and regularity for subsets of groups with finite VC-dimension." *arXiv preprint* arXiv:1802.04246 (2018).

18. Conway, John B. *A course in functional analysis*. Vol. 96. Springer Science & Business Media, 2013.

19. Diestel, Joseph. *Sequences and series in Banach spaces*. Vol. 92. Springer Science & Business Media, 2012.

20. Duret, Jean-Louis. "Weakly algebraically closed bodies that are not separately closed have the property of independence." *Model theory of algebra and arithmetic*. Springer, Berlin, Heidelberg, 1980. 136-162.

21. Ellis, David B., Robert Ellis, and Mahesh Nerurkar. "The topological dynamics of semigroup actions." *Transactions of the American Mathematical Society* (2001): 1279-1320.

22. Erdős, Paul. "Remarks on a theorem of Ramsay", *Bull. Res. Council Israel. Sect. F* **7F** (1957/1958), 21–24.

23. Erdős, Paul, and Daniel J. Kleitman. "On coloring graphs to maximize the proportion of multicolored k-edges." *Journal of Combinatorial Theory* 5.2 (1968): 164-169.

24. Erdős, P., and C. Ambrose Rogers. "The construction of certain graphs." *Canadian Journal of Mathematics* 14 (1962): 702-707.

25. Gannon, Kyle. "Local Keisler measures and NIP formulas." *The Journal of Symbolic Logic* 84.3 (2019): 1279-1292.

26. García, Darío, Alf Onshuus, and Alexander Usvyatsov. "Generic stability, forking, and -forking." *Transactions of the American Mathematical Society* 365.1 (2013): 1-22.

27. Glasner, E. "On tame dynamical systems." *Colloquium Mathematicum*. Vol. 105. No. 2. 2006.

28. Glicksberg, Irving. "Weak compactness and separate continuity." *Pacific Journal of Mathematics* 11.1 (1961): 205-214.

29. Glicksberg, Irving. ”Convolution semigroups of measures.” *Pacific Journal of Mathematics* 9 (1959): 51-67.

30. Hrushovski, Ehud. “Pseudo-finite fields and related structures.” *Model theory and applications* 11 (1991): 151-212.

31. Hrushovski, Ehud, Ya?acov Peterzil, and Anand Pillay. “Groups, measures, and the NIP.” *Journal of the American Mathematical Society* 21.2 (2008): 563-596.

32. Hrushovski, Ehud, and Anand Pillay. “On NIP and invariant measures.” *arXiv preprint* arXiv:0710.2330 (2007).

33. Hrushovski, Ehud, Anand Pillay, and Pierre Simon. “Generically stable and smooth measures in NIP theories.” *Transactions of the American Mathematical Society* 365.5 (2013): 2341-2366.

34. Ibarlucía, Tomás. “The dynamical hierarchy for Roelcke precompact Polish groups.” *Israel Journal of Mathematics* 215.2 (2016): 965-1009.

35. Kadets, Vladimir. *A Course in Functional Analysis and Measure Theory*. Cham: Springer International Publishing, 2018.

36. Keisler, H. Jerome. “Measures and forking.” *Annals of Pure and Applied Logic* 34.2 (1987): 119-169.

37. Khanaki, Karim. “NIP formulas and Baire 1 definability.” *arXiv preprint arXiv:1703.08731* (2017).

38. Khanaki, Karim, and Anand Pillay. ”Remarks on the NIP in a model.” *Mathematical Logic Quarterly* 64.6 (2018): 429-434.

39. Kim, Jeong Han. “The Ramsey number $R(3,t)$ has order of magnitude $t^2/\log t$.” *Random Structures & Algorithms* 7.3 (1995): 173-207.

40. Laskowski, Michael C. “Vapnik-Chervonenkis classes of definable sets.” *Journal of the London Mathematical Society* 2.2 (1992): 377-384.

41. Malliaris, Maryanthe, and Anand Pillay. “The stable regularity lemma revisited.” *Proceedings of the American Mathematical Society* 144.4 (2016): 1761-1765.

42. Malliaris, Maryanthe, and Saharon Shelah. “Regularity lemmas for stable graphs.” *Transactions of the American Mathematical Society* 366.3 (2014): 1551-1585.

43. Nešetřil, Jaroslav and Vojtěch Rödl. “The Ramsey property for graphs with forbidden complete subgraphs.”, *Journal of Combinatorial Theory, Series B* 20.3 (1976): 243-249.

44. Nešetřil, Jaroslav and Vojtěch Rödl. “Ramsey theorem for classes of hypergraphs with forbidden complete subhypergraphs.” *Czechoslovak Mathematical Journal* 29.2 (1979): 202-218.

45. Newelski, Ludomir. “Model theoretic aspects of the Ellis semigroup.” *Israel Journal of Mathematics* 190.1 (2012): 477-507.

46. Newelski, Ludomir. “On type definable subgroups of a stable group.” *Notre Dame Journal of Formal Logic* 32.2 (1991): 173-187.

47. Newelski, Ludomir. “Topological dynamics of definable group actions.” *The Journal of Symbolic Logic* 74.1 (2009): 50-72.

48. Pillay, Anand. *An introduction to stability theory*, Oxford Logic Guides, vol. 8, The Clarendon Press Oxford University Press, New York, 1983.

49. Pillay, Anand. “Topological dynamics and definable groups.” *The Journal of Symbolic Logic* 78.2 (2013): 657-666.

50. Pillay, Anand, and Predrag Tanovic. “Generic stability, regularity, and quasiminimality.” *Models, logics, and higher-dimensional categories* 53 (2011): 189-211.

51. Poizat, Bruno. *Stable groups.* Vol. 87. American Mathematical Soc., 2001.

52. Pym, John S. ”Idempotent probability measures on compact semitopological semigroups.” *Proceedings of the American Mathematical Society* 21.2 (1969): 499-501.

53. Pym, John S. ”Weakly separately continuous measure algebras.” *Mathematische Annalen* 175.3 (1967): 207-219.

54. Rao, KPS Bhaskara, and M. Bhaskara Rao. *Theory of charges: a study of finitely additive measures.* Academic Press, 1983.

55. Rudin, Walter. “Idempotent measures on Abelian groups.” *Pacific Journal of Mathematics* 9.1 (1959): 195-209.

56. Semadeni, Zbigniew. *Banach spaces of continuous functions.* Vol. 1. PWN Polish Scientific Publishers, 1971.

57. Shelah, Saharon. “Toward classifying unstable theories.” *Annals of Pure and Applied Logic* 80 (1996), no. 3, 229–255.

58. Shelah, Saharon.“Classification theory for elementary classes with the dependence property—a modest beginning”, *Scientiae Mathematicae Japonicae.* **59** (2004), no. 2, 265–316, Special issue on set theory and algebraic model theory.

59. Simon, Pierre. *A guide to NIP theories.* Cambridge University Press, 2015.

60. Simon, Pierre. “Distal and non-distal NIP theories.” *Annals of Pure and Applied Logic* 164.3 (2013): 294-318.

61. Simon, Pierre. “Invariant types in NIP theories.” *Journal of Mathematical Logic* 15.02 (2015): 1550006.

62. Simon, Pierre. “Rosenthal compacta and NIP formulas.” *arXiv preprint arXiv:1407.5761* (2014).

63. Starchenko, Sergei. “NIP, Keisler measures and combinatorics.” *Séminaire Bourbaki* (2016): 68.

64. Usvyatsov, Alexander. “On generically stable types in dependent theories.” *The Journal of Symbolic Logic* 74.1 (2009): 216-250.

65. Vapnik, Vladimir N., and A. Ya Chervonenkis. “On the uniform convergence of relative frequencies of events to their probabilities.” *Measures of complexity.* Springer, Cham, 2015. 11-30.

66. Wagner, Frank, and Frank Olaf Wagner. *Stable groups.* Vol. 240. Cambridge University Press, 1997.

67. Wendel, J. G. “Haar measure and the semigroup of measures on a compact group.” *Proceedings of the American Mathematical Society* 5.6 (1954): 923-929.

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