MEASURES AND STABILITY IN A MODEL

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Abstract. We prove a local version of Fubini’s theorem when a formula is stable in a model. This follows quickly from the observation that if a formula is stable in a model, then every local Keisler measure is the sum of (at most countably many) “weighted” types.

1. Introduction

We prove what is stated in the abstract. We begin with some notation. Let $x, y$ be tuples and let $\varphi(x; y)$ be a partitioned formula in a language $L$ with variables $x$ and parameters $y$. Let $\varphi^*(x; y)$ be the same formula as $\varphi(x; y)$, but with exchanged roles for the variables and parameters. We recall the definition of stable in a model:

**Definition 1.1.** A formula $\varphi(x; y)$ is stable in an $L$-structure $M$ if for any two sequences $(a_n)_{n \in \mathbb{N}}, (b_m)_{m \in \mathbb{N}}$ where $a_n \in M^x$ and $b_m \in M^y$, we have that

$$\lim_{m} \lim_{n} \varphi(a_n, b_m) = \lim_{n} \lim_{m} \varphi(a_n, b_m),$$

provided both limits exist (where $\varphi(a_n, b_m) = 1$ if $M \models \varphi(a_n, b_m)$ and $\varphi(a_n, b_m) = 0$ otherwise).

Let $S_{\varphi}(M)$ be the space of $\varphi$-types with parameters from $M$. Let $B_{\varphi}(M)$ be the Boolean algebra of definable subsets of $M$ generated by $\{\varphi(x, b) : b \in M\}$. We will routinely identify definable sets with the formulas which define them. A $\varphi$-formula is an element of $B_{\varphi}(M)$. Likewise, we have analogous definitions for $S_{\varphi^*}(M)$ and $B_{\varphi^*}(M)$. A $\varphi^*$-definition for a type $p$ in $S_{\varphi^*}(M)$ is a $\varphi^*$-formula, $d_p^\varphi(y)$, such that for each $b \in M^y$, $\varphi(x, b) \in p$ if and only if $M \models d_p^\varphi(b)$. Finally, we let $M_{\varphi}(M)$ and $M_{\varphi^*}(M)$ denote the spaces of finitely additive probability measures on $B_{\varphi}(M)$ and $B_{\varphi^*}(M)$ respectively. We recall that we can identify a measure in each of these spaces canonically with a regular Borel probability measure on their corresponding type space, e.g. $M_{\varphi}(M)$ is in canonical correspondence with regular Borel probability measures on $S_{\varphi}(M)$.

In [1], Ben Yaacov established a surprising connection between functional analysis and local stability. In particular, he proved the following using Grothendieck’s double limit theorem [2]:

**Theorem 1.2** ([1]). Assume that $\varphi(x; y)$ is stable in $M$, $p \in S_{\varphi}(M)$, and $q \in S_{\varphi^*}(M)$. Then $p$ has a $\varphi^*$-definition $d_p^\varphi(y)$, $q$ has a $\varphi$-definition $d_q^\varphi(x)$, and $d_p^\varphi(y) \in q$ if and only if $d_q^\varphi(x) \in p$.

It is natural to ask whether the statement above has an analogue for local Keisler measures. However, we will show that in this context, finitely additive probability measures are simply
“sums of types”. Recall that Keisler showed in [3] that if a formula \( \varphi(x; y) \) is \( k \)-stable for some \( k \), i.e. there do not exist \( a_1, ..., a_k, b_1, ..., b_k \) so that \( M \models \varphi(a_i, b_j) \) if and only if \( i < j \), then every finitely additive probability measure on \( \mathbb{B}_\varphi(M) \) is at most a countable sum of “weighted” types. From Theorem 1.2 and an application of the Sobczyk-Hammer Decomposition Theorem, we prove the following.

**Theorem 1.3.** Let \( \varphi(x; y) \) be stable in \( M \) and assume that \( \mu \in \mathcal{M}_\varphi(M) \). Then \( \mu = \sum_{i \in I} r_i \delta_{p_i} \) where \( I \) is some initial segment of \( \mathbb{N} \), each \( p_i \) is in \( S_\varphi(M) \), \( \delta_{p_i} \) is the corresponding Dirac measure at \( p_i \), each \( r_i \) is a positive real number (strictly greater than 0), and \( \sum_{i \in I} r_i = 1 \).

To be clear, a formula \( \varphi \) is stable in \( M \) if and only if \( \varphi^* \) is stable in \( M \). Therefore, Theorem 1.3 can also be applied to \( \mathcal{M}_\varphi^*(M) \). We note that from this description of measures in this context, we have almost for free the following corollary,

**Corollary 1.4 (Local Fubini).** Assume that \( \varphi(x; y) \) is stable in \( M \). Let \( \mu \in \mathcal{M}_\varphi(M) \) and \( \nu \in \mathcal{M}_\varphi^*(M) \). Let \( F_\mu^\varphi : S_\varphi(M) \to \mathbb{R} \) via \( F_\mu^\varphi(q) = \mu(d_\varphi^p(x)) \). Let \( F_\nu^\varphi^* : S_\varphi(M) \to \mathbb{R} \) via \( \nu(d_\varphi^p(y)) \). Then the maps \( F_\mu^\varphi \) and \( F_\nu^\varphi^* \) are measurable and

\[
\int_{p \in S_\varphi(M)} F_\nu^\varphi^*(p) d\mu = \int_{q \in S_\varphi^*(M)} F_\mu^\varphi(q) d\nu,
\]

where we have identified \( \mu \) and \( \nu \) with their corresponding regular Borel measures on \( S_\varphi(M) \) and \( S_\varphi^*(M) \) respectively.

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## 2. Local Measures and Stability in a Model

The proof of Theorem 1.3 uses the Sobczyk-Hammer Decomposition theorem for positive, bounded, finitely additive charges. We will state the theorem for finitely additive probability measures. Before referencing this theorem, we establish a convention and recall two kinds of measures.

**Remark 2.1.** Throughout this section, we will say that \( \mathbb{B} \) is a Boolean algebra on \( X \) if \( \mathbb{B} \subset \mathcal{P}(X) \) and \( \mathbb{B} \) is a Boolean algebra under the standard interpretation of of union, intersection, complement, etc. We also remark that \( X \) and \( \emptyset \) are elements of \( \mathbb{B} \).

**Definition 2.2.** Let \( \mathbb{B} \) be a Boolean algebra on a set \( X \) and \( \mu \) be a finitely additive probability measure on \( \mathbb{B} \).

1. We say that \( \mu \) is strongly continuous on \( \mathbb{B} \) if for all \( \epsilon > 0 \) there exist \( F_1, ..., F_n \in \mathbb{B} \) such that \( \{ F_i \}_{i=1}^n \) form a partition of \( X \) and for each \( i \), \( \mu(F_i) < \epsilon \).
2. We say that \( \mu \) is 0-1 valued on \( \mathbb{B} \) if for every \( F \) in \( \mathbb{B} \), \( \mu(F) = 0 \) or \( \mu(F) = 1 \).

**Theorem 2.3 (Sobczyk-Hammer Decomposition Theorem [4]).** Let \( \mathbb{B} \) be a Boolean algebra on \( X \) and \( \mu \) be a finitely additive probability measure on \( \mathbb{B} \). Then, there exists an initial segment \( I \) of \( \mathbb{N} \), a sequence of finitely additive probability measures \( (\mu_i)_{i \in I} \), and a sequence of positive real numbers \( (r_i)_{i \in I} \) (with some \( r_i \) potentially 0), with the following properties,

1. \( \mu_0 \) is strongly continuous on \( \mathbb{B} \),
2. \( \mu_i \) is 0-1 valued on \( \mathbb{B} \) for every \( i \geq 1 \),

\( I \) need not be a proper initial segment. \( I = \{0, ..., n \} \) for some \( n \) or \( I = \mathbb{N} \)
(iii) \( \sum_{i \in I} r_i = 1 \), and
(iv) \( \mu = \sum_{i \in I} r_i \mu_i = \sum_{i \in I, r_i \neq 0} r_i \mu_i \).

Further, the decomposition in (iv) is unique.

The Sobczyk-Hammer Decomposition theorem allows us to decompose any finitely additive probability measure into a single strongly continuous measure and at most countably many 0-1 valued measures. We will show that if \( \varphi(x; y) \) is stable in \( M \), then there do not exist any strongly continuous measures on \( \mathbb{B}_\varphi(M) \). Therefore, every finitely additive probability measure will be the “weighted sum” of at most countably many types.

2.1. Proof of Theorem 1.3.

**Definition 2.4** (2-Tree). Let \( \mathbb{B} \) be a Boolean algebra on a set \( X \). We say that \( \mathbb{B} \) has a 2-tree if there exists \( T \in \mathcal{P}(\mathbb{B}) \) such that \( (T, \supseteq) \) is an infinite, complete, binary tree, and if \( A, C \in T, \ A \not\supseteq C, \) and \( C \not\supseteq A \), then \( A \cap C = \emptyset \).

**Fact 2.5.** Let \( \mathbb{B} \) be a Boolean algebra on a set \( X \) and assume that \( \mathbb{B} \) has a 2-tree. Then \( |\text{Ult}(\mathbb{B})| \geq 2^{\aleph_0} \) where \( \text{Ult}(\mathbb{B}) \) is the set of ultrafilters on \( \mathbb{B} \).

**Proof.** Let \( \gamma \) be a path in \( T \) and let \( A_\gamma = \{ B \in T : B \in \gamma \} \). Clearly, \( A_\gamma \) has the finite intersection property (since if \( B, C \in A_\gamma \), then either \( B \subset C \) or \( C \subset B \)). Then, \( A_\gamma \) can be extended to an ultrafilter over \( \mathbb{B} \). For each path \( \gamma \), let \( U_\gamma \) be an ultrafilter extending \( A_\gamma \).

Now, assume that \( \delta, \gamma \) are two different paths in \( T \). Assume that \( U_\gamma = U_\delta = U \). Since \( \gamma, \delta \) are two separate paths, there exists \( A \in \gamma \) and \( B \in \delta \) such that \( A \not\subset B \) and \( B \not\subset A \). Then \( A \cap B = \emptyset \) and therefore \( U \) cannot extend both \( A_\gamma \) and \( A_\delta \). Therefore, we have at least \( 2^{\aleph_0} \) many ultrafilters on \( \mathbb{B} \).

**Theorem 2.6.** Let \( \mathbb{B} \) be a Boolean algebra on a set \( X \). Assume that there exists a strongly continuous measure \( \mu \) over \( \mathbb{B} \). Then \( \mathbb{B} \) has a 2-tree.

**Proof.** Using \( \mu \), we will build a 2-tree. We build this tree in steps:

Stage 0: Let \( T_0 = \{ X \} \).

Stage \( n + 1 \): We construct a tree of height \( n + 1 \). Assume that \( T_n \) is a (complete) binary tree of height \( n \) such that for each \( A \in T_n \), \( \mu(A) > 0 \). Assume furthermore that if \( A, B \in T \) and \( A \not\supseteq B \) and \( B \not\supseteq A \), then \( A \cap B = \emptyset \). We will construct \( T_{n+1} \) by adding two children to each leaf. Let \( \mathbb{L}_n \) be the collection of leaves on \( T_n \). By assumption, each node of our tree has positive measure, therefore for each \( L \in \mathbb{L}_n \), \( \mu(L) > 0 \). Let \( \epsilon = \min\{\mu(L) : L \in \mathbb{L}_n\} \). Now, since \( \mu \) is strongly continuous, there exist \( H_1, \ldots, H_m \in \mathbb{B} \) such that \( \mathbb{H} = \{H_1, \ldots, H_m\} \) partitions \( X \) and \( \mu(H) < \epsilon \) for each \( H \in \mathbb{H} \). Now fix a leaf \( L_i \). Consider \( L_i \cap \mathbb{H} = \{L_i \cap H_j : H_j \in \mathbb{H}\} \). We notice that \( L_i \cap \mathbb{H} \) forms a partition of \( L_i \). Therefore, we have that

\[
0 < \mu(L_i) = \sum_{K \in L_i \cap \mathbb{H}} \mu(K) = \mu \left( \bigcup_{K \in L_i \cap \mathbb{H}} K \right) = \sum_{K \in L_i \cap \mathbb{H}} \mu(K).
\]

Hence, there exists \( K_r \in L_i \cap \mathbb{H} \) such that \( \mu(K_r) > 0 \). Furthermore, we note that

\[
\mu(K_r) = \mu(L_i \cap H_r) \leq \mu(H_r) < \epsilon \leq \frac{L_i}{2}.
\]

By the above, we note that \( \mu(K_r) < \mu(L_i) \). Therefore there must exist some \( K_l \in L_i \cap \mathbb{H} \) such that \( K_l \neq K_r \) and \( \mu(K_l) > 0 \). We now add \( K_r, K_l \) as children for \( L_i \). Let \( T_{n+1} \) be the tree constructed after repeating this process for each \( L \in \mathbb{L}_n \). Clearly, \( T_{n+1} \) is a binary tree of height \( n + 1 \) such that for each \( A \in T_{n+1}, \mu(A) > 0 \).
Now let $T = \bigcup_{n \geq 0} T_n$. $T$ is clearly a 2-tree by construction.

**Definition 2.7.** Let $\text{Red}_\varphi(M)$ be the reduct of $M$ to language $L_\varphi = \{ \varphi \}$. Then, we say that a subset $N$ of $M$ is a $\varphi$-substructure of $M$, written $N \prec_\varphi M$, if $\text{Red}_\varphi(N) \prec \text{Red}_\varphi(M)$.

**Theorem 2.8.** Assume that there exists a strongly continuous measure over $B$. By Theorem 2.6 and Fact 2.5, we know that there exists a countable subalgebra $B \subset C$ in the infinite binary tree). Let $C \subset M$ such that for each $B \in B$, $b_1, \ldots, b_n$ in $C$ such that $B$ is an element of the boolean algebra generated by $\{ \varphi(x; b_i) : i \leq n \}$. Notice that since $B_0$ is countable, we can choose $C$ to be countable. By the Downward Löwenheim-Skolem theorem, there exists $N \prec_\varphi M$ such that $C \subset N$ and $|N| = \aleph_0$. Then,

$$2^{\aleph_0} \leq |\text{Ult}(B_\varphi(C))| \leq |\text{Ult}(B_\varphi(N))| = |\varphi(N)|.$$

However, by stability, every $\varphi$-type over $N$ is definable by a $\varphi^*$-formula with parameters from $N$. Since $|N| = \aleph_0$, there are only countably many $\varphi^*$-formulas. Therefore, not every $\varphi$-type is definable. Hence, $\varphi(x; y)$ is unstable in $N$. Since $N \prec_\varphi M$, by definition we have $N \subset M$ and so $\varphi(x; y)$ is unstable in $M$. □

**Corollary 2.9.** Let $\varphi(x; y)$ be stable in $M$ and let $\mu$ be a finitely additive probability measure on $B_\varphi$. Then there exists an initial segment $I$ of $N$ such that $\mu = \sum_{i \in I} r_i \delta_p$, where $r_i = 1$, and each $r_i > 0$.

**Proof.** By the Spector-Heckman Decomposition Theorem, any finitely additive measure on $B_\varphi$ is the sum of a strongly continuous measure and countably many 0-1 valued measures. Since there are no strongly continuous measures on $B_\varphi$, every measure is the “weighted” sum of at most countably 0-1 valued measures. Finally, notice that a 0-1 valued measure is of the form $\delta_p$ for some $p \in S_\varphi(M)$. □

### 2.2. Proof of Corollary 1.4

In this subsection, we prove the local version of Fubini’s theorem.

**Proposition 2.10.** Assume that $\varphi(x; y)$ is stable in $M$. Then the maps $F_\varphi^\varphi, F_\varphi^\varphi^*$ as defined in Corollary 1.4 are well defined and measurable. In particular, they are continuous.

**Proof.** By symmetry, we only need to show the proposition for $F_\varphi^\varphi$. By Theorem 1.3, $\mu = \sum_{i \in I} r_i \delta_p$. Since every type is definable, we know that for each $p \in S_\varphi(M)$, the map $F_\varphi^\varphi : S_\varphi^*(M) \to \mathbb{R}$ is continuous. Notice that $F_\varphi^\varphi = \sum_{i \in I} r_i F_\varphi^\varphi$. If $I = \{0, \ldots, n\}$, then $F_\varphi^\varphi$ is clearly continuous. If $I = \mathbb{N}$, let $g_N = \sum_{i = 1}^N r_i F_\varphi^\varphi$. Then, each $g_N$ is continuous and the sequence $(g_N)_{N \in \mathbb{N}}$ converges uniformly to $F_\varphi^\varphi$, so $F_\varphi^\varphi$ is continuous. □

**Proposition 2.11.** Assume that $\varphi(x; y)$ is stable in $M$, $p \in S_\varphi(M)$, and $\nu \in \mathfrak{M}_\varphi^*(M)$. Then,

$$\int_{q \in S_\varphi^*(M)} F_\varphi^\varphi(q) d\nu = \int_{r \in S_\varphi(M)} F_\varphi^\varphi^*(r) d\delta_p$$

**Proof.** We compute:

$$\int_{r \in S_\varphi(M)} F_\varphi^\varphi^*(r) d\delta_p = F_\varphi^\varphi^*(p) = \nu(d_\varphi^\varphi^*(y))$$
Likewise;

\[
\int_{q \in S_{\varphi^*}(M)} F_{\delta_p}(q) d\nu = \nu(\{ q \in S_{\varphi^*}(M) : F_{\delta_p}(q) = 1 \}) = \nu(\{ q \in S_{\varphi^*}(M) : \delta_q(d^p_{\varphi^*}(y)) = 1 \})
\]

\[
= \nu(\{ q \in S_{\varphi^*}(M) : d^p_{\varphi^*}(x) \in p \}) = \nu(\{ q \in S_{\varphi^*}(M) : d^p_{\varphi^*}(y) \in q \}) = \nu(d^p_{\varphi^*}(y)).
\]

\[
\square
\]

**Theorem 2.12.** Assume that \( \varphi(x; y) \) is stable in \( M \). Let \( \mu \in \mathcal{M}_{\varphi}(M) \) and \( \nu \in \mathcal{M}_{\varphi^*}(M) \).

Then,

\[
\int_{q \in S_{\varphi^*}(M)} F_{\mu}^\varphi(q) d\mu = \int_{p \in S_{\varphi}} F_{\nu}^{\varphi^*}(p) d\mu
\]

Proof. By stability in \( M \), \( \mu = \sum_{i=1}^r r_i \delta_{p_i} \). Then, we compute;

\[
\int_{q \in S_{\varphi^*}(M)} F_{\mu}^\varphi(q) d\nu = \lim_{N \to \infty} \int_{q \in S_{\varphi^*}(M)} \sum_{i=1}^N r_i F_{\mu}^\varphi(p) d\nu = \lim_{N \to \infty} \sum_{i=1}^N r_i \int_{p \in S_{\varphi}(M)} F_{\nu}^{\varphi^*}(p) d\delta_{p_i}
\]

\[
= \lim_{N \to \infty} \int_{p \in S_{\varphi}(M)} F_{\nu}^{\varphi^*}(p) d\left( \sum_{i=1}^N r_i \delta_{p_i} \right) = \int_{p \in S_{\varphi}(M)} F_{\nu}^{\varphi^*}(p) d\mu
\]

\[
\square
\]

**REFERENCES**


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