MATH 7410 (TOTAL POSITIVITY): HOMEWORK #2

• The homework is due *at 3pm on Tuesday, November 25th, 2025* on Gradescope: `https://www.gradescope.com/courses/1133762`. No late homework is accepted.
• Feel free to collaborate and use any sources. For each problem, all sources and collaborators must be clearly listed.
• Please use LaTeX. Pictures can be drawn by hand and scanned.
• Solve each of the below problems.

Because $\mathbb{R} \subset \mathbb{C}$, we naturally have $\mathrm{Gr}_{\mathbb{R}}(k,n) \subset \mathrm{Gr}_{\mathbb{C}}(k,n)$ consisting of those subspaces $V \subset \mathbb{C}^n$ contained inside $\mathbb{R}^n \subset \mathbb{C}^n$. Also, recall that the cyclic shift map $\sigma : \mathrm{Gr}(k,n) \to \mathrm{Gr}(k,n)$ takes a matrix with columns $(u_1,\dots,u_n)$ to a matrix with columns $((-1)^{k-1}u_n, u_1, \dots, u_{n-1})$.

Problem 1.

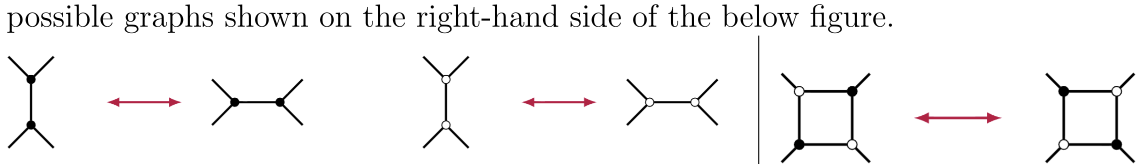
(i) Suppose that $V \in \mathrm{Gr}_{\mathbb{C}}(k,n)$ is the row span of a complex $k \times n$ matrix M. Find a necessary and sufficient condition for V to belong to $\mathrm{Gr}_{\mathbb{R}}(k,n)$ in terms of the maximal minors of M.

(ii) Find all *cyclically symmetric points* inside $\mathrm{Gr}_{\mathbb{C}}(k,n)$, i.e., points $V \in \mathrm{Gr}_{\mathbb{C}}(k,n)$ such that $\sigma(V) = V$.

(iii) Which of these points belong to $\mathrm{Gr}_{\mathbb{R}}(k,n)$? [Mandatory: experiment on a computer and make an educated guess. Optional: If you can, give a proof.]

Problem 2. Find the number of d-dimensional cells in $\mathrm{Gr}_{\geq 0}(2,n)$ for all n and d.

Problem 3. Recall that for a given bounded affine permutation f, all reduced plabic graphs for f are related by contraction-uncontraction and square moves, shown below. Call two plabic graphs *CU-equivalent* if they are related by a sequence of only contraction-uncontraction moves. For each $n \geq 4$, find the number C_n of CU-equivalence classes of reduced plabic graphs for the top cell $f = f_{2,n}$ in $\mathrm{Gr}_{\geq 0}(2,n)$. For example, $C_2 = 2$, with the two possible graphs shown on the right-hand side of the below figure.



contraction-uncontraction moves | square move

For an element $M \in \mathrm{Gr}(k,n)$ represented by a matrix with columns $u_1,\dots,u_n$, we have defined u_i for $i \in \mathbb{Z}$ by the condition that $u_{i+n} = u_i$. Recall that $f_V : \mathbb{Z} \to \mathbb{Z}$ is given by

$$f_M(i) = \min\{j \geq i \mid u_i \in \mathrm{Span}(u_{i+1},\dots,u_j)\}.$$

Problem 4. Prove that f_M is a bounded affine permutation.

For a linear subspace $M \subset \mathbb{R}^n$, we denote by $M^\perp$ its orthogonal complement with respect to the standard scalar product on $\mathbb{R}^n$.

Problem 5.

(i) Express the Plücker coordinates of $M^\perp \in \mathrm{Gr}(n-k,n)$ in terms of that of $M \in \mathrm{Gr}(k,n)$.

(ii) Express $f_{M^\perp}$ in terms of f_M.