MATH 7410 (TOTAL POSITIVITY): HOMEWORK #1

• The homework is due in class on *Tuesday, September 30th, 2025* on Gradescope: `https://www.gradescope.com/courses/1133762`. No late homework is accepted.
• Feel free to collaborate and use any sources. For each problem, all sources and collaborators must be clearly listed.
• Please use LaTeX. Pictures can be drawn by hand and scanned.
• Solve each of the below problems.

Problem 1. Recall that $\leq$ denotes the Bruhat order on $\mathfrak{S}_n$.[1] For $a, b \in [n]$ and $w \in \mathfrak{S}_n$, let $\hat{r}_{a,b}(w) := |\{i \geq a \mid w(i) \leq b\}|$. Show that for $v, w \in \mathfrak{S}_n$, we have $v \leq w$ if and only if $\hat{r}_{a,b}(v) \leq \hat{r}_{a,b}(w)$ for all $a, b \in [n]$.

Problem 2. Let $w = s_{i_1} \cdots s_{i_l}$ be a reduced word, and let $M := x_{i_1}(t_1) \cdots x_{i_l}(t_l)$ for some $t_1, \ldots, t_l \in \mathbb{R}_{>0}$.
(a) Express t_l in terms of minors of M.
(b) Deduce that the map $(\mathbb{R}_{>0})^l \to U^w_{>0}$ sending $(t_1, \ldots t_l) \mapsto x_{i_1}(t_1) \cdots x_{i_l}(t_l)$ is injective.

Problem 3. Let $G := \mathrm{GL}_n$ be the set of invertible matrices and let $B_- \subset \mathrm{GL}_n$ be the set of lower-triangular matrices (with nonzero entries on the diagonal). In this problem, you will prove the *Bruhat decomposition* of G. Recall that $\hat{r}_{ij}(M)$ denotes the rank of the NE-aligned submatrix $M_{[i,n],[j]}$ of M, and that $\hat{r}_{ij}(w) := \hat{r}_{ij}(P_w)$ counts the number of 1's in the corresponding NE-aligned rectangle of the permutation matrix P_w of w.
(a) Show that any matrix $M \in G$ belongs to $B_- P_w B_- := \{b_1 P_w b_2 \mid b_1, b_2 \in B_-\}$ for some $w \in \mathfrak{S}_n$.
(b) Show that for each $w \in \mathfrak{S}_n$, we have

$$B_- P_w B_- = \{M \in G \mid \hat{r}_{ij}(M) = \hat{r}_{ij}(w)\}$$

(c) Conclude that

$$G = \bigsqcup_{w \in \mathfrak{S}_n} B_- P_w B_-.$$

Problem 4. Let $M \in U_{\geq 0}$. Let $w \in \mathfrak{S}_n$ be a permutation such that $M \in B_- P_w B_-$ (so w exists and is unique by Problem 3).
(a) Let $i \in [n-1]$ be such that $\ell(s_i w) < \ell(w)$. Using Problem 2(a), show that there exists $t > 0$ such that $x_i(-t) M \in U_{\geq 0} \cap B_- P_{s_i w} B_-$.
(b) Conclude that any $M \in U_{\geq 0}$ can be written as a product of the generators $x_i(t)$ for $t > 0$.[2]
(c) In class, we showed that $U^w_{>0} \subset B_- P_w B_-$. Conclude that $U^w_{>0} = U_{\geq 0} \cap B_- P_w B_-$.

[1] You are allowed to use any of the three characterizations of the Bruhat order whose equivalence we proved in class.
[2] This is the Loewner–Whitney theorem that I mentioned in class.